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CONVERGENCE OF MULTISPLITTING METHODS WITH DIFFERENT WEIGHTING SCHEMES†

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Abstract. In this paper, we first introduce a special type of multisplitting method with different weighting scheme, and then we provide convergence results of multisplitting methods with different weighting schemes corresponding to both the AOR-like multisplitting and the SSOR-like multisplitting.

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1. Introduction

In this paper, we consider multisplitting methods with different weighting schemes for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse nonsingular matrix.

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive), and $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. For two vectors x, $y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). These definitions carry immediately over to matrices. For a square matrix A, $\mathrm{diag}(A)$ denotes a diagonal matrix whose diagonal part coincides with the diagonal part of A. Let $\rho(A)$ denote the *spectral radius* of a square matrix A. Varga [9] showed that for any two square matrices A and B, $|A| \leq B$ implies $\rho(A) \leq \rho(B)$.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if $a_{ij} \leq 0$ for $i \neq j$ and A is nonsingular with $A^{-1} \geq 0$. The *comparison matrix* $\langle A \rangle = (\alpha_{ij})$ of a matrix

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$A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j \end{cases}.$$

A matrix A is called an *H-matrix* if $\langle A \rangle$ is an M-matrix.

A representation $A = M - N$ is called a *splitting* of A if M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$. It is well known that if $A = M - N$ is a regular splitting of A, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$ [1, 9]. A splitting $A = M - N$ is called an *H-compatible splitting* of A if $\langle A \rangle = \langle M \rangle - |N|$. It was shown in [5] that if A is an H-matrix and $A = M - N$ is an H-compatible splitting of A, then $\rho(M^{-1}N) < 1$.

This paper is organized as follows. In Section 2, we introduce a special type of multisplitting method with different weighting scheme for solving the linear system (1), and then we provide convergence results of multisplitting methods with different weighting schemes corresponding to both the AOR-like multisplitting and the SSOR-like multisplitting. Lastly, some concluding remarks are withdrawn.

2. Multisplitting method with different weighting schemes

In this section, we study convergence of a special type of multisplitting method with different weighting schemes for solving the linear system (1).

Let $(M_k, N_k, E_k),\ k = 1, 2, \cdots, \ell$, be a multisplitting of A. Given a parameter $\lambda \in [0, 1]$ and an initial vector x_0, multisplitting method with different weighting schemes (depending on λ) for solving $Ax = b$ is defined by [11]

$$\begin{aligned} x_{i+1} &= H_\lambda x_i + G_\lambda b \\ &= x_i + G_\lambda(b - Ax_i),\ i = 0, 1, 2, \cdots, \end{aligned} \tag{2}$$

where

$$G_\lambda = \sum_{k=1}^{\ell} {E_k}^\lambda {M_k}^{-1} {E_k}^{1-\lambda} \text{ and } H_\lambda = I - G_\lambda A. \tag{3}$$

Here, ${E_k}^\lambda$ denotes the diagonal matrix obtained from E_k by replacing all diagonal entries by their λ-th power when for $\lambda \neq 0$, and ${E_k}^0 := I$. The case $\lambda = 1$ is the *multisplitting method with postweighting* which is usually called the multisplitting method and has been extensively studied in the literature, see [2, 3, 4, 7, 8, 10, 12]. The case $\lambda = \frac{1}{2}$ is called the *multisplitting method with symmetric weighting.* As is pointed out in [11], symmetric weighting is the appropriate choice when using certain multisplittings as preconditioners for the conjugate gradient method, provided A is symmetric. The case $\lambda = 0$ is called the *multisplitting method with preweighting* [6].

We first introduce a special type of multisplitting (M_k, N_k, E_k), $k = 1, 2, \cdots, l$, of A which is described below. For simplicity of exposition, we assume that $\ell = 3$.

Let A be partitioned into

$$A = \begin{pmatrix} A_1 & -C_{12} & -C_{13} & -C_{14} \\ -C_{21} & A_2 & -C_{23} & -C_{24} \\ -C_{31} & -C_{32} & A_3 & -C_{34} \\ -C_{41} & -C_{42} & -C_{43} & A_4 \end{pmatrix}, \tag{4}$$

where A_i's are square matrices. Let $A_k = B_k - C_k$ $(1 \le k \le \ell+1)$ be a splitting of A_k. Let

$$\begin{aligned}
M_1 &= \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ -C_{41} & 0 & 0 & B_4 \end{pmatrix}, \quad E_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 I \end{pmatrix}, \\
M_2 &= \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & -C_{42} & 0 & B_4 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_2 I \end{pmatrix}, \\
M_3 &= \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & -C_{43} & B_4 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & e_3 I \end{pmatrix}, \\
N_k &= M_k - A \ \ (1 \le k \le 3),
\end{aligned} \tag{5}$$

where $\sum_{k=1}^{\ell} e_k = 1$. Using this multisplitting (M_k, N_k, E_k), $k = 1, 2, 3, \cdots, \ell$, of A, G_λ and H_λ are of the form

$$\begin{aligned}
G_\lambda &= \sum_{k=1}^{\ell} {E_k}^{\lambda} {M_k}^{-1} {E_k}^{1-\lambda} \\
&= \begin{pmatrix} {B_1}^{-1} & 0 & 0 & 0 \\ 0 & {B_2}^{-1} & 0 & 0 \\ 0 & 0 & {B_3}^{-1} & 0 \\ {e_1}^{\lambda} {B_4}^{-1} C_{41} {B_1}^{-1} & {e_2}^{\lambda} {B_4}^{-1} C_{42} {B_2}^{-1} & {e_3}^{\lambda} {B_4}^{-1} C_{43} {B_3}^{-1} & {B_4}^{-1} \end{pmatrix},
\end{aligned}$$

$$H_\lambda = I - G_\lambda A = \begin{pmatrix} {B_1}^{-1} C_1 & {B_1}^{-1} C_{12} & {B_1}^{-1} C_{13} & {B_1}^{-1} C_{14} \\ {B_2}^{-1} C_{21} & {B_2}^{-1} C_2 & {B_2}^{-1} C_{23} & {B_2}^{-1} C_{24} \\ {B_3}^{-1} C_{31} & {B_3}^{-1} C_{32} & {B_3}^{-1} C_3 & {B_3}^{-1} C_{34} \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix},$$

where

$$\beta_i = \sum_{k=1,k\neq i}^{\ell} {e_k}^{\lambda} {B_4}^{-1} C_{4,k} {B_k}^{-1} C_{k,i}$$
$$+ (1-{e_i}^{\lambda}) {B_4}^{-1} C_{4,i} + {e_i}^{\lambda} {B_4}^{-1} C_{4,i} {B_i}^{-1} C_i \text{ for } i = 1,2,\cdots \ell,$$

$$\beta_4 = \sum_{k=1}^{\ell} {e_k}^{\lambda} {B_4}^{-1} C_{4,k} {B_k}^{-1} C_{k,4} + {B_4}^{-1} C_4.$$

Theorem 2.1 ([4, 11]). *Let $(M_k, N_k, E_k), k = 1, 2, \cdots, \ell$, be a multisplitting of A with M_k and E_k defined as in (5), $G_\lambda = \sum_{k=1}^{\ell} {E_k}^{\lambda} {M_k}^{-1} {E_k}^{1-\lambda}$ and $H_\lambda = I - G_\lambda A$.*

(a) *If $A^{-1} \geq 0$, $A_k = B_k - C_k$ is a weak regular splitting of A_k and $C_{ij} \geq 0$, then $\rho(H_\lambda) < 1$ for all $\lambda \in [0, 1]$*

(b) *If A is an H-matrix and $A_k = B_k - C_k$ is an H-compatible splitting of A_k, then $\rho(H_\lambda) < 1$ for all $\lambda \in [0, 1]$.*

We now provide a convergence result of multisplitting method with different weighting schemes corresponding to the AOR-like multisplitting of the form (5) when A is an H-matrix.

Theorem 2.2. *Assume that A is an H-matrix with $A = D - F$, where $D = \mathrm{diag}(A)$. Let (M_k, N_k, E_k) $(1 \leq k \leq \ell)$ be a multisplitting of A with M_k and E_k defined as in (5), where*

$$B_k = \frac{1}{\omega}(D_k - \gamma L_k), \ \ C_k = \frac{1}{\omega}\left((1-\omega)D_k + (\omega-\gamma)L_k + \omega V_k\right), \tag{6}$$

$D_k = \mathrm{diag}(A_k)$, L_k is a strictly lower triangular matrix and V_k is a general matrix satisfying $V_k = D_k - L_k - A_k$. If $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$ and $\langle A_k \rangle = |D_k| - |L_k| - |V_k|$, then for all $\lambda \in [0, 1]$,

$$\rho(H_\lambda) < 1,$$

where $G_\lambda = \sum_{k=1}^{\ell} {E_k}^{\lambda} M_k^{-1} {E_k}^{1-\lambda}$, $H_\lambda = I - G_\lambda A$ and $\alpha = \rho(|D|^{-1}|F|)$.

Proof. We consider the first case where $0 < \omega \leq 1$. Since $\langle A_k \rangle = |D_k| - |L_k| - |V_k|$, the corresponding coefficients of $(\omega - \gamma)L_k$ and ωV_k have the same sign for $k = 1, 2, \cdots, \ell + 1$. From equation (6), one obtains for $k = 1, 2, \cdots, \ell + 1$,

$$\begin{aligned}\langle B_k \rangle - |C_k| &= \langle \frac{1}{\omega}(D_k - \gamma L_k) \rangle - |\frac{1}{\omega}\left((1-\omega)D_k + (\omega-\gamma)L_k + \omega V_k\right)| \\ &= \frac{1}{\omega}(|D_k| - \gamma |L_k|) - \frac{1}{\omega}\left((1-\omega)|D_k| + (\omega-\gamma)|L_k| + \omega |V_k|\right) \\ &= |D_k| - |L_k| - |V_k| = \langle A_k \rangle.\end{aligned}$$

Hence, $A_k = B_k - C_k$ is an H-compatible splitting of A_k for $k = 1, 2, \cdots, \ell + 1$. By Theorem 2.1, $\rho(H_\lambda) < 1$ for $0 < \omega \leq 1$. Next we consider the case where

$1 < \omega < \frac{2}{1+\alpha}$. For $k = 1, 2, \cdots, \ell+1$, let

$$\tilde{C_k} = \frac{1}{\omega}\left((\omega-1)D_k + (\omega-\gamma)L_k + \omega V_k\right),$$
$$\tilde{A_k} = B_k - \tilde{C_k}.$$

Then, it can be easily seen that for $k = 1, 2, \cdots, \ell+1$,

$$\tilde{A_k} = \frac{2-\omega}{\omega}D_k - L_k - V_k.$$

Let $\tilde{A} = \frac{2-\omega}{\omega}D - F$. Then $\langle \tilde{A} \rangle = \frac{2-\omega}{\omega}|D| - |F|$ is a regular splitting of $\langle \tilde{A} \rangle$. Since $1 < \omega < \frac{2}{1+\alpha}$, $\rho\left(\frac{\omega}{2-\omega}|D|^{-1}|F|\right) = \frac{\omega}{2-\omega}\rho(|D|^{-1}|F|) = \frac{\omega\alpha}{2-\omega} < 1$. Hence, $\langle \tilde{A} \rangle^{-1} \geq 0$. Since $A_k = D_k - L_k - V_k$, $\tilde{A_k}$ is clearly a block diagonal components of $\tilde{A}$. Notice that for $k = 1, 2, \cdots, \ell+1$,

$$\begin{aligned}\langle B_k \rangle - |\tilde{C_k}| &= \frac{1}{\omega}(|D_k| - \gamma|L_k|) - \frac{1}{\omega}((\omega-1)|D_k| + (\omega-\gamma)|L_k| + \omega|V_k|) \\ &= \frac{2-\omega}{\omega}|D_k| - |L_k| - |V_k| = \langle \tilde{A_k} \rangle.\end{aligned}$$

Note that $\langle \tilde{A} \rangle$ can be written as

$$\langle \tilde{A} \rangle = \begin{pmatrix} \langle \tilde{A_1} \rangle & -|C_{1,2}| & \cdots & -|C_{1,\ell+1}| \\ -|C_{2,1}| & \langle \tilde{A_2} \rangle & \cdots & -|C_{2,\ell+1}| \\ \vdots & \vdots & \ddots & \vdots \\ -|C_{\ell+1,1}| & -|C_{\ell+1,2}| & \cdots & \langle \tilde{A}_{\ell+1} \rangle \end{pmatrix}$$

Let for $k = 1, 2, \cdots, \ell$,

$$\tilde{M_k} = \begin{pmatrix} \langle B_1 \rangle & & 0 & & \cdots & & 0 \\ & \ddots & & & & & \\ 0 & & \langle B_k \rangle & & & & \\ & & 0 & \ddots & & & \vdots \\ \vdots & & \vdots & & \ddots & & \\ & & 0 & & & & \\ 0 & \cdots & -|C_{\ell+1,k}| & 0 & \cdots & 0 & \langle B_{\ell+1} \rangle \end{pmatrix} \quad \text{and} \quad \tilde{N_k} = \tilde{M_k} - \langle \tilde{A} \rangle.$$

Then $(\tilde{M_k}, \tilde{N_k}, E_k)$, $k = 1, 2, \cdots \ell$, is a multisplitting of $\langle \tilde{A} \rangle$ of the form (5). Since $\langle \tilde{A} \rangle^{-1} \geq 0$ and $\langle \tilde{A_k} \rangle = \langle B_k \rangle - |\tilde{C_k}|$ is a regular splitting of $\langle \tilde{A_k} \rangle$ for $k = 1, 2, \cdots, \ell+1$, $\rho(\tilde{H_\lambda}) < 1$ from Theorem 2.1, where

$$\tilde{H_\lambda} = \begin{pmatrix} \langle B_1 \rangle^{-1}|\tilde{C_1}| & \langle B_1 \rangle^{-1}|C_{1,2}| & \cdots & \langle B_1 \rangle^{-1}|C_{1,\ell}| & \langle B_1 \rangle^{-1}|C_{1,\ell+1}| \\ \langle B_2 \rangle^{-1}|C_{2,1}| & \langle B_2 \rangle^{-1}|\tilde{C_2}| & \cdots & \langle B_2 \rangle^{-1}|C_{2,\ell}| & \langle B_2 \rangle^{-1}|C_{2,\ell+1}| \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle B_\ell \rangle^{-1}|C_{\ell,1}| & \langle B_\ell \rangle^{-1}|C_{\ell,2}| & \cdots & \langle B_\ell \rangle^{-1}|\tilde{C_\ell}| & \langle B_\ell \rangle^{-1}|C_{\ell,\ell+1}| \\ \tilde{\beta}_1 & \tilde{\beta}_2 & \cdots & \tilde{\beta}_\ell & \tilde{\beta}_{\ell+1} \end{pmatrix},$$

$$\tilde{\beta}_i = \sum_{k=1,k\neq i}^{\ell} {e_k}^{\lambda}\langle B_{\ell+1}\rangle^{-1}|C_{\ell+1,k}|\langle B_k\rangle^{-1}|C_{k,i}| + (1-{e_i}^{\lambda})\langle B_{\ell+1}\rangle^{-1}|C_{\ell+1,i}|$$
$$+ {e_i}^{\lambda}\langle B_{\ell+1}\rangle^{-1}|C_{\ell+1,i}|\langle B_i\rangle^{-1}|\tilde{C}_i| \text{ for } i = 1,2,\cdots \ell,$$
$$\tilde{\beta}_{\ell+1} = \sum_{k=1}^{\ell} {e_k}^{\lambda}\langle B_{\ell+1}\rangle^{-1}|C_{\ell+1,k}|\langle B_k\rangle^{-1}|C_{k,\ell+1}| + \langle B_{\ell+1}\rangle^{-1}|\tilde{C}_{\ell+1}|.$$

Since B_k is an H-matrix for $1 \leq k \leq \ell+1$, one obtains

$$|{B_k}^{-1}| \leq \langle B_k\rangle^{-1} \quad \text{and} \quad |C_k| \leq |\tilde{C}_k|.$$

Using these inequalities, $|H_\lambda| \leq \tilde{H}_\lambda$ is obtained. Thus, $\rho(H_\lambda) < 1$ for $1 < \omega < \frac{2}{1+\alpha}$. Therefore, $\rho(H_\lambda) < 1$ for $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$. □

If $\gamma = \omega$ in Theorem 2.2, then Theorem 2.2 reduces to a convergence result of multisplitting method with different weighting schemes corresponding to the SOR-like multisplitting of the form (5) when A is an H-matrix.

Note that if A is an M-matrix, then A is an H-matrix. We easily obtain the following corollary which is a convergence result of multisplitting method with different weighting schemes corresponding to the AOR-like multisplitting of the form (5) when A is an M-matrix, respectively.

Corollary 2.3. *Assume that A is an M-matrix with $A = D - F$, where $D = \operatorname{diag}(A)$. Let $(M_k, N_k, E_k), k = 1,2,\cdots,\ell$, be a multisplitting of A with M_k and E_k defined as in (5), where*

$$B_k = \frac{1}{\omega}(D_k - \gamma L_k), \quad C_k = \frac{1}{\omega}\left((1-\omega)D_k + (\omega-\gamma)L_k + \omega V_k\right),$$

$D_k = \operatorname{diag}(A_k)$, L_k is a nonnegative strictly lower triangular matrix and V_k is a nonnegative general matrix satisfying $V_k = D_k - L_k - A_k$. If $0 < \gamma \leq \omega < \frac{2}{1+\alpha}$, then for all $\lambda \in [0,1]$,

$$\rho(H_\lambda) < 1,$$

where $G_\lambda = \sum_{k=1}^{\ell} {E_k}^{\lambda}{M_k}^{-1}{E_k}^{1-\lambda}$, $H_\lambda = I - G_\lambda A$ and $\alpha = \rho(D^{-1}F)$.

Proof. Since $L_k \geq 0$ and $V_k \geq 0$, $\langle A_k\rangle = A_k = D_k - L_k - V_k = |D_k| - |L_k| - |V_k|$. From Theorem 2.2, the proof is complete. □

We next provide a convergence result of multisplitting method with different weighting schemes corresponding to the SSOR-like multisplitting of the form (5) when A is an H-matrix.

Theorem 2.4. *Assume that A is an H-matrix with $A = D - F$, where $D = \operatorname{diag}(A)$. Let $(M_k, N_k, E_k), k = 1,2,\cdots,\ell$, be a multisplitting of A with M_k and E_k defined as in (5), where*

$$\begin{aligned} B_k &= \frac{1}{\omega(2-\omega)}(D - \omega L_k){D_k}^{-1}(D_k - \omega V_k), \\ C_k &= \frac{1}{\omega(2-\omega)}\big((1-\omega)D_k + \omega L_k\big){D_k}^{-1}\big((1-\omega)D_k + \omega V_k\big), \end{aligned} \tag{7}$$

$D_k = \mathrm{diag}(A)$, L_k is a strictly lower triangular matrix and V_k is a general matrix satisfying $V_k = D_k - L_k - A_k$. If $0 < \omega < \frac{2}{1+\alpha}$ and $\langle A_k \rangle = |D_k| - |L_k| - |V_k|$, then for all $\lambda \in [0,1]$,

$$\rho(H_\lambda) < 1,$$

where $\alpha = \rho(|D|^{-1}|F|)$, $G_\lambda = \sum_{k=1}^{\ell} {E_k}^{\lambda} M_k^{-1} {E_k}^{1-\lambda}$ and $H_\lambda = I - G_\lambda A$.

Proof. We consider the first case where $0 < \omega \leq 1$. From the assumption, one obtains for $k = 1, 2, \cdots, \ell+1$,

$$\begin{aligned}\langle A_k \rangle =& \frac{1}{\omega(2-\omega)}(|D_k| - \omega|L_k|)|D_k|^{-1}(|D_k| - \omega|V_k|) \\ &- \frac{1}{\omega(2-\omega)}\big((1-\omega)|D_k| + \omega|L_k|\big)|D_k|^{-1}\big((1-\omega)|D_k| + \omega|V_k|\big).\end{aligned}$$

For $k = 1, 2, \cdots, \ell+1$, let

$$\begin{aligned}\tilde{B}_k &= \frac{1}{\omega(2-\omega)}(|D_k| - \omega|L_k|)|D_k|^{-1}(|D_k| - \omega|V_k|), \\ \tilde{C}_k &= \frac{1}{\omega(2-\omega)}\big((1-\omega)|D_k| + \omega|L_k|\big)|D_k|^{-1}\big((1-\omega)|D_k| + \omega|V_k|\big).\end{aligned}$$

Then $\langle A_k \rangle = \tilde{B}_k - \tilde{C}_k$ is a regular splitting of $\langle A_k \rangle$ for $k = 1, 2, \cdots, \ell+1$. Let for $k = 1, 2, \cdots, \ell$,

$$\tilde{M}_k = \begin{pmatrix} \tilde{B}_1 & & 0 & & \cdots & & 0 \\ & \ddots & & & & & \\ 0 & & \tilde{B}_k & & & & \\ & & 0 & \ddots & & & \vdots \\ \vdots & & \vdots & & \ddots & & \\ & & 0 & & & & \\ 0 & \cdots & -|C_{\ell+1,k}| & 0 & \cdots & 0 & \tilde{B}_{\ell+1} \end{pmatrix} \quad \text{and} \quad \tilde{N}_k = \tilde{M}_k - \langle A \rangle.$$

Then $(\tilde{M}_k, \tilde{N}_k, E_k)$, $k = 1, 2, \cdots, \ell$, is a multisplitting of $\langle A \rangle$ of the form (5). Since $\langle A \rangle^{-1} \geq 0$, $\rho(\tilde{H}_\lambda) < 1$ from Theorem 2.1, where

$$\tilde{H}_\lambda = \begin{pmatrix} \tilde{B}_1^{-1}\tilde{C}_1 & \tilde{B}_1^{-1}|C_{1,2}| & \cdots & \tilde{B}_1^{-1}|C_{1,\ell}| & \tilde{B}_1^{-1}|C_{1,\ell+1}| \\ \tilde{B}_2^{-1}|C_{2,1}| & \tilde{B}_2^{-1}\tilde{C}_2 & \cdots & \tilde{B}_2^{-1}|C_{2,\ell}| & \tilde{B}_2^{-1}|C_{2,\ell+1}| \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{B}_\ell^{-1}|C_{\ell,1}| & \tilde{B}_\ell^{-1}|C_{\ell,2}| & \cdots & \tilde{B}_\ell^{-1}\tilde{C}_\ell & \tilde{B}_\ell^{-1}|C_{\ell,\ell+1}| \\ \tilde{\beta}_1 & \tilde{\beta}_2 & \cdots & \tilde{\beta}_\ell & \tilde{\beta}_{\ell+1} \end{pmatrix}, \tag{8}$$

$$\begin{aligned}\tilde{\beta}_i = &\sum_{k=1,k\neq i}^{\ell} e_k{}^{\lambda}\tilde{B}_{\ell+1}^{-1}|C_{\ell+1,k}|\tilde{B}_k^{-1}|C_{k,i}| + (1-e_i{}^{\lambda})\tilde{B}_{\ell+1}^{-1}|C_{\ell+1,i}| \\ &+ e_i{}^{\lambda}\tilde{B}_{\ell+1}^{-1}|C_{\ell+1,i}|\tilde{B}_i^{-1}\tilde{C}_i \ \text{ for } \ i=1,2,\cdots \ell, \\ \tilde{\beta}_{\ell+1} = &\sum_{k=1}^{\ell} e_k{}^{\lambda}\tilde{B}_{\ell+1}^{-1}|C_{\ell+1,k}|\tilde{B}_k^{-1}|C_{k,\ell+1}| + \tilde{B}_{\ell+1}^{-1}\tilde{C}_{\ell+1}.\end{aligned}$$

Since A_k is an H-matrix, $D_k-\omega L_k$ and $D_k-\omega V_k$ are H-matrices for $k=1,2,\cdots,\ell+1$. Hence one obtains

$$\begin{aligned}|(D_k-\omega L_k)^{-1}| &\le (|D_k|-\omega|L_k|)^{-1}, \\ |(D_k-\omega V_k)^{-1}| &\le (|D_k|-\omega|V_k|)^{-1}, \\ |B_k{}^{-1}| \le \tilde{B}_k^{-1} \ &\text{ and } \ |C_k| \le \tilde{C}_k.\end{aligned}$$

Using these inequalities, $|H_\lambda| \le \tilde{H}_\lambda$ is obtained. Therefore, $\rho(H_\lambda)<1$ for $0<\omega\le 1$. Next we consider the case where $1<\omega<\frac{2}{1+\alpha}$. Let

$$\hat{C}_k = \frac{1}{\omega(2-\omega)}\big((\omega-1)|D_k|+\omega|L_k|\big)|D_k|^{-1}\big((\omega-1)|D_k|+\omega V_k\big).$$

Then one obtains for $k=1,2,\cdots,\ell+1$,

$$\tilde{B}_k-\hat{C}_k = \frac{\omega}{2-\omega}\left(\frac{2-\omega}{\omega}|D_k|-|L_k|-|V_k|\right).$$

Let $\tilde{A}=|D|-\frac{\omega}{2-\omega}|F|$ and $\tilde{A}_k=\frac{2-\omega}{\omega}|D_k|-|L_k|-|V_k|$ for $k=1,2,\cdots,\ell+1$. Since $1<\omega<\frac{2}{1+\alpha}$, $\rho\left(|D|^{-1}\frac{\omega}{2-\omega}|F|\right)=\frac{\omega}{2-\omega}\rho(|D|^{-1}|F|)=\frac{\omega\alpha}{2-\omega}<1$. Thus, $\tilde{A}^{-1}\ge 0$. Note that $\tilde{A}$ can be written as

$$\tilde{A} = \begin{pmatrix} \frac{\omega}{2-\omega}\tilde{A}_1 & -\frac{\omega}{2-\omega}|C_{1,2}| & \cdots & -\frac{\omega}{2-\omega}|C_{14}| \\ -\frac{\omega}{2-\omega}|C_{21}| & \frac{\omega}{2-\omega}\tilde{A}_2 & \cdots & -\frac{\omega}{2-\omega}|C_{24}| \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\omega}{2-\omega}|C_{\ell+1,1}| & -\frac{\omega}{2-\omega}|C_{\ell+1,2}| & \cdots & \frac{\omega}{2-\omega}\tilde{A}_{\ell+1} \end{pmatrix}.$$

Let for $k=1,2,\cdots,\ell$,

$$M_k^\star = \begin{pmatrix} \tilde{B}_1 & & & 0 & & & \cdots & 0 \\ & \ddots & & & & & & \\ 0 & & & \tilde{B}_k & & & & \\ & & & 0 & & \ddots & & \vdots \\ \vdots & & & \vdots & & & \ddots & \\ & & & 0 & & & & \\ 0 & \cdots & & -\frac{\omega}{2-\omega}|C_{\ell+1,k}| & 0 & \cdots & 0 & \tilde{B}_{\ell+1} \end{pmatrix} \quad \text{and} \quad N_k^\star = M_k^\star - \tilde{A}.$$

Then $(M_k^\star, N_k^\star, E_k)$, $k = 1, 2, \cdots, \ell$, is a multisplitting of $\tilde{A}$ of the form (5). Since $\frac{\omega}{2-\omega}\tilde{A}_k = \tilde{B}_k - \hat{C}_k$ is a regular splitting of $\frac{\omega}{2-\omega}\tilde{A}_k$ for $k = 1, 2, \cdots, \ell+1$, $\rho(H_\lambda^\star) < 1$ from Theorem 2.1, where

$$H_\lambda^\star = \frac{\omega}{2-\omega}\begin{pmatrix} \frac{2-\omega}{\omega}\tilde{B}_1^{-1}\hat{C}_1 & \tilde{B}_1^{-1}|C_{1,2}| & \cdots & \tilde{B}_1^{-1}|C_{1,\ell}| & \tilde{B}_1^{-1}|C_{1,\ell+1}| \\ \tilde{B}_2^{-1}|C_{2,1}| & \frac{2-\omega}{\omega}\tilde{B}_2^{-1}\hat{C}_2 & \cdots & \tilde{B}_2^{-1}|C_{2,\ell}| & \tilde{B}_2^{-1}|C_{2,\ell+1}| \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{B}_\ell^{-1}|C_{\ell,1}| & \tilde{B}_\ell^{-1}|C_{\ell,2}| & \cdots & \frac{2-\omega}{\omega}\tilde{B}_\ell^{-1}\hat{C}_\ell & \tilde{B}_\ell^{-1}|C_{\ell,\ell+1}| \\ \beta_1^\star & \beta_2^\star & \cdots & \beta_\ell^\star & \beta_{\ell+1}^\star \end{pmatrix},$$

$$\begin{aligned} \beta_i^\star =& \frac{\omega}{2-\omega}\sum_{k=1,k\neq i}^{\ell} {e_k}^\lambda \tilde{B}_{\ell+1}^{-1}|C_{\ell+1,k}|\tilde{B}_k^{-1}|C_{k,i}| + (1-{e_i}^\lambda)\tilde{B}_{\ell+1}^{-1}|C_{\ell+1,i}| \\ &+ {e_i}^\lambda \tilde{B}_{\ell+1}^{-1}|C_{\ell+1,i}|\tilde{B}_i^{-1}\hat{C}_i \ \text{ for } \ i = 1, 2, \cdots \ell, \\ \beta_{\ell+1}^\star =& \frac{\omega}{2-\omega}\sum_{k=1}^{\ell} {e_k}^\lambda \tilde{B}_{\ell+1}^{-1}|C_{\ell+1,k}|\tilde{B}_k^{-1}|C_{k,\ell+1}| + \tilde{B}_{\ell+1}^{-1}\hat{C}_{\ell+1}. \end{aligned}$$

Since $|B_k^{-1}| \le \tilde{B}_k^{-1}$, $|C_k| \le \hat{C}_k$ and $\frac{\omega}{2-\omega} > 1$, $|H_\lambda| \le H_\lambda^\star$ is obtained. Thus, $\rho(H_\lambda) < 1$ for $1 < \omega < \frac{2}{1+\alpha}$. Therefore, $\rho(H_\lambda) < 1$ for all $0 < \omega < \frac{2}{1+\alpha}$. $\square$

The following corollary for an M-matrix A can be directly obtained from Theorem 2.4.

Corollary 2.5. *Assume that A is an M-matrix with $A = D - F$, where $D = \operatorname{diag}(A)$. Let $(M_k, N_k, E_k), k = 1, 2, \cdots, \ell$, be a multisplitting of A with M_k and E_k defined as in (5), where $B_k = \frac{1}{\omega(2-\omega)}(D_k - \omega L_k){D_k}^{-1}(D_k - \omega V_k)$, $C_k = \frac{1}{\omega(2-\omega)}\big((1-\omega)D_k + \omega L_k\big){D_k}^{-1}\big((1-\omega)D_k + \omega V_k\big)$, $D_k = \operatorname{diag}(A_k)$, L_k is a nonnegative strictly lower triangular matrix and V_k is a nonnegative general matrix satisfying $V_k = D_k - L_k - A_k$. If $0 < \omega < \frac{2}{1+\alpha}$, then for all $\lambda \in [0, 1]$,*

$$\rho(H_\lambda) < 1,$$

where $\alpha = \rho(D^{-1}F)$, $G_\lambda = \sum_{k=1}^{\ell} {E_k}^\lambda M_k^{-1}{E_k}^{1-\lambda}$ and $H_\lambda = I - G_\lambda A$.

3. Concluding remarks

In this paper, we provided convergence results of a special type of multisplitting methods with different weighting schemes corresponding to both the AOR-like multisplitting and the SSOR-like multisplitting. Future work will include numerical experiments for these multisplitting methods in order to find an optimal parameter λ.

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