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INERTIAL EFFECT ON CONVECTIVE FLOW IN A PASSIVE MUSHY LAYER

DAMBARU BHATTA*, DANIEL N. RIAHI AND MALLIKARJUNAIAH S. MUDDAMALLAPPA

Abstract. Here we consider the inertial effect in a horizontal mushy layer during solidification of a binary alloy. Using perturbation technique, we obtain two systems, one of zero order and the other of first order. We consider a mushy layer with an impermeable mush-liquid interface and of constant permeability. The analysis reveals that the effect of inertial parameter is stabilizing in the sense that the critical Rayleigh number at the onset of motion increases by the inertial effect.

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1. Introduction

Convection plays a very important role during the solidification process of binary alloys. The final form of the solidified alloy contains impurities known as freckles. When a binary alloy is cooled from below, due to the temperature difference at the bottom, the solidification front becomes morphologically unstable. The result is a mixture of solid-liquid contents better known as mushy layer. This layer is sandwiched between a solid layer at the bottom and a liquid layer at the top. The convective flow with in the mushy layer influences the formation of thread like structures in the final form of the solidified alloy. Therefore, it is of great importance to have fundamental understanding of the freckle formation and means of controlling it. A fairly large number of, theoretical as well as experimental, studies have been devoted to predicting the chimney formation during the solidification process. This kind of heat transfer can be seen in solidification of binary alloys. Many previous studies [1-7] have examined in detail

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*Corresponding author.

about the mechanism of freckle formation during the solidification of multi component alloys. Fowler [1] developed mathematical analysis of freckle formation to predict the criterion for freckling. This prediction is equivalent to the classical Rayleigh number condition for convective instability. Vives and Perry [2] carried out an experimental investigation of solidification of tin and aluminium alloys under the influence of externally imposed magnetic field. They reported that the stationary magnetic field decreases the superheat and increases the rate of solidification. Tait et. al. [3] observed the hexagonal pattern of convection just when the system becomes unstable during their experimental work.

Worster [4, 5] applied linear stability analysis for the two layer model and concluded that the mushy layer mode is responsible for the development of chimneys. Amberg and Homsy [6] studied the simplified mushy layer model with constant permeability. They carried out a weakly nonlinear analysis of simplified mushy layer model that was proposed by Worster [5]. A near eutectic approximation was applied and the limit of large far-field temperature was considered. Such asymptotic limits allowed them to examine the dynamics of mushy layer. A weakly nonlinear analysis of simplified mushy layer model that was proposed in [6] was carried out by Anderson and Worster [7]. A near eutectic approximation was applied and the limit of large far-field temperature was considered. Such asymptotic limits allowed them to examine the dynamics of mushy layer. They also considered the limit of large Stefan number, which enabled them to reach a domain for the existence of the oscillatory mode of convection. Chen [8] carried out laboratory experiments on water-ammonium chloride solution. Riahi [9] studied asymptotically the nonlinear compositional convection in chimneys under the externally imposed magnetic field in vertical direction. It was found that for sufficiently large values of Chandrasekhar number, convection and volume flux in the chimneys decrease with increasing Chandrasekhar number.

Drazin and Reid [10] have presented varous studies done by many researchers. They presented methods and results of thermal convection, rotating and curved flows and parallel shear fows. Development of asymptotic theory of Orr- Sommerfeld equation, applications of linear stability theory and nonlinear thoery of hydrodynamic stability have been presented. In another recent development by Okhuysen and Riahi [11], a weakly nonlinear analysis of buoyant convection in two-layer model was considered. They predicted subcritical down-hexagonal pattern for the case of reactive mushy layer. Authors investigated linear marginal stabilities for magneto-convection cases [12, 13]. However, extensive investigations of the inertial effects on solidification process have not been undertaken until recently. The objective of this paper is to model the convection in the mushy layer model, incorporating the inertial effects. By using both analytical and numerical techniques the governing partial differential equations are solved for the primary disturbance variables. Our analysis revealed that the inertial effects are stabilizing in the sense that the critical Rayleigh number increases as inertial parameter increases.

2. Governing system for the mushy layer

We consider a system governing the mushy layer of thickness d which is cooled from below as shown in the figure 1.

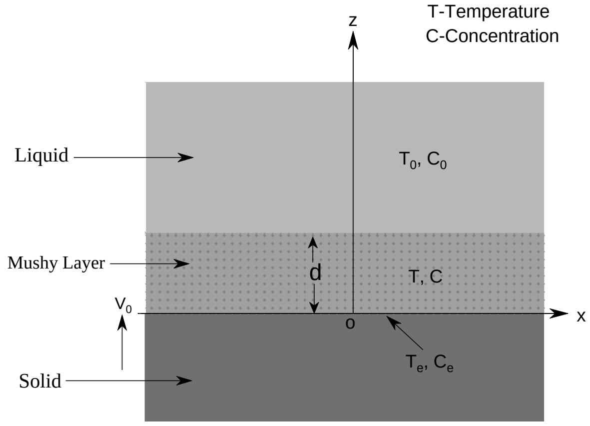

FIGURE 1. Geometry

We use the model proposed by Worster [4]. This system is given by

$$\begin{aligned} \frac{\mu}{\Pi}\overrightarrow{U} + \frac{1}{1-\Phi}\left\{\frac{\partial}{\partial t} + \frac{\overrightarrow{U}\,.\,\nabla}{1-\Phi}\right\}\overrightarrow{U} &= -\nabla p - (\rho-\rho_0)\,g\,\overrightarrow{k} \\ \nabla.\overrightarrow{U} &= 0 \\ \frac{\partial T}{\partial t} + \overrightarrow{U}.\nabla T &= \kappa\nabla^2 T + \frac{l_h}{\gamma}\frac{\partial \Phi}{\partial t} \\ \chi\frac{\partial C}{\partial t} + \overrightarrow{U}.\nabla C &= (C - C_s)\,\frac{\partial \Phi}{\partial t}\,. \end{aligned} \tag{1}$$

Here the equations represent conservation of momentum, conservation of mass, conservation of heat and conservation of solute respectively. Here t, T, κ, γ, l_h, represent time, temperature, thermal diffusivity of the liquid, specific heat of the liquid, latent heat per unit mass respectively. Here $\overrightarrow{U} = U\vec{i} + V\vec{j} + W\vec{k}$ is the liquid flux where U, V are used to denote horizontal components, W denotes the vertical component of $\overrightarrow{U}$ and $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors along x, y, z directions. Also Φ stands for the local solid volume fraction, i.e., $\Phi = 1 - \chi$ where χ is the local liquid volume fraction. C is the composition of the liquid and C_s is the composition of the solid phase. Here μ is used for dynamic viscosity of the liquid, p represents the dynamic pressure, ρ is the density of the liquid and g denotes the acceleration due to gravity. Permeability $\Pi = \Pi(\chi)$ is a function of the local liquid volume fraction, χ.

The boundary conditions are

$$\begin{aligned} T = T_e, \quad W = 0 \qquad\qquad & \text{at} \quad z = 0 \\ T = T_0, \quad \Phi = W = 0 \qquad & \text{at} \quad z = d\,. \end{aligned}$$

Here T_0 denotes the temperature at the mush-liquid interface (at $z = d$), and T_e and C_e represent eutectic temperature and eutectic concentration (at the solid-mush interface, $z = 0$) respectively.

2.1. Nondimensionalization. We nondimensionalize the system in a frame moving with the solidification front at constant speed V_0 and use the following scalings: velocity scale is V_0, i.e., $\overrightarrow{\mathcal{U}} = \frac{\overrightarrow{U}}{V_0}$, length scale is $\frac{\kappa}{V_0}$, time scale is $\frac{\kappa}{V_0^2}$, pressure scale is $\frac{\kappa\mu}{\Pi_0}$, $\Theta = \frac{T-T_0}{\Delta T}$, $\mathcal{K} = \frac{\Pi_0}{\Pi}$ where $\Delta T = T_0 - T_e$,$\Delta C = C_0 - C_e$ and $\Pi_0$is a reference Π. The nondimensional constants appearing in the derivation are Rayleigh number, $\mathcal{R} = \frac{\beta g \Pi_0 \Delta C}{V_0 \mu}$, Stefan number, $\mathcal{S} = \frac{l_h}{\gamma \Delta T}$, concentration ratio, $\mathcal{C} = \frac{C_s - C_0}{\Delta C}$ and inertial parameter, $\mathcal{I} = \frac{\Pi_0}{\mu}$.

Nondimensional system can be expressed as

$$\begin{aligned} \mathcal{K}\overrightarrow{\mathcal{U}} + \nabla\mathcal{P} + \mathcal{R}\Theta\overrightarrow{k} + \frac{\mathcal{I}}{1-\Phi}\left\{\frac{\partial}{\partial t} - \frac{\partial}{\partial z} + \frac{\overrightarrow{\mathcal{U}}\,.\nabla}{1-\Phi}\right\}\overrightarrow{\mathcal{U}} &= \overrightarrow{0} \\ \nabla.\overrightarrow{\mathcal{U}} &= 0 \\ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)[\Theta - \mathcal{S}\Phi] + \overrightarrow{\mathcal{U}}.\nabla\Theta &= \nabla^2\Theta \\ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)[(1-\Phi)\,\Theta + \mathcal{C}\Phi] + \overrightarrow{\mathcal{U}}.\nabla\Theta &= 0 \end{aligned} \tag{2}$$

with boundary conditions:

$$\begin{aligned} \Theta = -1,\ \mathcal{W} = 0 \qquad & \text{at} \quad z = 0 \\ \Theta = \Phi = \mathcal{W} = 0 \qquad & \text{at} \quad z = \delta \end{aligned}$$

where $\mathcal{W}$ denotes the vertical component of $\overrightarrow{\mathcal{U}}$. For this study, we take permeability as constant, i.e., $\mathcal{K} = 1$.

3. Solution Procedure

Assuming solutions of the form

$$\begin{aligned} \Theta(x,y,z,t) &= \theta_b(z) + \epsilon\theta\,(x,y,z,t) \\ \Phi(x,y,z,t) &= \phi_b(z) + \epsilon\phi\,(x,y,z,t) \\ \overrightarrow{\mathcal{U}}(x,y,z,t) &= \overrightarrow{0} + \epsilon\overrightarrow{u}\,(x,y,z,t) \\ \mathcal{P}(x,y,z,t) &= p_b(z) + \epsilon p\,(x,y,z,t) \end{aligned} \tag{3}$$

where θ_b, ϕ_b, p_b are solutions to the steady basic state system (system with no flow) and θ, ϕ, $\overrightarrow{u}$, p are perturbation solutions. ϵ is the perturbation parameter.

3.1. Basic State Solutions. Using (3) in (2) and setting $\epsilon = 0$, we obtain steady basic state system as

$$\frac{d^2\theta_b}{dz^2} + \frac{d\theta_b}{dz} - \mathcal{S}\frac{d\phi_b}{dz} = 0 \tag{4}$$

$$(1-\phi_b)\frac{d\theta_b}{dz} + (\mathcal{C}-\theta_b)\frac{d\phi_b}{dz} = 0 \tag{5}$$

$$\frac{dp_b}{dz} + \mathcal{R}\theta_b = 0 \tag{6}$$

with boundary conditions:

$$\theta_b = -1 \quad \text{at} \quad z = 0$$
$$\theta_b = \phi_b = 0 \quad \text{at} \quad z = \delta.$$

Solutions θ_b and ϕ_b are, respectively, given by (as presented by Worster [5])

$$z = \frac{r_1-\mathcal{C}}{r_1-r_2}\ln\left[\frac{1+r_1}{r_1-\theta_b}\right] + \frac{\mathcal{C}-r_2}{r_1-r_2}\ln\left[\frac{1+r_2}{r_2-\theta_b}\right] \tag{7}$$

and

$$\phi_b = \frac{\theta_b}{\theta_b-\mathcal{C}} \tag{8}$$

where r_1, r_2 are given by

$$r_1 = \frac{\mathcal{C}+\mathcal{S}+\theta_\infty+\sqrt{(\mathcal{C}+\mathcal{S}+\theta_\infty)^2-4\mathcal{C}\theta_\infty}}{2}$$
$$r_2 = \frac{\mathcal{C}+\mathcal{S}+\theta_\infty-\sqrt{(\mathcal{C}+\mathcal{S}+\theta_\infty)^2-4\mathcal{C}\theta_\infty}}{2}.$$

and θ_∞ is the non-dimensional temperature far away from mush/liquid interface. Thickness of the layer can be determined as

$$\delta = \frac{r_1-\mathcal{C}}{r_1-\beta}\ln\left[\frac{1+r_1}{r_1}\right] + \frac{\mathcal{C}-r_2}{r_1-r_2}\ln\left[\frac{1+r_2}{r_2}\right].$$

3.2. Linear Perturbed System. After linearization, the perturbed system is given by

$$\overrightarrow{u} + \nabla p + \mathcal{R}\theta\hat{k} + \frac{\mathcal{I}}{1-\phi_b}\left\{\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right\}\overrightarrow{u} = \overrightarrow{0} \tag{9}$$

$$\nabla.\overrightarrow{u} = 0 \tag{10}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z} - \nabla^2\right)\theta - \mathcal{S}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\phi + w\frac{d\theta_b}{dz} = 0 \quad (11)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\{(1-\phi_b)\,\theta + (\mathcal{C} - \theta_b)\,\phi\} + w\frac{d\theta_b}{dz} = 0 \quad (12)$$

with boundary conditions

$$\theta = w = 0 \quad \text{at} \quad z = 0$$
$$\theta = \phi = w = 0 \quad \text{at} \quad z = \delta.$$

where w is the vertical component of $\overrightarrow{u}$, i.e., $\overrightarrow{u} = (u,\, v,\, w)$.

3.3. Elimination of the Pressure term in Perturbed System. Now we eliminate the term involving p in the equation (9) by considering the third component of the double curl of each term in that equation.

Third component of $\nabla \times \nabla \times \overrightarrow{u}$ is given by

$$\begin{aligned} & u_{xz} + v_{yz} - w_{xx} - w_{yy} \\ &= [u_x + v_y]_z - w_{xx} - w_{yy} \\ &= -w_{zz} - w_{xx} - w_{yy} \qquad \text{as} \quad u_x + v_y + w_z = 0 \\ &= -\nabla^2 w \end{aligned}$$

Here ∇^2 denotes 3-D Laplacian. Similarly, for the third component of $\nabla \times \nabla \times (\mathcal{R}\theta\hat{k})$, we have

$$-\mathcal{R}\,[\theta_{xx} + \theta_{yy}] = -\mathcal{R}(\Delta_2\theta)$$

where Δ_2 is used to denote 2-D Laplacian.

Third component of $\nabla \times \nabla \times \left[\frac{\mathcal{I}}{1-\phi_b}\left\{\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right\}\overrightarrow{u}\right]$ is

$$\begin{aligned} & \mathcal{I}\left[\left\{\frac{(\partial_t - \partial_z)\,u}{1-\phi_b}\right\}_{xz} + \left\{\frac{(\partial_t - \partial_z)\,v}{1-\phi_b}\right\}_{yz} - \left\{\frac{(\partial_t - \partial_z)\,w}{1-\phi_b}\right\}_{xx} - \left\{\frac{(\partial_t - \partial_z)\,w}{1-\phi_b}\right\}_{yy}\right] \\ &= \mathcal{I}\left[\left\{\frac{1}{1-\phi_b}(\partial_t - \partial_z)(u_x + v_y)\right\}_z - \frac{1}{1-\phi_b}(\partial_t - \partial_z)\{w_{xx} + w_{yy}\}\right] \\ &= -\mathcal{I}\left[\left\{\frac{1}{1-\phi_b}(\partial_t - \partial_z)\,w_z\right\}_z + \frac{1}{1-\phi_b}(\partial_t - \partial_z)\{w_{xx} + w_{yy}\}\right] \\ &= -\frac{\mathcal{I}}{1-\phi_b}\left[(\partial_t - \partial_z)\left(\nabla^2 w\right) + \frac{\phi_b'}{1-\phi_b}(\partial_t - \partial_z)\,w_z\right] \end{aligned}$$

Thus after elimination of the pressure term, the perturbed system can be expressed as

$$\nabla^2 w + \mathcal{R}\,(\Delta_2\theta) + \frac{\mathcal{I}}{1-\phi_b}\left\{\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\nabla^2 w + \frac{\phi_b'}{1-\phi_b}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\frac{\partial w}{\partial z}\right\} = 0 \quad (13)$$

$$\left(\nabla^2 + \frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\theta - \mathcal{S}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\phi - \theta_b' w = 0 \tag{14}$$

$$\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\left[\left(\theta_b - \mathcal{C}\right)\phi - \left(1 - \phi_b\right)\theta\right] + \theta_b' w = 0 \tag{15}$$

Assuming $\mathcal{I}$ to be small and using the normal mode approach [2, 17] for two dimensional case, we can write

$$\begin{aligned} w(x,z,\mathcal{I},t) &= \text{Re}\left[w_0(z)\, e^{i\alpha x+\sigma t}\right] = \text{Re}\left[\left\{w_{00}(z) + \mathcal{I} w_{01}(z)\right\} e^{i\alpha x+\sigma t}\right] \\ \theta(x,z,\mathcal{I},t) &= \text{Re}\left[\theta_0(z)\, e^{i\alpha x+\sigma t}\right] = \text{Re}\left[\left\{\theta_{00}(z) + \mathcal{I} \theta_{01}(z)\right\} e^{i\alpha x+\sigma t}\right] \\ \phi(x,z,\mathcal{I},t) &= \text{Re}\left[\phi_0(z)\, e^{i\alpha x+\sigma t}\right] = \text{Re}\left[\left\{\phi_{00}(z) + \mathcal{I} \phi_{01}(z)\right\} e^{i\alpha x+\sigma t}\right] \end{aligned}$$

and $\mathcal{R} = \mathcal{R}_{00} + \mathcal{I}\mathcal{R}_{01}$ to obtain two systems: one of order $\mathcal{I}^0$ and the other of order $\mathcal{I}^1$.

System of order $\mathcal{I}^0$ is given by

$$\left(D^2 - \alpha^2\right) w_{00} - \alpha^2 \mathcal{R}_{00}\theta_{00} = 0 \tag{16}$$

$$\left(D^2 + D - \alpha^2 - \sigma\right)\theta_{00} - \mathcal{S}\left(D - \sigma\right)\phi_{00} - \theta_b' w_{00} = 0 \tag{17}$$

$$\left(D - \sigma\right)\left[\left(\theta_b - \mathcal{C}\right)\phi_{00} - \left(1 - \phi_b\right)\theta 0_0\right] + \theta_b' w_{00} = 0 \tag{18}$$

where $D = \partial_z$ .This system can be written as

$$L\, q_{00} = 0 \tag{19}$$

with

$$L = \begin{bmatrix} D^2 - \alpha^2 & -\alpha^2\mathcal{R}_{00} & 0 \\ -\theta_b' & D^2 + D - \alpha^2 - \sigma & -\mathcal{S}\left(D - \sigma\right) \\ \theta_b' & -\left(1 - \phi_b\right)\left(D - \sigma\right) + \phi_b' & \left(\theta_b - \mathcal{C}\right)\left(D - \sigma\right) + \theta_b' \end{bmatrix} \tag{20}$$

and $q_{00} = \left[\begin{array}{ccc} w_{00}, & \theta_{00}, & \phi_{00} \end{array}\right]^T$. Here T is used to denote the transpose.

System of order $\mathcal{I}^1$ can be written as

$$\begin{aligned} \left(D^2 - \alpha^2\right) w_{01} - \alpha^2\mathcal{R}_{00}\theta_{01} &= \alpha^2\mathcal{R}_{01}\theta_{00} + \frac{\phi_b'}{\left(1 - \phi_b\right)^2} D\left(D - \sigma\right) w_{00} \\ &+ \frac{1}{1 - \phi_b}\left(D^2 - \alpha^2\right)\left(D - \sigma\right) w_{00} \end{aligned} \tag{21}$$

$$\left(D^2 + D - \alpha^2 - \sigma\right)\theta_{01} - \mathcal{S}\left(D - \sigma\right)\phi_{01} - \theta_b' w_{01} = 0 \tag{22}$$

$$\left(D - \sigma\right)\left[\left(\theta_b - \mathcal{C}\right)\phi_{01} - \left(1 - \phi_b\right)\theta_{01}\right] + \theta_b' w 0_1 = 0 \tag{23}$$

which can be written as

$$L\, q_{01} = T_{00} \tag{24}$$

with

$$T_{00} = \begin{bmatrix} \alpha^2 \mathcal{R}_1 \theta_0 + \frac{1}{1-\phi_b} \left\{ D^2 - \alpha^2 + \frac{\phi_b'}{1-\phi_b} D \right\} (D - \sigma) w_0 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

and $q_{01} = [w_{01},\ \theta_{01},\ \phi_{01}]^T$. We want to obtain the numerical solutions of the systems given by (19) and (24). To solve the system given by (24), we need to compute $\mathcal{R}_{01}$ first, so we introduce adjoint system L_a for the operator L.

3.4. Adjoint System. Now we define the linear adjoint operator L_a of the linear operator L as

$$< Lq_{00},\ q_a > \quad = \quad < q_{00},\ L_a q_a > \tag{26}$$

where L is given by (20), $q_a = \begin{bmatrix} w_a, & \theta_a, & \phi_a \end{bmatrix}^T$ and $< f,\ g > = \int_0^\delta fg dz$.

To obtain the adjoint system, we multiply the equations (16), (17) and (18) by w_a, θ_a and ϕ_a respectively, and then integrate with respect to z from $z = 0$ to $z = \delta$. These yield

$$\begin{aligned} &\int_0^\delta w_a \left\{ \left(D^2 - \alpha^2\right) w_{00} - \alpha^2 \mathcal{R}_{00} \theta_{00} \right\} dz \\ &+ \int_0^\delta \theta_a \left\{ \left(D^2 + D - \alpha^2 - \sigma\right) \theta_{00} - \mathcal{S} (D - \sigma) \phi_{00} - \theta_b' w_{00} \right\} dz \\ &+ \int_0^\delta \phi_a \left\{ (D - \sigma) \left[(\theta_b - \mathcal{C}) \phi_{00} - (1 - \phi_b) \theta_{00} \right] + \theta_b' w_{00} \right\} dz = 0 \end{aligned} \tag{27}$$

Boundary conditions for the adjoint system are

$$\begin{aligned} &w_a = \theta_a = \phi_a = 0 \qquad \text{at} \quad z = 0 \\ &w_a = \theta_a = 0 \qquad \text{at} \quad z = \delta. \end{aligned}$$

Integration by parts on (27), and use of the boundary conditions yield the adjoint system as

$$\begin{aligned} &\left\{ D^2 - \alpha^2 \right\} w_a + \theta_b' (\phi_a - \theta_a) = 0 \\ &\left\{ D^2 - D - \alpha^2 - \sigma \right\} \theta_a - \alpha^2 \mathcal{R}_{00} w_a + (1 - \phi_b)(D + \sigma) \phi_a = 0 \\ &\mathcal{S} (D + \sigma) \theta_a + (\mathcal{C} - \theta_b)(D + \sigma) \phi_a = 0 \end{aligned} \tag{28}$$

and, in turn, we can write adjoint operator L_a as

$$L_a \quad = \quad \begin{bmatrix} D^2 - \alpha^2 & -\theta_b' & \theta_b' \\ -\alpha^2 \mathcal{R}_{00} & D^2 - D - \alpha^2 - \sigma & (1 - \phi_b)(D + \sigma) \\ 0 & \mathcal{S}(D + \sigma) & (\mathcal{C} - \theta_b)(D + \sigma) \end{bmatrix} \tag{29}$$

3.4.1. Computation of $\mathcal{R}_{01}$. To obtain the dependent variables w_{01}, θ_{01} and ϕ_{01} appearing on for the order $\mathcal{I}^1$, we need to find $\mathcal{R}_{01}$ once we obtain the solutions w_{00}, θ_{00} and ϕ_{00} for the order $\mathcal{I}^0$. We multiply the equations (21), (22) and (23) by w_a, θ_a and ϕ_a respectively, and then integrate with respect to z from $z = 0$ to $z = \delta$ to obtain

$$\begin{aligned}
&\int_0^\delta w_a \left\{ \left(D^2 - \alpha^2\right) w_{01} - \alpha^2 \mathcal{R}_{00} \theta_{01} \right\} dz \\
&\quad + \int_0^\delta \theta_a \left\{ \left(D^2 + D - \alpha^2 - \sigma\right) \theta_{01} - \mathcal{S} \left(D - \sigma\right) \phi_{01} - \theta_b' w_{01} \right\} dz \\
&\quad + \int_0^\delta \phi_a \left\{ \left(D - \sigma\right) \left[\left(\theta_b - \mathcal{C}\right) \phi_{01} - \left(1 - \phi_b\right) \theta_{01} \right] + \theta_b' w_{01} \right\} dz \\
&\qquad = \int_0^\delta w_a \left\{ \alpha^2 \mathcal{R}_{01} \theta_{00} + \frac{\phi_b'}{(1 - \phi_b)^2} D \left(D - \sigma\right) w_{00} \right\} dz \\
&\qquad\quad + \int_0^\delta w_a \left\{ \frac{1}{1 - \phi_b} \left(D - \sigma\right) \left(D^2 - \alpha^2\right) w_{00} \right\} dz
\end{aligned}$$

Successive integration by parts and use of boundary conditions yield

$$\begin{aligned}
&\int_0^\delta w_{01} \left\{ \left(D^2 - \alpha^2\right) w_a - \theta_b' \theta_a + \theta_b' \phi_a \right\} dz \\
&\quad + \int_0^\delta \theta_{01} \left\{ -\alpha^2 \mathcal{R}_{00} \theta_a + \left(D^2 - D - \alpha^2 - \sigma\right) \theta_a + \left(1 - \phi_b\right) \left(D + \sigma\right) \phi_a \right\} dz \\
&\quad + \int_0^\delta \phi_{01} \left\{ \mathcal{S}(D + \sigma) \theta_a + \left(\mathcal{C} - \theta_b\right) \left(D + \sigma\right) \phi_a \right\} dz \\
&\qquad = \int_0^\delta w_a \left\{ \alpha^2 \mathcal{R}_{01} \theta_{00} + \frac{\phi_b'}{(1 - \phi_b)^2} D \left(D - \sigma\right) w_{00} \right\} dz \\
&\qquad\quad + \int_0^\delta w_a \left\{ \frac{1}{1 - \phi_b} \left(D - \sigma\right) \left(D^2 - \alpha^2\right) w_{00} \right\} dz
\end{aligned}$$

Thus, we obtain an expression for $\mathcal{R}_1$ as

$$\mathcal{R}_{01} = \frac{\int_0^\delta \frac{w_a}{1 - \phi_b} \left\{ \frac{\phi_b'}{1 - \phi_b} D \left(D - \sigma\right) w_{00} + \left(D - \sigma\right) \left(D^2 - \alpha^2\right) w_{00} \right\} dz}{-\alpha^2 \int_0^\delta w_a \theta_{00} dz} \qquad (30)$$

4. Numerical Results

For marginal stability results, we set $\sigma = 0$. For our computations, we chose the parameters which are used by experimentalists [8]. These parameters are 3.2 for Stefan number, 9.0 for concentration ratio, 0.1 for far-field temperature. First we obtain the basic state solutions by performing computation on the equations

(7) and (8). Once we obtain the basic state solutions, then to determine stability results, we solve the linear stability problem (order ϵ^0) numerically using the fourth-order Runge-Kutta in combination of shooting method [14]. In order to solve the equations (16), (17), and (18), we first convert these equations into a system of five first-order ordinary differential equations. The linear problem of order $\mathcal{I}^0$ yields 1.71 and 18.5220 as critical wavenumber, α and critical Rayleigh number, $\mathcal{R}_{00}$. After obtaining the critical wavenumber and critical Rayleigh number, we solve the adjoint system given by (28) numerically so that the computed value of $\mathcal{R}_{01}$ can be found from the equation (30). Then we obtain the marginal stability curves (where $\mathcal{R} = \mathcal{R}_{00} + \mathcal{I}\mathcal{R}_{01}$) for various values of the inertial parameter, $\mathcal{I}$. Figure 2 presents the graphs of marginal stability curves for $\mathcal{I} = 0.0$, 0.001, 0.005 and 0.01. It can be seen from the figure 2 that presence of the inertial effect is stabilizing in the sense that the critical Rayleigh number increases with $\mathcal{I}$.

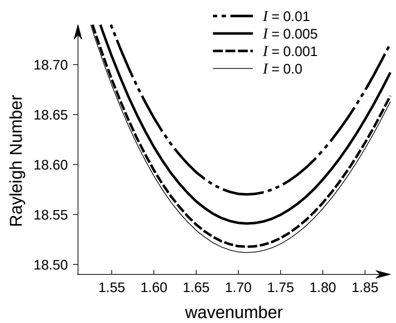

FIGURE 2. Marginal Stability Curves

We solve the system of order $\mathcal{I}^1$ given by (21), (22) and (23) numerically. This yields the solutions w_{01}, θ_{01} and ϕ_{01}. Figures 3, 4 and 5 present the dependent variables $w_0 = w_{00} + \mathcal{I}w_{01}$, $\theta_0 = \theta_{00} + \mathcal{I}\theta_{01}$ and $\phi_0 = \phi_{00} + \mathcal{I}\phi_{01}$ for different values of $\mathcal{I}$. As values of $\mathcal{I}$ increases, the vertcal velocity component decreases, thus $\mathcal{I}$ has stabilizing effect. The results presented in the figure 3 indicate that presence of inertal term reduces the non-uniformity of the vertical velocity for the neutral stability flow.

The results shown in the figure 4 indicate that the inertial effect increases the amount of non-uniformity in the temperature of the fluid in the mushy layer.

The results presented in the figure 5 reinforcethose [resented in the figure 4 and indicate more non-uniformity o=in the sloid fraction due to inertial effect.

5. Conclusion

We investigated the effect of inertial parameter in a horizontal mushy layer arising in binary alloy solidification problems. Using perturbation technique, we obtained two systems: one of order $\mathcal{I}^0$ and other of order $\mathcal{I}^1$. Then we

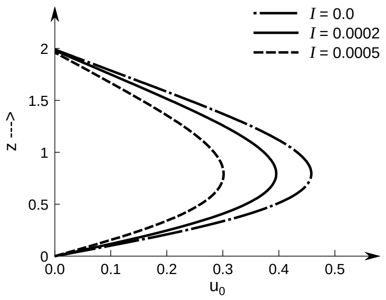

FIGURE 3. Linear vertical velocity component

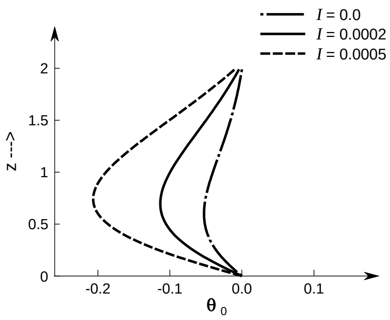

FIGURE 4. Linear temperature component

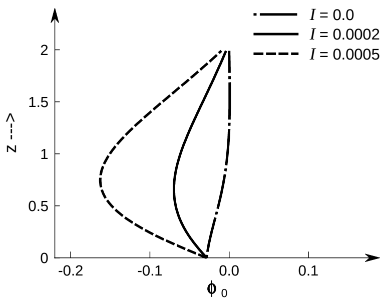

FIGURE 5. Linear solid volume fraction

obtained the system which is adjoint to the system of order $\mathcal{I}^0$. For the case we investigated, we conclude from our numerical results that the inertial parameter is stabilizing in the sense that the critical Rayleigh number increases by the inertial effect. In addition, presence of the inertial effect tends to increase the amount of non-uniformity in the tempertaure and the solid fraction but decreases such non-uniformity in the vertical velocity in the mushy zone.

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Dambaru Bhatta received his PhD from Dalhousie University, Halifax, Canada. Currently he is a faculty in the Department of Mathematics, The University of Texas-Pan American, TX, USA. His research interests include applied mathematics, computational mathematics, numerical solutions of differential equations, fluid flow, stability analysis.

Department of Mathematics, The University of Texas-Pan American, 1201 W. University Drive, Edinburg, TX 78541, USA.
e-mail : bhattad@utpa.edu

Daniel N. Riahi received Ph.D. from Florida State University, USA. Currently he is a professor at The University of Texas-Pan American, TX, USA. His research interests include applied mathematics, applied mechanics, engineering mathematics, hydrodynamic stability, computational mathematics.

Department of Mathematics, The University of Texas-Pan American, 1201 W. University Drive, Edinburg, TX 78541, USA.
e-mail : driahi@utpa.edu

Mallikarjunaiah S. Muddamallappa is pursuing a Ph.D. at the Department of Mathematics, Texas A & M University, College Station, TX, USA.