$h-$STABILITY OF PERTURBED DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

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ABSTRACT. In this paper, we investigate $h-$stability of the nonlinear perturbed differential systems using the the notion of t_∞-similarity.

1. Introduction

As is traditional in a pertubation theory of nonlinear differential equations, the behavior of solutions of a perturbed equation is determined in terms of the behavior of solutions of an unperturbed equation. Using the variation of constants formula and the integral inequalities, we study the qualitative behavior of the solutions of perturbed nonlinear system of differential equations:

Pinto [13] introduced h-stability(hS) which is an important extention of the notions of exponential asymptotic stability and uniform Lipschitz stability. Also, he obtained some properties about asymptotic behavior of solutions of perturbed h-systems, some general results about asymptotic integration and gave some important examples in [14].

Choi and Ryu [3] obtained hS of the solutions of the various differential systems and Volterra integro-differential systems. Recently, Choi et al. [4] and Goo and Ry [7,8] dealt with results for hS of nonlinear differential systems via -similarity. Goo et al. [9,10] investigated hS for the nonlinear Volterra integro-differential system.

In this paper, we investigate $h-$stability of the nonlinear perturbed differential systems using the the notion of t_∞-similarity.

2. Preliminaries

We consider the nonlinear differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where $f \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$.

Let $x(t) = x(t, t_0, x_0)$ denote the unique solution of (1) through (t_0, x_0) in $\mathbb{R}^+ \times \mathbb{R}^n$ such that $x(t_0, t_0, x_0) = x_0$. Also, we consider the associated variational systems around the zero solution of (1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \; v(t_0) = v_0 \tag{2}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \; z(t_0) = z_0. \tag{3}$$

The fundamental matrix solution $\Phi(t, t_0, x_0)$ of (3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix solution of (2). The symbol $|.|$ denotes arbitrary vector norm in $\mathbb{R}^n$.

We recall some notions of h-stability [13] and the notion of t_∞-similarity [5].

Definition 2.1. The system (1) (the zero solution $x = 0$ of (1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c \, |x_0| \, h(t) \, h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough.

Definition 2.2. The system (1) (the zero solution $x = 0$ of (1)) is called *h-stable (hS)* if there exist $\delta > 0$ such that (1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [5].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)| dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{4}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [4,11]. We give some related properties that we need in the sequal.

Lemma 2.1 ([15]). *The linear system*

$$x' = A(t)x, \ x(t_0) = x_0, \tag{5}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{6}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (5).

We need Alekseev formula to compare between the solutions of (1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \ y(t_0) = y_0, \tag{7}$$

where $g \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (7) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.2. *If $y_0 \in \mathbb{R}^n$, for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 2.3 ([3]). *If the zero solution of (1) is hS, then the zero solution of (2) is hS.*

Theorem 2.4 ([4]). *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. Then the solution $v = 0$ of (2) is hS if and only if the solution $z = 0$ of (3) is hS.*

3. Main results

In this section, we investigate hS for the nonlinear perturbed differential systems. Now, we examine the properties of hS for the perturbed system of (1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds, \ y(t_0) = y_0, \tag{8}$$

where $g \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$ and $g(t, 0) = 0$

Theorem 3.1. *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1) is hS with the increasing function h, and g in (8) satisfies*

$$\left| \int_{t_0}^s g(\tau, y(\tau)) d\tau \right| \leq a(s)|y(s)| + b(s) \int_{t_0}^s c(\tau)|y(\tau)|d\tau, \ t \geq t_0 \geq 0,$$

where $a, b, c \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\int_{t_0}^\infty [a(s) + b(s) \int_{t_0}^s c(\tau)d\tau]ds < \infty$. Then, the solution $y = 0$ of (8) is hS.

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$. By Theorem 2.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (2) is hS. Therefore, by Theorem 2.4, the solution $z = 0$ of (3) is hS. By Lemma 2.1, Lemma 2.2 and the increasing property of the function h, we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left| \int_{t_0}^s g(\tau, y(\tau)) d\tau \right| ds$$

$$\leq c_1 |y_0| h(t) \, h(t_0)^{-1}$$

$$+ \int_{t_0}^t c_2 h(t) \, h(s)^{-1} [a(s)|y(s)| + b(s) \int_{t_0}^s c(\tau)|y(\tau)|d\tau] ds.$$

Set $u(t) = |y(t)| h(t)^{-1}$. Then, by Gronwall's inequality, we obtain

$$|y(t)| \leq c_1 |y_0| h(t) \, h(t_0)^{-1} e^{c_2 \int_{t_0}^t [a(s)+b(s) \int_{t_0}^s c(\tau)d\tau]ds}$$

$$\leq c |y_0| h(t) \, h(t_0)^{-1}, \quad c = c_1 e^{c_2 \int_{t_0}^\infty [a(s)+b(s) \int_{t_0}^s c(\tau)d\tau]ds}$$

It follows that $y = 0$ of (8) is hS. Hence, the proof is complete. $\square$

Remark 3.1. In the linear case, we can obtain that if the zero solution $x = 0$ of (5) is hS, then the perturbed system

$$y' = A(t)y + \int_{t_0}^t g(s, y(s))ds, y(t_0) = y_0,$$

is also hS under the same hypotheses in Theorem 3.1 except the condition of t_∞-similarity.

Remark 3.2. Letting $b(s) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 3.3 in [8].

To investigate an h-system of (8), we need the following lemma.

Lemma 3.2. *Let $u, p, q, w, r \in C(\mathbb{R}^+, \mathbb{R}^+)$ and suppose that, for some $c \geq 0$, we have*

$$u(t) \leq c + \int_{t_0}^t p(s) \int_{t_0}^s [q(\tau)u(\tau) + w(\tau) \int_{t_0}^\tau r(a)u(a)da]d\tau ds, \ t \geq t_0. \quad (9)$$

Then

$$u(t) \leq c \exp\left(\int_{t_0}^t p(s) \int_{t_0}^s [q(\tau) + w(\tau) \int_{t_0}^\tau r(a)da]d\tau ds \right), \ t \geq t_0. \quad (10)$$

Proof. Setting $v(t) = c + \int_{t_0}^t p(s) \int_{t_0}^s [q(\tau)u(\tau) + w(\tau) \int_{t_0}^\tau r(a)u(a)da]d\tau ds$, we have $v(t_0) = c$ and

$$v'(t) = p(t) \int_{t_0}^t [q(s)u(s) + w(s) \int_{t_0}^s r(a)u(a)da]ds$$

$$\leq p(t) \int_{t_0}^t [q(s) + w(s) \int_{t_0}^s r(a)da]v(s)ds \qquad (11)$$

$$\leq [p(t) \int_{t_0}^t [q(s) + w(s) \int_{t_0}^s r(a)da]ds]v(t),\ t \geq t_0,$$

since $v(t)$ is nondecreasing and $u(t) \leq v(t)$. It follows from the Gronwall inequality that (11) yields the estimate (10). $\qquad\square$

Theorem 3.3. *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $x = 0$ of (1) is an h-system with a positive continuous function h and g in (8) satisfies*

$$|g(t,y)| \leq \lambda(t)|y| + \beta(t) \int_{t_0}^t \gamma(s)|y(s)|ds,\ t \geq t_0,\ y \in \mathbb{R}^n,$$

where $\lambda, \beta, \gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with

$$\int_{t_0}^\infty \frac{1}{h(s)} \int_{t_0}^s [h(\tau)\lambda(\tau) + \beta(\tau) \int_{t_0}^\tau h(r)\gamma(r)dr]d\tau ds < \infty, \qquad (12)$$

for all $t_0 \geq 0$, then the solution $y = 0$ of (8) is an h-system.

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$. By Theorem 2.3, since the solution $x = 0$ of (1) is an h-system, the solution $v = 0$ of (2) is an h-system. Therefore, by Theorem 2.4, the solution $z = 0$ of (3) is an h-system. By Lemma 2.1 and Lemma 2.2, we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \int_{t_0}^s |g(\tau, y(\tau))|d\tau ds$$

$$\leq c_1|y_0|h(t)\,h(t_0)^{-1}$$

$$+ \int_{t_0}^t c_2 \frac{h(t)}{h(s)}[\int_{t_0}^s h(\tau)\lambda(\tau)\frac{|y(\tau)|}{h(\tau)} + \beta(\tau) \int_{t_0}^\tau h(r)\gamma(r)\frac{|y(r)|}{h(r)}dr d\tau]ds.$$

Setting $u(t) = |y(t)|h(t)^{-1}$ and using Lemma 3.2, we obtain

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1}e^{c_2 \int_{t_0}^t \frac{1}{h(s)} \int_{t_0}^s [h(\tau)\lambda(\tau) + \beta(\tau) \int_{t_0}^\tau h(r)\gamma(r)dr]d\tau ds}$$

$$\leq c|y_0|h(t)\,h(t_0)^{-1},\ t \geq t_0,$$

where $c = c_1 e^{c_2 \int_{t_0}^\infty \frac{1}{h(s)} \int_{t_0}^s [h(\tau)\lambda(\tau) + \beta(\tau) \int_{t_0}^\tau h(r)\gamma(r)dr]d\tau ds}$. It follows that $y = 0$ of (8) is an h-system. Hence, the proof is complete. $\qquad\square$

Remark 3.3. Letting $\beta(t) = 0$ in Theorem 3.3, we have the similar result as that of Theorem 2.5 in [7].

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REFERENCES

1. V. M. Alekseev, An estimate for the perturbations of the solutions of ordinary differential equations, Vestn. Mosk. Univ. Ser. I. Math. Mech., 2 (1961), 28-36.
2. S. K. Choi and N. J. Koo, $h-$stability for nonlinear perturbed systems, Ann. of Diff. Eqs. 11 (1995), 1–9.
3. S. K. Choi and H. S. Ryu, $h-$stability in differential systems, Bull. Inst. Math. Acad. Sinica 21 (1993), 245–262.
4. S. K. Choi and N. J. Koo and H.S. Ryu, h-stability of differential systems via t_∞-similarity, Bull. Korean. Math. Soc. 34 (1997), 371–383.
5. R. Conti, Sulla t_∞-similitudine tra matricie l'equivalenza asintotica dei sistemi differenziali lineari, Rivista di Mat. Univ. Parma 8 (1957), 43–47.
6. S. Elaydi and R. R. M. Rao, Lipschitz stability for nonlinear Volterra integro-differential systems, Appl. Math. Computations 27 (1988), 191–199.
7. Y .H. Goo , h-stability of the nonlinear differential systems via t_∞-similarity, J. Chungcheong Math. Soc. 23 (2010), 383-389.
8. Y .H. Goo and D. H. Ry , h-stability of the nonlinear perturbed differential systems , J. Chungcheong Math. Soc. 23 (2010), 827-834.
9. Y .H. Goo and D. H. Ry, h-stability for perturbed integro-differential systems, J. Chungcheong Math. Soc. 21 (2008), 511–517.
10. Y. H. Goo, M. H. Ji and D. H. Ry, h-stability in certain integro-differential equations, J. Chungcheong Math. Soc. 22 (2009), 81–88.
11. G. A. Hewer, Stability properties of the equation by -similarity, J. Math. Anal. Appl. 41 (1973), 336–344.
12. V. Lakshmikantham and S. Leela, Differential and Integral Inequalities: Theory and Applications Vol., Academic Press, New York and London, 1969.
13. M. Pinto, Perturbations of asymptotically stable differential systems, Analysis 4 (1984), 161–175.
14. M. Pinto, Asymptotic integration of a system resulting from the perturbation of an h-system , J. Math. Anal. Appl. 131 (1988), 194–216.
15. M. Pinto, Stability of nonlinear differential systems, Applicable Analysis 43 (1992), 1–20.

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