J. Appl. Math. & Informatics Vol. **30**(2012), No. 3 - 4, pp. 561 - 570
Website: http://www.kcam.biz

ON FUNCTIONS DEFINED BY ITS FOURIER TRANSFORM

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Abstract. Fourier transform is well known for trigonometric systems. It is also a very useful tool for the construction of wavelets. The method of constructing wavelets has evolved as times went by. We review some methods. Then we do some calculations on wavelets defined by its Fourier transform.

AMS Mathematics Subject Classification : 40A30, 42C15, 42C40, 65H05.
Key words and phrases : Fourier transform, scaling function, wavelet, distribution.

1. Introduction

Wavelets are a fairly simple mathematical tool with a great variety of possible applications. An Orthonormal wavelet is a function $\psi \in L^2(R)$ such that the system $\{\psi_j k = 2^j/2\psi(2^j x + k), j, k \in Z\}$ is an orthonormal basis of $\psi \in L^2(R)$. There are two classes of equalities, involving the Fourier Transform, $\hat{\psi}$, of ψ that characterized orthonormal wavelets. The first one is

$$\sum_{k\in Z} \hat{\psi}(2^j(\xi + 2l\pi))\overline{\hat{\psi}(\xi + 2l\pi)} = \delta_0 j \tag{1}$$

for a.e $\xi \in R$ whenever $j \geq 0$. The second class of equation is

$$\text{(i)} \quad \sum_{k\in Z} |\hat{\psi}(2^j\xi)|^2 = 1 \tag{2}$$

$$\text{(ii)} \quad \sum_{k\in Z} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + (2m+1)2\pi))} = 0 \tag{3}$$

a.e. whenever $m \in Z$. The first one is equivalent to the orthonomality of the system and the second one is equivalent to the completeness of this system $\{\psi_{jk}\}$ in $L^2(R)$. When ψ is band-limited, the sums in the first and second equalities

Received November 20, 2011. Revised January 20, 2012. Accepted January 25, 2012.
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are finite for each ξ. A function is said to be band-limited if its Fourier transform has compact support. Lemarie and Meyer [10] has constructed a band-limited wavelet having a Fourier transform that is infinitely differentiable.

The great value of orthogonality is to make expansion coefficient easy to compute. The prototype of orthonormal wavelet is the Haar function, which is piecewise constant. The defect in piecewise constant is that they are very poor at approximation. Representing a smooth function requires higher order of differentiability.

The construction of wavelet rather begins with a function ϕ called scaling function. There is a well-known method for constructing compactly supported wavelet bases in $L^2(R)$. It starts with the two - scale difference (or refinement, or dilation) equation

$$\phi(x) = \sum_{k=M}^{N} 2c_k \phi(2x-k), \tag{4}$$

where, at this point, the only condition on the complex numbers c_k is that $\sum c_k = 1$. In order to solve the this equation we first define the trigonometric polynomial

$$m_0(z) = \sum_k = M^N c_k e^i kz.$$

Then the inverse Fourier transform ϕ of the entire function

$$A(z) = \prod_{j=1}^{\infty} m_0(2^{-j}z)$$

is a solution of (4). In general, ϕ is a distribution with compact support contained in the interval $[M, N]$. If ϕ (or equivalently A) is in $L^2(R)$ and m_0 satisfies Cohen's criterion [18, Def. 5.2], then ϕ is a scaling function of multiresolution analysis. Corhen's criterion is satisfied if m_0 does not have zeros in $[-\pi/2, \pi/2]$. Then a standard definition leads to the associated wavelet and the corresponding wavelet basis of $L^2(R)$.

So far we talked about single scaling function. A method using vector valued function appeared. The idea was first introduced in [4,5]. It starts with r functions, $\phi^1, \phi^2, \cdots, \phi^r$ and we store them in a vector $\Phi(x) = (\phi^1(x)\phi^2(x)\cdots\phi^r(x))^T$. In this case Φ satisfies a matrix refinement equation

$$\Phi(x) = \sum_{k=0}^{N} C_k \Phi(2x-k), \tag{5}$$

where C_k are $r \times r$ matrices.

Here we want to mention about wavelets based on prolate spheroidal wave function [20]. The continuous prolate shperoidal wave functions (PSWFs) are those that are most highly localized simultaneously in both the time and frequency domain. This fact was discovered by Slepian and his collaborators and was presented in a series of articles [11], [12], [14]-[16] about forty years ago. Since then the study of PSWFs has been an active area of research in both electrical engineering and mathematics. The PSWFs are concentrated on the interal $[-\tau, \tau]$ and, of course depend on the two parameters σ and τ. There are several ways of characterizing PSAFs , one of which is as the eigenfucntions of an integral operator

$$\frac{\sigma}{\pi}\int_{\tau}^{\tau} \varphi_{n,\sigma,\tau}(x)S(\frac{\sigma}{\pi}(t-x))dx = \lambda_{n,\sigma,\tau}(t), \tag{6}$$

where $S(t) = \frac{\sin \pi t}{\pi t}$, the parameter comes from the interval of concentration and the parameter comes from the support of the Fourier transform. PSWFs are closely related to the Fourier transform. Indeed, the Fourier transform of $\varphi_{n,\sigma,\tau}$ is given by

$$\hat{\varphi}_{n,\sigma,\tau}(\omega) = (-1)^n\sqrt{\frac{2\pi\tau}{\sigma}}\lambda_{n,\sigma,\tau}\varphi_{n,\sigma,\tau}(\frac{\tau\omega}{\sigma})\chi_\sigma(\omega), \tag{7}$$

where $\chi_\sigma(\omega)$ is the characteristic function of $[-\sigma, \sigma]$. Therefore the inverse Fourier transform gives us still another formula

$$\varphi_{n,\sigma,\tau}(x) = (-1)^n\sqrt{\frac{2\pi\tau}{\sigma}}\lambda_{n,\sigma,\tau}\frac{1}{2\sigma}\int_{\tau}^{\tau} \varphi_{n,\sigma,\tau}(\frac{\tau\omega}{\sigma})d\omega. \tag{8}$$

In short, we may classify multiresolution analysis into three categories as follows; orthogonal bases, semi-orthogonal bases, nonorthogonal biorthogonal bases.

2. Overshoot of wavelet expansions at discontinuity

The overshoot of an approximation to a function near a discontinuity is called Gibbs phenomenon, which has been recognized for over a century. When a function is represented by the trigonometric series, one can see that its graph exhibits an overshoot or downshoot near the point of jump discontinuity of the function. At the beginning, this undesirable phenomenon was understood as the reason that the series expansion was approximated by a finite sum out of infinite series. To the contrary of the earlier guess, the overshoot (or downshoot) can not not be removed. Instead, the ratio of overshoot to the jump converges to a certain constant, the Gibbs constant, as the partial sum is taken to infinite series. But it is not unique to the trigonometric series. It has been shown to exist for many other approximations. In most cases, it is an undesirable aspect of the approximation since it involves oscillations near the discontinuity. Foster [3] and Richard [13] demonstrated a Gibbs phenomenon using piecewise linear continuous and spline functions respectively. The same holds for approximations based on Fourier transforms, Lagrange polynomial and trigonometric interpolation.

We shall try to discuss the characteristics that lead to Gibbs phenomenon in the case of Fourier series.

Let $f(x) = \pi/2\mathrm{sgn}(x) - x/2, |x| < \pi$. Then the Fourier series of $f(x)$ is

$$f_n(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \sum_{n=1}^{\infty} \int_0^x \cos nt dt \tag{9}$$

$$= \lim_{n\to\infty} \int_0^x \left(\frac{1}{2} + \sum_{k=1}^{n} \cos kt \right) dt - \frac{x}{2} \tag{10}$$

$$= \lim_{n\to\infty} \left(\pi \int_0^x D_n(t) dt \right) - \frac{x}{2}, \tag{11}$$

where D_n is the Dirichlet kernel given by

$$D_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi} \frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}.$$

We suppose f satisfies a left and right Lipschitz condition at 0. Then the series converges to 0 at 0. We investigate the partial sums of the series as $x \longrightarrow 0^+$. Then we have

$$\pi \int_0^x D_n(t) = \int_0^x \left(\frac{\sin nt \cos nt}{2\sin\frac{1}{2}} + \frac{\cos nt}{2} \right) \tag{12}$$

$$= \int_0^x \frac{\sin nt}{t} dt + \int_0^x \sin nt \left(\frac{\cos t/2}{2\sin t/2} \right) dt + \int_0^x \frac{\cos nt}{2} dt, \tag{13}$$

for $x > 0$. The second and third integral on the last line of (13) converges uniformly to 0. This leaves only the last integral, which may be written as

$$I(nx) = \int_0^n x \frac{\sin t}{t} dt \longrightarrow \int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}, \quad \text{for fixed} x > 0.$$

If we take the sequence $x_n = \frac{\pi}{n}$, the integral becomes

$$I(nx_n) = \int_0^{\pi} \frac{\sin t}{t} dt > \frac{\pi}{2}.$$

Hence the partial sums of the given function at x_n are given by

$$S_n(x_n) = I(nx_n) - \frac{x_n}{2} + \epsilon_n \longrightarrow \int_0^{\pi} \frac{\sin t}{t} dt,$$

which converges to a value $> f(0^+)$.

Gibbs phenomenon is also found in wavelet expansions involving both orthogonal and sampling series of wavelets [9] as well as continuous wavelet transforms [6]. Gibbs phenomenon in wavelet expansions was studied by Kelly [5]. Kelly showed that Daubechies' compactly supported wavelets exhibit this phenomenon at the origin and computed the size of them by using computer. It has also been shown by Shim and Volkmer [17] that a Gibbs phenomenon occurs virtually all types of continuous orthogonal wavelets. It also occurs at approximations of vector valued wavelet expansion, which was shown by Ruch [2]. We also discuss the characteristics that lead to Gibbs phenomenon in wavelet expansions.

Each wavelet system has an associated "multiresolution analysis" consisting of a nested sequence $\{V_m\}$ of subspaces of $L^2(R)$ where the space V_m is the closed linear span of $\{\phi(2^m t-n)\}_{n\in Z}$. A function f in $L^2(R)$ can be approximated by its projection $P_m f$ onto V_m;

$$(P_m f)(x) = f_m(x) = \int_{-\infty}^{\infty} q_{mn}(x,y)f(y)dy \tag{14}$$

$$= \sum_{n\in Z} < \phi_{mn}, f > \phi_{mn}(x-n), \tag{15}$$

where

$$q(x,y) = \sum_{n\in Z} \phi(x-n)\phi(y-n), \phi_{mn}(x) = 2^{m/2}\phi(2^m x-n). \tag{16}$$

For the uniform convergence of this series on the interval of continuity, we refer the work of Walter [19, p. 12, pp. 116-128]. The necessary and sufficient condition[9] for Gibbs phenomenon on the right (or on the left) to exist is

$$G(x) := \int_0^{\infty} q(a,y)dy > 1 \quad \text{for some} \quad x > 0 \tag{17}$$

$$(\text{or} \quad \int_0^{\infty} q(x,y)dy < 0 \quad \text{for some} \quad x < 0). \tag{18}$$

We list some of the results based on this criteria as follows;

Theorem 1 ([9])**.** Let $\phi \in S_r$ be a scaling function and $f : R \to R$ be a square integrable bounded function with jump discontinuity at 0. Then the wavelet expansion of f shows a Gibbs' phenomenon at the right hand side of 0 if and only if there is an $a > 0$ such that $\int_0^{\infty} q(a,t)dt > 1$ and it shows a Gibbs' phenomenon at the left hand side of 0 if and only if there is an $a < 0$ such that $\int_0^{\infty} q(a,t)dt < 0$.

Theorem 2 ([17])**.** Let ϕ be a continuous scaling function which is differentiable at a dyadic number with nonvanishing derivative there, and which satisfies $|\phi(x)| \leq K(1+|x|)^{-\beta}, \beta > 3$ for $x \in R$. Then the corresponding wavelet expansion shows a Gibbs phenomenon at the right hand side or left hand side of 0.

Theorem 3 ([2]). Let $\Phi = (\phi^1, \cdots, \phi^A)^T$ be a continuous, compactly supported scaling vector with polynomial accuracy at least 2. If the multiresolution analysis is orthogonal or Φ has a dual biorthogonal basis Φ^* that is compactly supported, then the corresponding wavelet expansion shows a Gibbs' phenomenon at least one side of 0.

3. Calculations over functions defined by the Fourier transform

Some times we may be interested in calculating the maximun size of overshoot when Gibbs phenomenon occurs. It depends on the choice of x in the formula (17). Some wavelets are defined by its Fourier transform. Cardinal B-spline is of them. In this case, we need to convert the formula in terms of the Fourier transform. The k-th order cardinal B-spline $N^{[k]}$ is defined as the k-th fold convolution of the characteristic function of the interval $[0,1]$ for $k = 2, 3, \cdots$. These functions are not scaling function in the sense of section 2 because the orthogonal condition is not satisfied. The corresponding orthogonalized scaling function $\phi^{[k]}$ leads to Battle-Lemarie wavelets. These are defined as the function whose Fourier transform is given by

$$\hat{\phi}^{[k]}(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^k \sigma_k(\omega)^{-\frac{1}{2}}, \tag{19}$$

where (see [1, p. 216])

$$\sigma_k(\omega) = \left(\sin\frac{\omega}{2}\right)^{2k} \sum_n (\frac{\omega}{2} + n\pi)^{-2k}. \tag{20}$$

We note that the space $V_0 = V_0^{[k]}$ consists of all square integrable and $k-2$ times continuously differentiable functions that agree with a polynimial function of degree at most $k-1$ on each interval $[n, n+1]$ for $n \in Z$. Now we have

$$\hat{\phi}^{[k]}(\omega) = e^{-i\omega k/2} \left(\frac{\sin\omega/2}{\omega/2}\right)^k \sigma_k(\omega)^{-\frac{1}{2}} \tag{21}$$

$$= e^{-i\omega k/2} \left(\frac{2}{\omega}\right)^k \frac{1}{\sqrt{r_{2k}(\omega)}}, \tag{22}$$

where $r_k(\omega) = \sum_n \frac{1}{(\frac{\omega}{2}+n\pi)^k}$. By taking $x \in Z$ we have

$$\overline{\hat{q}^{[k]}}(x,\omega) = \overline{\hat{\phi}^{[k]}}(\omega) \sum_n \hat{\phi}^{[k]}(\omega - 2n\pi) e^{i\omega x}, \tag{23}$$

$$\frac{\overline{\hat{q}^{[k]}}(x,\omega)}{i\omega} = \overline{\hat{\phi}^{[k]}}(\omega) \sum_n \hat{\phi}^{[k]}(\omega - 2n\pi) \frac{\cos\omega x + i\sin\omega x}{i\omega}. \tag{24}$$

We also have

$$\sum_n \hat{\phi}^{[k]}(\omega - 2n\pi) = e^{-ik\omega/2}\sigma_k{}^{-\frac{1}{2}}(\omega)\sin^k\frac{\omega}{2}\sum_n \frac{1}{(\frac{\omega}{2}-n\pi)^k} \tag{25}$$

$$= e^{-ik\omega/2}\left(\sum_n \frac{1}{(\frac{\omega}{2}+n\pi)^{2k}}\right)^{-\frac{1}{2}}\sum_n \frac{1}{(\frac{\omega}{2}-n\pi)^k} \tag{26}$$

$$= e^{-ik\omega/2}\frac{r_k(\omega)}{\sqrt{r_{2k}(\omega)}}. \tag{27}$$

Now we take $Q_k(\omega)$ as

$$Q_k(\omega) := \overline{\hat{\phi}^{[k]}}(\omega)\sum_n \hat{\phi}^{[k]}(\omega - 2n\pi) = (\frac{2}{\omega})^k\frac{r_k(\omega)}{r_{2k}(\omega)}. \tag{28}$$

By observing

$$r_k(-\omega) = \sum_n \frac{1}{(-\frac{\omega}{2}+n\pi)^k} = (-1)^k\sum_n \frac{1}{(\frac{\omega}{2}-n\pi)^k} = (-1)^k r_k(\omega), \tag{29}$$

we can see $Q(\omega)$ is even function;

$$Q_k(-\omega) = (-1)^k(\frac{2}{\omega})^k\frac{(-1)^k r_k(\omega)}{r_{2k}(\omega)} = Q_k(\omega).$$

Here we impose one condition with $\phi \in S_r$, which is defined as

$$|\phi^(k)(x)| \le C_{pk}(1+|x|)^{-p}, k = 0, 1, \cdots, r; p \in Z, x \in R.$$

Then we have the Fourier transform formula for the equation (14) as follows.

Theorem 4. For scaling function $\phi \in S_r$ and $\hat{\phi}(\omega) \ge 0$, we have

$$\int_0^\infty q(x,y)dy = \frac{1}{2\pi}pv\int_{-\infty}^\infty \frac{\bar{\hat{q}}(x,\omega)}{i\omega}d\omega + \frac{1}{2}.$$

Proof. For a scaling function $\phi \in S_r$, its Fourier transform $\hat{\phi}$ satisfies $\hat{\phi}(2n\pi) = \delta_{0n}$ if $\hat{\phi}(0) \ge 0$ (see [19, p. 41]). Let h be the Heavyside functional

$$h(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0. \end{cases}$$

Then we claim that

$$\hat{h}(\omega) = \pi\delta(\omega) + \mathrm{pv}\frac{1}{i\omega},$$

where $\hat{h}$ is the Fourier transform of h, $\delta(\omega)$ is the delta functional and $\mathrm{pv}\frac{1}{i\omega}$ is the Cauchy principal value of $\int_{-\infty}^\infty \frac{\phi(\omega)}{i\omega}d\omega$ when it is applied to a test function

ϕ. This can be proved as follows. For any test function ϕ, i.e., infinitely smooth and rapidly decreasing functions, we have

$$\begin{aligned}< \hat{h}(\omega), \phi(\omega) > &=< h(t), \hat{\phi}(t) > \\ &= \int_{-\infty}^{\infty} h(t)\hat{\phi}(t)dt \\ &= \int_{0}^{\infty}\int_{-\infty}^{\infty} \phi(\omega)e^{-i\omega t}d\omega dt.\end{aligned}$$

By using the Fubini's theorem, the double integral in the final equality turns out

$$\begin{aligned}&\lim_{T\to\infty}\int_{0}^{T}\int_{-\infty}^{\infty} \phi(\omega)e^{-i\omega t}d\omega dt \\ &= \lim_{T\to\infty}\int_{-\infty}^{\infty} \phi(\omega)\left(\int_{0}^{T} e^{-i\omega t}dt\right)d\omega \\ &= \int_{-\infty}^{\infty} \frac{1}{i\omega}\phi(\omega)d\omega - \lim_{T\to\infty}\int_{-\infty}^{\infty} \frac{e^{-i\omega T}}{i\omega}\phi(\omega)d\omega \\ &= < \mathrm{Pv}\frac{1}{i\omega}, \phi > - \lim_{T\to\infty}\int_{-\infty}^{\infty} \frac{e^{-i\omega T}}{i\omega}\phi(\omega)d\omega.\end{aligned}$$

The last integral can be written as

$$\lim_{T\to\infty}\int_{-\infty}^{\infty}\left(\frac{\phi(\omega)}{i\omega}\cos T\omega d\omega + \frac{\phi(\omega)}{i\omega}\sin T\omega d\omega\right) \tag{30}$$

By the fact [4, p. 177], we have

$$\lim_{T\to\infty}\frac{\sin T\omega}{\pi\omega} \longrightarrow \delta(\omega),$$

and the second term in (30) converges to $\pi\phi(0)$. By the fact [7, p. 191], the first term in (30) converges to zero. Hence we have

$$\begin{aligned}< \hat{h}(\omega), \phi(\omega) > &=< \mathrm{pv}\frac{1}{i\omega}, \phi > + < \pi\delta(\omega), \phi(\omega) > \\ &=< \mathrm{pv}\frac{1}{i\omega} + \pi\delta(\omega), \phi(\omega) > .\end{aligned}$$

Therefore we have

$$\hat{h}(\omega) = \pi\delta(\omega) + \mathrm{Pv}\frac{1}{i\omega}.$$

Then we have, by parseval's equality,

$$\begin{aligned}\int_0^\infty q(x,y)dy &= \int_{-\infty}^\infty h(y)q(x,y)dy \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \hat{h}(\omega)\bar{\hat{q}}(x,\omega)d\omega \\ &= \frac{1}{2\pi} < \hat{h}(\cdot), \bar{\hat{q}}(x,\cdot) > .\end{aligned}$$

By taking the Fourier transform of $q(x,y)$ with respect to x and using the poisson summation formula , we obtain

$$\begin{aligned}\bar{\hat{q}}(x,\omega) &= \bar{\hat{\phi}}(\omega)\sum_n \phi(x-n)e^{i\omega n} \\ &= \bar{\hat{\phi}}(\omega)\sum_n \hat{\phi}(\omega-2\pi n)e^{i(\omega-2\pi n)x}.\end{aligned}$$

From the fact $\hat{\phi}(2n\pi) = \delta_{0n}$, we have $\bar{\hat{q}}(x,0) = 1$. By noticing $\hat{h}(\omega) = pv\left(\frac{1}{i\omega}\right) + \pi\delta(\omega)$, where pv is the Cauchy's principal value and δ is the delta functional, we obtain the result. $\square$

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