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ON INTUITIONISTIC FUZZY PRIME Γ-IDEALS OF Γ-LA-SEMIGROUPS

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Abstract. In this paper, we introduce and study the intuitionistic fuzzy prime (semi-prime) $\Gamma-$ideals of $\Gamma-$LA-semigroups and some interesting properties are investigated. The main result of the paper is: if $A = \langle \mu_A, \gamma_A \rangle$ is an IFS in $\Gamma-$LA-semigroup S, then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime (semi-prime) $\Gamma-$ideal of S if and only if for any $s, t \in [0,1]$, the sets $U(\mu_A, s) = \{x \in S : \mu_A(x) \geq s\}$ and $L(\gamma_A, t) = \{x \in S : \gamma_A(x) \leq t\}$ are prime (semi-prime) $\Gamma-$ideals of S.

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1. Introduction

The fundamental concept of fuzzy subset as a mapping from a non-empty set S to unit closed interval *i.e*, $f : S \longrightarrow [0,1]$, was introduced by L. A. Zadeh in 1965 [25]. In the following decades the study of fuzzy subset in algebraic structure has been started in the definitive paper of Rosenfeld 1971 [23]. Fuzzy subgroup and its important properties were defined and established by Rosenfeld [23]. In 1981, Kuroki introduced the concept of fuzzy semigroup in his paper [16]. The concept of a fuzzy ideal in semigroups was first developed by Kuroki. He studied fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals and fuzzy semiprime ideals of semigroups [16, 17, 18, 19, 20, 21, 22]. The concept of fuzzy interior ideals in a semigroup was introduced by Hong [10] and he obtained some related properties of such ideals. Recently, in [3] M. Aslam et.al., characterized $\Gamma-$LA-semigroup by the properties of generalized fuzzy Γ ideals.

The idea of intuitionistic fuzzy set was first published by K. T. Atanassov in his pioneer papers [5, 6], as generalization of the notion of fuzzy sets. Gau and Buehre in [9], presented the concept of vague sets. But, Burillo and Bustine in

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[8], have shown that the notion of vague sets coincides with that of intuitionistic fuzzy sets. R. Biswas in [7], introduced the notion of intuitionistic fuzzy subgroup of a group by using the notion of intuitionistic fuzzy sets and obtained some useful properties. Recently, K. H. Kim and Y. B. Jun in [14], introduced the notion of intuitionistic fuzzy ideal of semigroup and some basic properties have been investigated. K. H. Kim and J. G. Lee in [15], initiated the concept of intuitionistic fuzzy bi-ideal of semigroup and obtained some useful properties. In 2001, K. H. Kim and Y. B. Jun in [13], introduced the notion of intuitionistic fuzzy interior ideal of semigroup and some fundamental properties were investigated. Recently, M. Khan et.al., introduced the notion of intuitionistic fuzzy ideals in ordered semigroups and obtained some important results in [12]. The concept of LA-semigroup was first introduced by Kazim and Naseerudin [11]. Let S be a non-empty set. Then, $(S, *)$ is called an LA-semigroup, if S is closed and satisfies the identity $(x * y) * z = (z * y) * x$ for all $x, y, z \in S$, which is called left invertive law. Later, Q. Mushtaq and others have investigated the structure further and added many useful results to theory of LA-semigroups. Recently, S. Abdullah et. al intdroduced the notion of direct product intuitionistic fuzzy ideals of LA-semigroups in his papers [2, 1]. Recently, T. Shah and I. Rehman have introduced the concept of Γ-LA-semigroup [24]. Whereas the Γ-LA-semigroup is a generalization of LA-semigroup.

Our aim in this paper is to introduce and study the intuitionistic fuzzy prime (semi-prime) Γ-ideals of Γ-LA-semigroups by using the notion of intuitionistic fuzzy set. Main result of the paper is: if $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in Γ-LA-semigroup S, then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime (semi-prime) Γ-ideal of S if and only if for any $s, t \in [0, 1]$, the sets $U(\mu_A, s) = \{x \in S : \mu_A(x) \geq s\}$ and $L(\gamma_A, t) = \{x \in S : \gamma_A(x) \leq t\}$ are prime (semi-prime) Γ-ideals of S. Moreover if S is a Γ-LA-semigroup and $\emptyset \neq P \subseteq S$ is prime (semi-prime) Γ-ideal of S, then $A = \langle \mathcal{X}_P, \overline{\mathcal{X}_P} \rangle$ is an intuitionistic fuzzy prime (semi-prime) Γ-ideal of S.

2. Preliminaries

Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then S is called a Γ-LA-semigroup if it satisfying $x\gamma y \in S$ and $(x\beta y)\gamma z = (z\beta y)\gamma x$ for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$. A non-empty set U of a Γ-LA-semigroup S is said to be a subΓ-LA-semigroup S if $U\Gamma U \subseteq U$. A left (resp. right) Γ-ideal I of a Γ-LA-semigroup S is non-empty subset I of S such that $S\Gamma I \subseteq I$ (resp. $I\Gamma S \subseteq I$). If I is both a left and a right Γ-ideal of a Γ-LA-semigroup S, then I is called a Γ-ideal of S. A subΓ-LA-semigroup B of Γ-LA-semigroup S is called bi-Γ-ideal of S, if $(B\Gamma S)\Gamma B \subseteq B$. A Γ-ideal P of Γ-LA-semigroup S is said to be prime if $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, for all Γ-ideals A and B in S. A Γ-ideal P is called semi-prime if $I\Gamma I \subseteq P$ implies that $I \subseteq P$, for any Γ-ideals I of S. If every Γ-ideal of Γ-LA-semigroup S is a semi-prime, then S is said to be fully semiprime and if every Γ-ideal is prime, then S is called fully prime. An element a of a Γ-LA-semigroup S is called regular if for there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = (x\alpha a)\,\beta x$. If every element of a Γ-LA-semigroup S is left regular, then S is called left regular (See [24]).

A fuzzy set μ in a Γ-LA-semigroup S is called a fuzzy subΓ-LA-semigroup of S, if $\mu_A(x\gamma y) \geq \mu_A(x) \wedge \mu_A(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$. A fuzzy set μ in a Γ-LA-semigroup S is called a fuzzy bi-Γ-ideal, if $\mu_A(x\gamma y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\mu_A((x\gamma y)\beta z) \geq \mu_A(x) \wedge \mu_A(z)$ for all $x, y, z \in S$ and $\gamma, \beta \in \Gamma$. A fuzzy set μ in a Γ-LA-semigroup S is called fuzzy left (resp. right) Γ-ideal of S, if $\mu_A(x\gamma y) \geq \mu_A(y)$ (resp. $\mu_A(x\gamma y) \geq \mu_A(x)$) for all $x, y \in S$ and $\gamma \in \Gamma$. A fuzzy set μ in a Γ-LA-semigroup S is called fuzzy Γ-ideal of S, if μ is both fuzzy left Γ-ideal and fuzzy right Γ-ideal of a Γ-LA-semigroup S. A fuzzy Γ-ideal of S is called fuzzy prime (semi-prime) Γ-ideal of S if $\inf_{\gamma\in\Gamma} \mu\,(x\gamma y) = \max\{\mu\,(x)\,, \mu\,(y)\}$ $(\mu_A\,(x) \geq \mu_A\,(x\gamma x)\ )$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Definition 2.1 ([5, 6])**.** Let X be a non-empty fixed set. An intuitionistic fuzzy set (briefly, IFS) A is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \longrightarrow [0,1]$ and $\gamma_A : X \longrightarrow [1,0]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in S$. For the sake of simplicity, we use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

Definition 2.2 ([3])**.** An IFS $A = (\mu_A, \gamma_A)$ in S is called an intuitionistic fuzzy right (resp. left) Γ-ideal of S if satisfies $\mu_A(x\gamma y) \geq \mu_A(x)$ and $\gamma_A(x\gamma y) \leq \gamma_A(x)$ (resp. $\mu_A(x\gamma y) \geq \mu_A(y)$ and $\gamma_A(x\gamma y) \leq \gamma_A(y)$) for all $x, y \in S$ and $\gamma \in \Gamma$.

Lemma 2.1. *If $A = \langle\mu_A, \gamma_A\rangle$ and $B = \langle\mu_B, \gamma_B\rangle$ is any intuitionistic fuzzy right Γ-ideal of a regular Γ-LA-semigroup S, then $A\Gamma B = A \cap B$*

3. Intuitionistic Fuzzy Prime Γ-Ideals of Γ-LA-semigroups

Definition 3.1. Let $A = \langle\mu_A, \gamma_A\rangle$ be an IFS in Γ-LA-semigroup S. Then $A = \langle\mu_A, \gamma_A\rangle$ is called an intuitionistic fuzzy prime if

$(IFP1)$ $\inf_{\gamma\in\Gamma} \mu_A\,(x\gamma y) = \max\{\mu_A\,(x)\,, \mu_A\,(y)\}\,,$
$(IFP2)$ $\sup_{\gamma\in\Gamma} \gamma_A\,(x\gamma y) = \min\{\gamma_A\,(x)\,, \gamma_A\,(y)\}\,,\ \forall x, y \in S$ and $\gamma \in \Gamma$.

An intuitionistic fuzzy Γ-ideal $A = \langle\mu_A, \gamma_A\rangle$ of S is called an intuitionistic fuzzy prime Γ-ideal of S if it is an intuitionistic fuzzy prime.

Let $\mathcal{X}_P$ denote the characteristic function of a non-empty subset P of a Γ-LA-semigroup.

Theorem 3.1. *Let S be Γ-LA-semigroup and $\emptyset \neq P \subseteq S$ is prime Γ-ideal of S. Then, $A = \langle\mathcal{X}_P, \overline{\mathcal{X}_P}\rangle$ is an intuitionistic fuzzy prime Γ-ideal of S, where $\overline{\mathcal{X}_P} = 1 - \mathcal{X}_P$.*

Proof. Let $x, y \in S$ and $\gamma \in \Gamma$. If $x\Gamma y \subseteq P$, then $x \in P$ or $y \in P$. Thus we have

$$\begin{aligned}\inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) &= 1 \text{ and } \mathcal{X}_P(x) = 1 \text{ or } \mathcal{X}_P(y) = 1\\ \inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) &= 1 = \max\{\mathcal{X}_P(x), \mathcal{X}_P(y)\}\end{aligned}$$

and

$$\begin{aligned}1 - \inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) &= 0 \text{ and } 1 - \mathcal{X}_P(x) = 0 \text{ or } 1 - \mathcal{X}_P(y) = 0\\ \sup_{\gamma\in\Gamma} \overline{\mathcal{X}_P}(x\gamma y) &= 0 \text{ and } \overline{\mathcal{X}_P}(x) = 0 \text{ or } \overline{\mathcal{X}_P}(y) = 0\\ \sup_{\gamma\in\Gamma} \overline{\mathcal{X}_P}(x\gamma y) &= 0 = \min\left\{\overline{\mathcal{X}_P}(x), \overline{\mathcal{X}_P}(y)\right\}\end{aligned}$$

If $x\Gamma y \nsubseteq P$, then $x \notin P$ and $y \notin P$. Thus we have

$$\begin{aligned}\inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) &= 0,\ \mathcal{X}_P(x) = 0 \text{ and } \mathcal{X}_P(y) = 0\\ \inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) &= 0 = \max\{\mathcal{X}_P(x), \mathcal{X}_P(y)\}\end{aligned}$$

and

$$\begin{aligned}1 - \inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) &= 1,\ 1 - \mathcal{X}_P(x) = 1 \text{ and } \mathcal{X}_P(y) = 1\\ \sup_{\gamma\in\Gamma} \overline{\mathcal{X}_P}(x\gamma y) &= 1,\ \overline{\mathcal{X}_P}(x) = 1 \text{ and } \overline{\mathcal{X}_P}(y) = 1\\ \sup_{\gamma\in\Gamma} \overline{\mathcal{X}_P}(x\gamma y) &= 1 = \min\left\{\overline{\mathcal{X}_P}(x), \overline{\mathcal{X}_P}(y)\right\}\end{aligned}$$

Hence, $A = \langle \mathcal{X}_P, \overline{\mathcal{X}_P} \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S. □

Theorem 3.2. *Let P a non-empty subset of S. If $A = \langle \mathcal{X}_P, \overline{\mathcal{X}_P} \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S, then P is a prime Γ-ideal of S, where $\overline{\mathcal{X}_P} = 1 - \mathcal{X}_P$.*

Proof. Suppose that $A = \langle \mathcal{X}_P, \overline{\mathcal{X}_P} \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S. Let $x, y \in S$ such that $x\Gamma y \subseteq P$. Then, $\mathcal{X}_P(x\gamma y) = 1$ for all $\gamma \in \Gamma$. So $\inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) = 1$. Its follows from $(IFP1)$ that

$$1 = \inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) = \max\{\mathcal{X}_P(x), \mathcal{X}_P(y)\}.$$

Hence, $\mathcal{X}_P(x) = 1$ or $\mathcal{X}_P(y) = 1$, so $x \in P$ or $y \in P$. Thus, P is a prime. Now, from $(IFP1)$ that

$$\begin{aligned}0 &= 1 - \inf_{\gamma\in\Gamma} \mathcal{X}_P(x\gamma y) = \sup_{\gamma\in\Gamma} \overline{\mathcal{X}_P}(x\gamma y) = \min\left\{\overline{\mathcal{X}_P}(x), \overline{\mathcal{X}_P}(y)\right\}\\ 0 &= \min\{1 - \mathcal{X}_P(x), 1 - \mathcal{X}_P(y)\}\end{aligned}$$

and so $1 - \mathcal{X}_P(x) = 0$ or $1 - \mathcal{X}_P(y) = 0 \Rightarrow \mathcal{X}_P(x) = 1$ or $\mathcal{X}_P(y) = 1$, so $x \in P$ or $y \in P$. Thus, P is a prime. □

Lemma 3.3. *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S, then μ_A and $\overline{\gamma_A}$ are fuzzy prime Γ-ideals of S.*

Proof. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S, so for any $x, y \in S$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}
\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) &= \max\{\mu_A(x), \mu_A(y)\} \text{ and} \\
\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) &= \min\{\gamma_A(x), \gamma_A(y)\} \\
1 - \sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) &= 1 - \min\{\gamma_A(x), \gamma_A(y)\} \\
\inf_{\gamma\in\Gamma} \overline{\gamma_A}(x\gamma y) &= \max\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \\
\inf_{\gamma\in\Gamma} \overline{\gamma_A}(x\gamma y) &= \max\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}.
\end{aligned}$$

Hence, μ_A and $\overline{\gamma_A}$ are fuzzy prime Γ-ideals of S. □

Lemma 3.4. *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S, then $\overline{\mu_A}$ and γ_A are anti fuzzy prime Γ-ideals of S.*

Theorem 3.5. *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S, then $\Box A = \langle \mu_A, \overline{\mu_A} \rangle$ and $\Diamond A = \langle \overline{\gamma_A}, \gamma_A \rangle$ are intuitionistic fuzzy prime Γ-ideals of S.*

Proof. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S. So for any $x, y \in S$ and $\gamma \in \Gamma$, we have

$$\begin{aligned}
\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) &= \max\{\mu_A(x), \mu_A(y)\} \\
1 - \inf_{\gamma\in\Gamma} \mu_A(x\gamma y) &= 1 - \max\{\mu_A(x), \mu_A(y)\} \\
\sup_{\gamma\in\Gamma} (1 - \mu_A(x\gamma y)) &= \min\{1 - \mu_A(x), 1 - \mu_A(y)\} \\
\sup_{\gamma\in\Gamma} \overline{\mu_A}(x\gamma y) &= \min\{\overline{\mu_A}(x), \overline{\mu_A}(y)\}.
\end{aligned}$$

Hence, $\Box A = \langle \mu_A, \overline{\mu_A} \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S. Similarly, we have

$$\begin{aligned}
\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) &= \min\{\gamma_A(x), \gamma_A(y)\} \\
1 - \sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) &= 1 - \min\{\gamma_A(x), \gamma_A(y)\} \\
\inf_{\gamma\in\Gamma} (1 - \gamma_A(x\gamma y)) &= \max\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \\
\inf_{\gamma\in\Gamma} \overline{\gamma_A}(x\gamma y) &= \max\{\overline{\gamma_A}(x), \overline{\gamma_A}(y)\}.
\end{aligned}$$

Hence, $\Diamond A = \langle \overline{\gamma_A}, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideals of S. □

Theorem 3.6. *Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS in Γ-LA-semigroup S. Then, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S if and only if for any $s, t \in [0, 1]$, the sets $U(\mu_A, s) = \{x \in S : \mu_A(x) \geq s\} \neq \emptyset$ and $L(\gamma_A, t) = \{x \in S : \gamma_A(x) \leq t\} \neq \emptyset$ are prime Γ-ideals of S.*

Proof. Suppose that $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S. Let $s, t \in [0,1]$ such that $U(\mu_A, s)$ and $L(\mu_A, t)$ are non-empty. Now, let $x, y \in S$ such that $x\Gamma y \subseteq U(\mu_A, s)$. Then, $\mu_A(x\gamma y) \geq s$ for all $\gamma \in \Gamma$. This implies that $\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) \geq s$. Since, we have

$$\begin{aligned} s &\leq \inf_{\gamma\in\Gamma} \mu_A(x\gamma y) = \max\{\mu_A(x), \mu_A(y)\} \\ s &\leq \max\{\mu_A(x), \mu_A(y)\} \\ \mu_A(x) &\geq s \text{ or } \mu_A(y) \geq s. \end{aligned}$$

Hence, $x \in U(\mu_A, s)$ or $y \in U(\mu_A, s)$. Thus $U(\mu_A, s)$ is a prime Γ-ideal of S. Now, let $x\Gamma y \in L(\mu_A, t)$. Then, $\gamma_A(x) \leq t$ for all $\gamma \in \Gamma$. This implies that $\sup_{\gamma\in\Gamma} \gamma_A(x) \leq t$. Since

$$\begin{aligned} t &\geq \sup_{\gamma\in\Gamma} \mu_A(x\gamma y) = \min\{\gamma_A(x), \gamma_A(y)\} \\ t &\geq \min\{\gamma_A(x), \gamma_A(y)\} \\ \gamma_A(x) &\leq t \text{ or } \gamma_A(y) \leq t. \end{aligned}$$

Hence, $x \in L(\gamma_A, t)$ or $y \in L(\gamma_A, t)$. Thus, $L(\gamma_A, t)$ is a prime Γ-ideal of S.

Conversely, suppose that $U(\mu_A, s)$ and $L(\gamma_A, t)$ are prime Γ-ideals of S. Let $\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) = s$ (Since $\mu_A(x\gamma y) \in [0,1], \forall\gamma \in \Gamma$, so $\inf_{\gamma\in\Gamma} \mu_A(x\gamma y)$ exists). Then, $\mu_A(x\gamma y) \geq s$, $\forall\gamma \in \Gamma$. So $x\gamma y \in U(\mu_A, s), \forall\gamma \in \Gamma$. Since $U(\mu_A, s)$ is a prime, so $x \in U(\mu_A, s)$ or $y \in U(\mu_A, s) \Rightarrow \mu_A(x) \geq s$ or $\mu_A(y) \geq s$, we have

$$\max\{\mu_A(x), \mu_A(y)\} \geq s = \inf_{\gamma\in\Gamma} \mu_A(x\gamma y). \tag{1}$$

Since, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy Γ-ideals of S, so

$$\mu_A(x\gamma y) \geq \max\{\mu_A(x), \mu_A(y)\}, \forall\gamma \in \Gamma$$

$$\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) \geq \max\{\mu_A(x), \mu_A(y)\} \tag{2}$$

From 1 and 2, we have

$$\inf_{\gamma\in\Gamma} \mu_A(x\gamma y) = \max\{\mu_A(x), \mu_A(y)\}.$$

Now, let $\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) = t$ (Since $\gamma_A(x\gamma y) \in [0,1], \forall\gamma \in \Gamma$, so $\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y)$ exists). Then $\gamma_A(x\gamma y) \leq t$, $\forall\gamma \in \Gamma$, so $x\gamma y \in L(\gamma_A, t)$, $\forall\gamma \in \Gamma$. Since $L(\gamma_A, t)$ is a prime, so $x \in L(\gamma_A, t)$ or $y \in L(\gamma_A, t) \Rightarrow \gamma_A(x) \leq t$ or $\gamma_A(y) \leq t$.

$$\min\{\gamma_A(x), \gamma_A(y)\} \leq t = \sup_{\gamma\in\Gamma} \gamma_A(x\gamma y). \tag{3}$$

Since, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy Γ-ideals of S, so

$$\gamma_A(x\gamma y) \leq \min\{\gamma_A(x), \gamma_A(y)\} \ \forall\gamma \in \Gamma$$

$$\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) \leq \min\{\gamma_A(x), \gamma_A(y)\} \tag{4}$$

From 3 and 4, we have

$$\sup_{\gamma\in\Gamma} \gamma_A(x\gamma y) = \min\{\gamma_A(x), \gamma_A(y)\}$$

Hence $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S. □

Corollary 3.7. *Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS in Γ-LA-semigroup S. Then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S if and only if for any $s, t \in [0,1]$, the sets $U^s(\mu_A, s) = \{x \in S : \mu_A(x) > s\} \neq \emptyset$ and $L_t(\gamma_A, t) = \{x \in S : \gamma_A(x) < t\} \neq \emptyset$ are prime Γ-ideals of S.*

Corollary 3.8. *Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS in Γ-LA-semigroup S. Then, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S if and only if for any $s, t \in [0,1]$, the sets $P = \{x \in S : \mu_A(x) \geq s$ and $\gamma_A(x) \leq t\} \neq \emptyset$.*

Corollary 3.9. *Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS in Γ-LA-semigroup S. Then, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy prime Γ-ideal of S if and only if for any $s, t \in [0,1]$, the sets $P^{(s,t)} = \{x \in S : \mu_A(x) > s$ and $\gamma_A(x) < t\} \neq \emptyset$.*

4. Intuitionistic Fuzzy semi-prime Γ-Ideals

Definition 4.1. Let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in Γ-*LA-semigroup* S. Then $A = \langle \mu_A, \gamma_A \rangle$ is called an intuitionistic fuzzy semi-prime if

$(IFP3)$ $\mu_A(x) \geq \mu_A(x\gamma x)$,

$(IFP4)$ $\gamma_A(x) \leq \mu\gamma_A(x\gamma x)$ $\forall x \in S$ and $\gamma \in \Gamma$.

An intuitionistic fuzzy Γ-ideal is called an intuitionistic fuzzy semi-prime Γ-ideal of S if it is an intuitionistic fuzzy semi-prime.

Theorem 4.1. *Let S be a Γ-LA-semigroup and $\emptyset \neq T \subseteq S$ is a semi-prime Γ-ideal of S. Then, $A = \langle \mathcal{X}_T, \overline{\mathcal{X}_T} \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S, where $\overline{\mathcal{X}_T} = 1 - \mathcal{X}_T$.*

Proof. The proof follows from 3.1 □

Theorem 4.2. *Let T be a non empty subset of S. If $A = \langle \mathcal{X}_T, \overline{\mathcal{X}_T} \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S, then T is a semi-prime Γ-ideal of S, where $\overline{\mathcal{X}_T} = 1 - \mathcal{X}_T$.*

Proof. The proof follows from 3.2 □

Theorem 4.3. *For any intuitionistic fuzzy sub*Γ*-LA-semigroup $A = \langle \mu_A, \gamma_A \rangle$ of S. If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime, then $A(x) = A(x\gamma x)$ $\forall x \in S$ and $\gamma \in \Gamma$.*

Proof. Let $x \in S$. Then, since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy subΓ-LA-semigroup, we have

$$\begin{aligned} \mu_A(x) &\geq \mu_A(x\gamma x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x) \\ \mu_A(x) &= \mu_A(x\gamma x) \end{aligned}$$

and also, we have

$$\begin{aligned} \gamma_A(x) &\leq \gamma_A(x\gamma x) \leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x) \\ \gamma_A(x) &= \gamma_A(x\gamma x) \end{aligned}$$

This completes the proof.. □

Theorem 4.4. *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S, then $\Box A = \langle \mu_A, \overline{\mu_A} \rangle$ and $\Diamond A = \langle \overline{\gamma_A}, \gamma_A \rangle$ are intuitionistic fuzzy semi-prime Γ-ideals of S.*

Proof. The proof follows from 3.5 □

Lemma 4.5. *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S, then μ_A and $\overline{\gamma_A}$ are fuzzy prime Γ-ideals of S.*

Proof. Straightforward. □

Lemma 4.6. *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S, then $\overline{\gamma_A}$ and γ_A are anti fuzzy prime Γ-ideals of S.*

Theorem 4.7. *Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS in Γ-LA-semigroup S. Then, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S if and only if for any $s, t \in [0, 1]$, the sets $U(\mu_A, s) = \{x \in S : \mu_A(x) \geq s\} \neq \emptyset$ and $L(\gamma_A, t) = \{x \in S : \gamma_A(x) \leq t\} \neq \emptyset$ are semi-prime Γ-ideals of S.*

Proof. Suppose that $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S. Let $s, t \in [0, 1]$ such that $U(\mu_A, s)$ and $L(\gamma_A, t)$ are non-empty. Now let $x \in S$ such that $x\Gamma x \subseteq U(\mu_A, s)$. Then $\mu_A(x\gamma x) \geq s$ for all $\gamma \in \Gamma$. Since

$$\begin{aligned} \mu_A(x) &\geq \mu_A(x\gamma x) \geq s \\ \mu_A(x) &\geq s. \end{aligned}$$

Hence, $x \in U(\mu_A, s)$. Thus $U(\mu_A, s)$ is a semi-prime Γ-ideal of S. Now, let $x\Gamma x \subseteq L(\gamma_A, t)$. Then, $\gamma_A(x\gamma x) \leq t$ for all $\gamma \in \Gamma$. Since

$$\begin{aligned} \gamma_A(x) &\leq \gamma_A(x\gamma x) \leq t \\ \gamma_A(x) &\leq t. \end{aligned}$$

Hence, $x \in L(\gamma_A, t)$. Thus $L(\gamma_A, t)$ is semi-prime Γ-ideal of S.

Conversely, let $A = \langle \mu_A, \gamma_A \rangle$ be an IFS in S such that $U(\mu_A, s)$ and $L(\gamma_A, t)$ are semi-prime Γ-ideals of S. Let suppose $A = \langle \mu_A, \gamma_A \rangle$ is not intuitionistic fuzzy semi-prime Γ-ideal of S. Then, there exist $x_\circ \in S$ such that $\mu_A(x_\circ) < \mu_A(x_\circ \gamma x_\circ)$. Let

$$\begin{aligned} s_\circ &= \frac{1}{2}\left[\mu_A(x_\circ) + \mu_A(x_\circ \gamma x_\circ)\right]. \text{ Then} \\ \mu_A(x_\circ) &< s_\circ < \mu_A(x_\circ \gamma x_\circ). \end{aligned}$$

So, $x_\circ \gamma x_\circ \in U(\mu_A, s_\circ)$ but $x_\circ \notin U(\mu_A, s_\circ)$, a contradiction. Therefore, $\mu_A(x) \geq \mu_A(x\gamma x)$ for all $x \in S$. Similarly, $\gamma_A(x) \leq \gamma_A(x\gamma x)$ for all $x \in S$ and $\gamma \in \Gamma$. Hence, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S. □

Corollary 4.8. *Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS in Γ-LA-semigroup S. Then, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S if and only if for any $s, t \in [0, 1]$, the sets $U^s(\mu_A, s) = \{x \in S : \mu_A(x) > s\} \neq \emptyset$ and $L_t(\gamma_A, t) = \{x \in S : \gamma_A(x) < t\} \neq \emptyset$ are semi-prime Γ-ideals of S.*

Corollary 4.9. *Let $A = \langle \mu_A, \gamma_A \rangle$ be IFS in Γ-LA-semigroup S. Then, $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy semi-prime Γ-ideal of S if and only if for any $s, t \in [0,1]$, the set $U^{(s,t)} = \{x \in S : \mu_A(x) > s$ and $\gamma_A(x) < t\} \neq \emptyset$ is a semi-prime Γ-ideal of S.*

Definition 4.2. An element a of a Γ-*LA-semigroup* S is called left regular if for there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = (x\alpha x)\beta a$. If every element of a Γ-*LA-semigroup* S is left regular, then S is called left regular. Similarly, for right regular.

Theorem 4.10. *Let S be a left regular. Then, for every intuitionistic fuzzy right Γ-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, $A(x) = A(x\alpha x)$ $\forall x \in S$ and $\alpha \in \Gamma$.*

Proof. Let x be any element of S. Since S is a left regular, there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = (x\alpha x)\beta a$. Thus, we have

$$\begin{aligned} \mu_A(x) &= \mu_A((x\alpha x)\beta a) \geq \mu_A(x\alpha x) \geq \mu_A(x) \\ \mu_A(x) &= \mu_A(x\alpha x) \text{ and} \\ \gamma_A(x) &= \gamma_A((x\alpha x)\beta a) \leq \gamma_A(x\alpha x) \leq \gamma_A(x) \\ \gamma_A(x) &= \gamma_A(x\alpha x) \end{aligned}$$

Hence, $A(x) = A(x\alpha x)$ $\forall x \in S$ and $\alpha \in \Gamma$. □

Lemma 4.11. *Every intuitionistic fuzzy right Γ-ideal of a regular Γ-LA-semigroup S is Γ-idempotent.*

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be any intuitionistic fuzzy right Γ-ideal of S. Since S regular, so by Proposition 2.1

$$\begin{aligned} A\Gamma A &= A \cap A = A \\ A\Gamma A &= A \end{aligned}$$

□

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