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TRAVELING WAVES OF AN SIRS EPIDEMIC MODEL WITH SPATIAL DIFFUSION AND TIME DELAY†

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ABSTRACT. This paper is concerned with an SIRS epidemic model with spatial diffusion and time delay representing the length of the immunity period. By using a new cross iteration scheme and Schauder's fixed point theorem, we reduce the existence of traveling wave solutions to the existence of a pair of upper-lower solutions. By constructing a newfashioned pair of upper-lower solutions, we derive the existence of a traveling wave solution connecting the uninfected steady state and the infected steady state.

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1. Introduction

Mathematical models describing the population dynamics of infectious diseases have played an important role in better understanding epidemiological patterns and disease control for a long time. Let $S(t)$ denote the number of individuals who are susceptible to the disease, $I(t)$ the number of infected individuals who are infectious and are able to spread the disease by contact with susceptible individuals and $R(t)$ the number of individuals who have been removed from the possibility of infection through full immunity. In [12], Wen and Yang considered the following delayed SIRS epidemic model

$$\begin{aligned}\dot{S}(t) &= A - \mu S(t) - \beta S(t)I(t) + \gamma e^{-\mu\tau} I(t-\tau),\\ \dot{I}(t) &= \beta S(t)I(t) - (\mu+\gamma+\alpha)I(t),\\ \dot{R}(t) &= \gamma I(t) - \gamma e^{-\mu\tau} I(t-\tau) - \mu R(t),\end{aligned} \tag{1}$$

where the parameters $\mu, \alpha, A, \beta, \gamma$ are positive constants in which μ is the natural death rate of the population, α is the death rate of the infective individuals

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due to disease, A is the recruitment rate of the susceptible population, β is the contact rate, and γ is the recovery rate from the infected compartment. The term $\gamma e^{-\mu\tau}I(t-\tau)$ represents that an individual has survived natural death in a recovery pool before becoming susceptible again, where $\tau \geq 0$ is a constant representing the length of the immunity period. Sufficient conditions were derived in [12] for the global stability of an endemic equilibrium by using a Lyapunov functional approach.

We note that the spatial content of the environment has been ignored in (1). However, due to the large mobility of people within a country or even worldwide, spatially uniform models are not sufficient to give a realistic picture of a disease diffusion. For this reason, the spatial effects cannot be neglected in studying the spread of epidemics. In recent years, many investigators have introduced population movements into related equations for epidemic modeling in efforts to understand the most basic features of spatially distributed interactions (see, for example, [3, 7, 9, 14]), and in this situation the governing equations for the population densities are described by a system of reaction-diffusion equations. For nonlinear reaction-diffusion equations describing a variety of physical and biological phenomena, traveling wave solutions are important since in many situations they determine the long term behavior of other solutions, and account for phase transitions between different states of physical systems, propagation of patterns, and domain invasion of species in population biology. Since the pioneering work of Schaaf [10], traveling wave solutions of delayed reaction-diffusion equations have been widely investigated due to the significant applications in several areas (see, e.g., [5, 6, 8, 11, 13] and references therein). However, traveling wave solutions of epidemic models have been rarely studied. Gan et al. [1] studied the existence of traveling waves of a delayed SIRS epidemic model with reaction terms satisfying the partial quasi-monotonicity conditions. By using a cross iteration method, Schauder's fixed point theorem and constructing a pair of upper-lower solutions, the existence of a traveling wave solution were derived .

Motivated by the works of Wen and Yang [12] and Gan et al. [1], in the present paper, we are concerned with the effect of spatial diffusion and time delay due to immunity period on the dynamics of an SIRS epidemic model. To this end, we consider the following delayed partial differential equations

$$\begin{aligned}
\frac{\partial S}{\partial t} &= d_1\frac{\partial^2 S}{\partial x^2} + A - \mu S(x,t) - \beta S(x,t)I(x,t) + \gamma e^{-\mu\tau}I(x,t-\tau),\\
\frac{\partial I}{\partial t} &= d_2\frac{\partial^2 I}{\partial x^2} + \beta S(x,t)I(x,t) - (\mu+\gamma+\alpha)I(x,t),\\
\frac{\partial R}{\partial t} &= d_3\frac{\partial^2 R}{\partial x^2} + \gamma I(x,t) - \gamma e^{-\mu\tau}I(x,t-\tau) - \mu R(x,t),
\end{aligned} \tag{2}$$

where $S(t,x)$, $I(t,x)$ and $R(t,x)$ represent the densities of the susceptible, the infected and the removed at time t and location x, respectively. d_i $(i=1,2,3)$ denote the corresponding diffusion rates for the three populations, respectively.

Since the first two equations are independent of the last in (2), we need only to consider the following subsystem

$$\begin{aligned}\frac{\partial S}{\partial t} &= d_1\frac{\partial^2 S}{\partial x^2} + A - \mu S(x,t) - \beta S(x,t)I(x,t) + \gamma e^{-\mu\tau}I(x,t-\tau),\\ \frac{\partial I}{\partial t} &= d_2\frac{\partial^2 I}{\partial x^2} + \beta S(x,t)I(x,t) - (\mu+\gamma+\alpha)I(x,t).\end{aligned} \tag{3}$$

The paper is organized as follows. Section 2 is devoted to some preliminaries for traveling waves. In Section 3, we employ a new cross iteration method and Schauder's fixed point theorem in a profile set to obtain the existence of traveling wave solutions for a generalized system with the nonlinear reaction terms satisfying the mixed quasi-monotonicity. In Section 4, by constructing a pair of upper-lower solutions, we use the result derived in Section 3 to prove the existence of traveling wave solutions of system (2). We conclude this work in Section 5.

2. Preliminaries

Throughout this paper, we employ the usual notations for the standard ordering in $\mathbb{R}^2$. That is, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$, we denote $u \le v$ if $u_i \le v_i$; $u < v$ if $u \le v$ but $u \ne v$; and $u \ll v$ if $u \le v$ but $u_i \ne v_i$, $i = 1, 2$. We use $|\cdot|$ to denote the Euclidean norm and $\|\cdot\|$ the supremum norm in $C([-\tau, 0], \mathbb{R}^2)$.

In order to focus on the mathematical ideas and for the sake of simplicity, we consider the following general reaction-diffusion system with discrete delays

$$\begin{aligned}\frac{\partial S}{\partial t} &= d_1\frac{\partial^2 S}{\partial x^2} + f_1(S_t(x), I_t(x)),\\ \frac{\partial I}{\partial t} &= d_2\frac{\partial^2 I}{\partial x^2} + f_2(S_t(x), I_t(x)),\end{aligned} \tag{4}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$, $d_i > 0$, f_i: $C([-\tau, 0], \mathbb{R}^2) \to \mathbb{R}^1$ is continuous ($i = 1, 2$), and for any fixed $x \in \mathbb{R}$, $S_t(x) \in C([-\tau, 0], \mathbb{R}^1)$ is given by $S_t(x)(s) = S(t+s, x)$, $s \in [-\tau, 0]$, $I_t(x)$ is defined accordingly.

A traveling wave solution of (4) is a special solution of the form $S(x,t) = \phi(x+ct)$, $I(x,t) = \varphi(x+ct)$, where $(\phi, \varphi) \in C^2(\mathbb{R}, \mathbb{R}^2)$ is the profile of the wave that propagates through the one-dimensional spatial domain at a constant speed $c > 0$. Substituting $S(x,t) = \phi(x+ct)$, $I(x,t) = \varphi(x+ct)$ into (4) and denoting $x + ct$ by t, we obtain the corresponding wave equations

$$\begin{aligned}d_1\phi''(t) - c\phi'(t) + f_{c1}(\phi_t, \varphi_t) &= 0,\\ d_2\varphi''(t) - c\varphi'(t) + f_{c2}(\phi_t, \varphi_t) &= 0,\end{aligned} \tag{5}$$

where $\phi_t(\zeta) = \phi(\zeta + t)$, $\varphi_t(\zeta) = \varphi(\zeta + t)$, and the functions $f_{ci} : X_c = C([-c\tau, 0], \mathbb{R}^2) \to \mathbb{R}$ ($i = 1, 2$) are defined by

$$f_{ci}(\phi, \varphi) = f_i(\phi^c, \varphi^c), \phi^c(s) = \phi(cs), \varphi^c(s) = \varphi(cs), s \in [-\tau, 0].$$

We call that (4) has a traveling wave solution if and only if for some $c > 0$, (5) with the asymptotic boundary conditions

$$\begin{array}{l}\lim_{t\to-\infty}\phi(t)=\phi_-,\ \lim_{t\to-\infty}\varphi(t)=\varphi_-,\\ \lim_{t\to+\infty}\phi(t)=\phi_+,\ \lim_{t\to+\infty}\varphi(t)=\varphi_+\end{array}$$

has a solution on $\mathbb{R}^2$, where (ϕ_-,φ_-) and (ϕ_+,φ_+) are two equilibria of (5). Without loss of generality, we can assume

$$(\phi_-,\varphi_-)=(0,0),\ (\phi_+,\varphi_+)=(k_1,k_2). \tag{6}$$

Corresponding to (4), we make the following assumptions throughout the remainder of this paper:

(H1) $f_i(0,0)=f_i(k_1,k_2)=0,\ i=1,2.$

(H2) There exist constants $L_i>0$, such that

$|f_i(\phi_1,\varphi_1)-f_i(\phi_2,\varphi_2)|\le L_i\parallel \Phi-\Psi\parallel$

for $\Phi(s)=(\phi_1,\varphi_1)(s),\Psi(s)=(\phi_2,\varphi_2)(s)\in C([-\tau,0],\mathbb{R}^2)$ with $(0,0)\le(\phi_j(s),\varphi_j(s))\le(m_1,m_2)$, $s\in[-\tau,0]$, $m_i>k_i$ are positive constants, $i,j=1,2$.

(H3) The reaction terms satisfy the mixed quasi-monotonicity conditions, that is, there exist constants $\beta_1,\beta_2>0$ such that

$f_{c1}(\phi_1,\varphi_1(0),\varphi_2(-\tau))-f_{c1}(\phi_2,\varphi_2(0),\varphi_1(-\tau))+\beta_1[\phi_1(0)-\phi_2(0)]\ge 0,$

$f_{c2}(\phi_2,\varphi_1)-f_{c2}(\phi_1,\varphi_2)+\beta_2[\varphi_1(0)-\varphi_2(0)]\ge 0$

for $\Phi(s)=(\phi_1,\varphi_1)(s),\Psi(s)=(\phi_2,\varphi_2)(s)\in C([-\tau,0],\mathbb{R}^2)$ with $\mathbf{0}\le\Psi(s)\le\Phi(s)\le(m_1,m_2),\ s\in[-\tau,0]$. Here, we denote φ in f_{c1} by $(\varphi(0),\varphi(-\tau))$, since we treat them differently.

Now, we give the definition of upper and lower solutions of system (4).

Definition 1. The continuous functions $\overline{\Phi}(t)=(\overline{\phi},\overline{\varphi})(t)$ and $\underline{\Phi}(t)=(\underline{\phi},\underline{\varphi})(t)$ are called an upper solution and a lower solution of system (5), respectively, if there exist constants T_i $(i=1,2,\cdots,m)$, such that $\overline{\Phi}(t)$ and $\underline{\Phi}(t)$ are twice differential in $\mathbb{R}\setminus\{T_i:i=1,2,\cdots,m\}$ and they are essentially bounded on $\mathbb{R}^2$, and there hold

$$\begin{array}{ll}d_1\overline{\phi}''(t)-c\overline{\phi}'(t)+f_{c1}(\overline{\phi}_t,\overline{\varphi}_t(0),\underline{\varphi}_t(-\tau))\le 0 & \\ d_2\overline{\varphi}''(t)-c\overline{\varphi}'(t)+f_{c2}(\underline{\phi}_t,\overline{\varphi}_t)\le 0 & t\in\mathbb{R}\setminus\{T_i:i=1,2,\cdots,m\}\end{array}$$

and

$$\begin{array}{ll}d_1\underline{\phi}''(t)-c\underline{\phi}'(t)+f_{c1}(\underline{\phi}_t,\underline{\varphi}_t(0),\overline{\varphi}_t(-\tau))\ge 0 & \\ d_2\underline{\varphi}''(t)-c\underline{\varphi}'(t)+f_{c2}(\overline{\phi}_t,\underline{\varphi}_t)\ge 0 & t\in\mathbb{R}\setminus\{T_i:i=1,2,\cdots,m\}.\end{array}$$

Here, in order to give a new-style cross iteration scheme in Section 3, the definition of upper-lower solutions is different from those in the literature (see, for example, [1, 2, 6, 8, 11]).

3. Existence of traveling waves of system (4)

In this section, we apply Schauder's fixed point theorem to study the existence of traveling wave solutions of system (4) connecting the uninfected steady state and the infected steady state.

In what follows, we assume that there exist an upper solution $\overline{\Phi}(t) = (\overline{\phi}, \overline{\varphi})(t)$ and a lower solution $\underline{\Phi}(t) = (\underline{\phi}, \underline{\varphi})(t)$ of system (5) satisfying (P1)-(P3):

(P1) $\mathbf{0} \le \underline{\Phi} \le \overline{\Phi} \le M = (m_1, m_2)$;

(P2) $\lim_{t\to-\infty} \overline{\Phi}(t) = \mathbf{0}$, $\lim_{t\to\infty} \underline{\Phi}(t) = \lim_{t\to\infty} \overline{\Phi}(t) = K = (k_1, k_2)$,

(P3) $\overline{\Phi}'(t+) \le \overline{\Phi}'(t-)$, $\underline{\Phi}'(t+) \ge \underline{\Phi}'(t-)$, $t \in \mathbb{R}$.

Let

$$C_{[0,M]}(\mathbb{R}, \mathbb{R}^2) = \{\Phi(t) = (\phi, \varphi)(t) \in C(\mathbb{R}, \mathbb{R}^2) : \mathbf{0} \le \Phi(t) \le M,\ t \in \mathbb{R}\}.$$

We seek for traveling wave solutions of system (4) in the following profile set:

$$\Gamma = \left\{(\phi, \varphi)(t) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^2) :\ (\underline{\phi}, \underline{\varphi})(t) \le (\phi, \varphi)(t) \le (\overline{\phi}, \overline{\varphi})(t)\right\}.$$

It is easy to show that Γ is non-empty. In fact, $(\overline{\phi}, \overline{\varphi})(t) \in \Gamma$ by (P1).

For $(\phi, \varphi) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^2)$, and the constants $\beta_i > 0$ $(i = 1, 2)$ in (H3), define $H = (H_1, H_2) : C_{[0,M]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$\begin{aligned} H_1(\phi, \varphi)(t) &= f_{c1}(\phi_t, \varphi_t) + \beta_1 \phi(t), \\ H_2(\phi, \varphi)(t) &= f_{c2}(\phi_t, \varphi_t) + \beta_2 \varphi(t). \end{aligned}$$

Then system (5) can be rewritten as

$$\begin{aligned} d_1 \phi''(t) - c\phi'(t) - \beta_1 \phi(t) + H_1(\phi, \varphi)(t) = 0, \\ d_2 \varphi''(t) - c\varphi'(t) - \beta_2 \varphi(t) + H_2(\phi, \varphi)(t) = 0. \end{aligned}$$

Define

$$\lambda_{i1} = \frac{c - \sqrt{c^2 + 4\beta_i d_i}}{2d_i},\ \lambda_{i2} = \frac{c + \sqrt{c^2 + 4\beta_i d_i}}{2d_i}\ (i = 1, 2).$$

It is clear that

$$\lambda_{i1} < 0 < \lambda_{i2},\ d_i \lambda_{ij}^2 - c\lambda_{ij} - \beta_i = 0\ (i, j = 1, 2).$$

For $(\phi, \varphi) \in C_{[0,M]}(\mathbb{R}, \mathbb{R}^2)$, define $F = (F_1, F_2) : C_{[0,M]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$F_i(\phi, \varphi)(t) = \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})} \left[\int_{-\infty}^{t} e^{\lambda_{i1}(t-s)} + \int_{t}^{\infty} e^{\lambda_{i2}(t-s)}\right] H_i(\phi, \varphi)(s)\mathrm{d}s \quad (i = 1, 2).$$

Then F is well defined such that

$$d_i F_i(\phi, \varphi)''(t) - cF_i(\phi, \varphi)'(t) - \beta_i F_i(\phi, \varphi)(t) + H_i(\phi, \varphi)(t) = 0\ (i = 1, 2).$$

Thus, a fixed point of F is a solution of (5), which is a traveling wave solution of (4) connecting $\mathbf{0} = (0, 0)$ with $\boldsymbol{K} = (k_1, k_2)$ if it satisfies (6).

In the following, we introduce a topology in $C(\mathbb{R}, \mathbb{R}^2)$. Let $0 < \mu < \min\{-\lambda_{i1}, \lambda_{i2}, i = 1, 2\}$ and equip $C(\mathbb{R}, \mathbb{R}^2)$ with the exponential decay norm defined by

$$|\Phi|_\mu = \sup_{t\in\mathbb{R}} e^{-\mu|t|} |\Phi(t)|_{\mathbb{R}^2}.$$

Define

$$B_\mu(\mathbb{R}, \mathbb{R}^2) = \{\Phi \in C(\mathbb{R}, \mathbb{R}^2) : |\Phi|_\mu < \infty\}.$$

Then it is easy to check that $(B_\mu(\mathbb{R}, \mathbb{R}^2), |\cdot|_\mu)$ is a Banach space.

By (H3), it is not difficult to verify that the operators H and F admit the following properties.

Lemma 1. *Assume that* (H3) *holds. Then*

$$\begin{gathered} H_1(\phi_2, \varphi_2(0), \varphi_1(-\tau))(t) \le H_1(\phi_1, \varphi_1(0), \varphi_2(-\tau))(t), \\ H_2(\phi_1, \varphi_2)(t) \le H_2(\phi_2, \varphi_1)(t); \\ F_1(\phi_2, \varphi_2(0), \varphi_1(-\tau))(t) \le F_1(\phi_1, \varphi_1(0), \varphi_2(-\tau))(t), \\ F_2(\phi_1, \varphi_2)(t) \le F_2(\phi_2, \varphi_1)(t) \end{gathered}$$

for $\Phi(s) = (\phi_1, \varphi_1)(s), \Psi(s) = (\phi_2, \varphi_2)(s) \in C([-\tau, 0], \mathbb{R}^2)$ *with* $\mathbf{0} \le \Psi(s) \le \Phi(s) \le (m_1, m_2),\ s \in [-\tau, 0]$.

Lemma 2. *Assume that* (H2) *holds. Then* $F = (F_1, F_2)$ *is continuous with respect to the norm* $|\cdot|_\mu$ *in* $B_\mu(\mathbb{R}, \mathbb{R}^2)$.

The proof of Lemma 2 is similar to those in Huang and Zou [2] and Li et al. [4], we therefore omit it here.

Lemma 3. *Assume that* (H3) *holds, then* $F\Gamma \subset \Gamma$.

Proof. For any $\Phi = (\phi, \varphi) \in \Gamma$, by Lemma 1, we have that

$$\begin{aligned} &F_1(\underline{\phi}, \underline{\varphi}(0), \overline{\varphi}(-\tau))(t) \le F_1(\phi, \varphi(0), \varphi(-\tau))(t) \le F_1(\overline{\phi}, \overline{\varphi}(0), \underline{\varphi}(-\tau))(t), \\ &F_2(\overline{\phi}, \underline{\varphi})(t) \le F_2(\phi, \varphi)(t) \le F_2(\underline{\phi}, \overline{\varphi})(t). \end{aligned}$$

Now, we need only to prove that

$$\begin{aligned} &\underline{\phi} \le F_1(\underline{\phi}, \underline{\varphi}(0), \overline{\varphi}(-\tau))(t) \le F_1(\overline{\phi}, \overline{\varphi}(0), \underline{\varphi}(-\tau))(t) \le \overline{\phi}, \\ &\underline{\varphi} \le F_2(\overline{\phi}, \underline{\varphi})(t) \le F_2(\underline{\phi}, \overline{\varphi})(t) \le \overline{\varphi}. \end{aligned} \tag{7}$$

Without loss of generality, we assume that $\overline{\phi}(t)$ is continuously differentiable in $R \setminus \{T_i : i = 1, 2, \cdots, m\}$ with $T_1 < T_2 < \cdots < T_m$. Denote $T_0 = -\infty$ and $T_{m+1} = \infty$. For $t \in (T_k, T_{k+1})$, $0 \le k \le m$, according to the definitions of F and the upper-lower solutions, we have

$$\begin{aligned} &F_1(\overline{\phi}, \overline{\varphi}(0), \underline{\varphi}(-\tau))(t) \\ =&\frac{1}{d_1(\lambda_{12} - \lambda_{11})} \left[\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{\infty} e^{\lambda_{12}(t-s)} \right] H_1(\overline{\phi}, \overline{\varphi}(0), \underline{\varphi}(-\tau))(s)\mathrm{d}s \\ \le&\frac{1}{d_1(\lambda_{12} - \lambda_{11})} \left[\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{\infty} e^{\lambda_{12}(t-s)} \right] (\beta_1 \overline{\phi} + c\overline{\phi}' - d_1 \overline{\phi}'')(s)\mathrm{d}s \\ =&\overline{\phi}(t) + \frac{1}{\lambda_{12} - \lambda_{11}} \left[\sum_{j=1}^{k} e^{\lambda_{11}(t-T_j)} (\overline{\phi}'(T_j+) - \overline{\phi}'(T_j-)) \right. \\ &\left. \qquad + \sum_{j=k+1}^{m} e^{\lambda_{12}(t-T_j)} (\overline{\phi}'(T_j+) - \overline{\phi}'(T_j-)) \right] \\ \le&\overline{\phi}(t). \end{aligned}$$

Obviously, $F_1(\overline{\phi}, \overline{\varphi}(0), \underline{\varphi}(-\tau))(t) \le \overline{\phi}(t)$ for all $t \in \mathbb{R}$ in view of the continuity of $F_1(\overline{\phi}, \overline{\varphi}(0), \underline{\varphi}(-\tau))(t)$ and $\overline{\phi}(t)$. By a similar argument, we can prove that (7) holds for $t \in R$. This completes the proof. □

Using a similar argument as those in Huang and Zou [2] and Li et al. [4], one can obtain the following result.

Lemma 4. *Assume that* (H2) *and* (H3) *hold. Then* $F : \Gamma \to \Gamma$ *is compact.*

We now give the following main theorem.

Theorem 1. *Assume that* (H1)-(H3) *hold, and assume further that system* (5) *has an upper and a lower solution* $\overline{\Phi}(t) = (\overline{\phi}, \overline{\varphi})(t)$, $\underline{\Phi}(t) = (\underline{\phi}, \underline{\varphi})(t)$ *satisfying* (P1)-(P3). *Then for any* $c > 0$, *system* (4) *has a traveling wave solution satisfying* (6).

Proof. Based on Lemmas 2, 3 and 4, using Schauder's fixed point theorem, we know that there exists a fixed point $(\phi^*(t), \varphi^*(t)) \in \Gamma$, which is a solution of (5).

Further, by (P2) and the inequality

$$\mathbf{0} \le (\underline{\phi}, \underline{\varphi}) \le (\phi^*, \varphi^*) \le (\overline{\phi}, \overline{\varphi}) \le (m_1, m_2),$$

we see that

$$\lim_{t\to-\infty}(\phi^*(t), \varphi^*(t)) = (0,0), \ \lim_{t\to\infty}(\phi^*(t), \varphi^*(t)) = (k_1, k_2).$$

Therefore, the fixed point $(\phi^*(t), \varphi^*(t))$ satisfies the asymptotic boundary condition (6). The proof is completed. □

4. Existence of traveling waves of system (2)

In this section, we establish the existence of traveling wave solutions for system (2) using the result in Section 3.

It is easy to show that system (3) ((2)) always has a semi-trivial steady state $E_0(\lambda/\mu, 0)$ ($E_0(\lambda/\mu, 0, 0)$). If $R_0 = \beta A/[\mu(\mu+\gamma+\alpha)] > 1$, system (3) ((2)) has a unique positive steady state $E^*(S^*, I^*)$ ($E^*(S^*, I^*, R^*)$), where $S^* = (\mu+\gamma+\alpha)/\beta$, $I^* = [\beta A - \mu(\mu+\gamma+\alpha)]/[\beta(\mu+\alpha+\gamma(1-e^{-\mu\tau}))]$ ($R^* = \gamma(1-e^{-\mu\tau})I^*/\mu$).

Letting $d_1 = d_2 = D$, and making a change of variables $\widetilde{N} = A/\mu - (S+I)$, $\widetilde{I} = I$, dropping the tildes, then system (3) is equivalent to the following system

$$\begin{aligned} \frac{\partial N}{\partial t} &= D\frac{\partial^2 N}{\partial x^2} - \mu N(x,t) - \gamma e^{-\mu\tau} I(x, t-\tau) + (\gamma+\alpha) I(x,t), \\ \frac{\partial I}{\partial t} &= D\frac{\partial^2 I}{\partial x^2} + [A/\mu - (\mu+\gamma+\alpha)] I(x,t) - \beta N(x,t) I(x,t) - \beta I^2(x,t). \end{aligned} \tag{8}$$

It is easy to show that if $R_0 > 1$, system (8) has two steady states $(0,0)$, (k_1, k_2), where $k_1 = A/\mu - S^* - I^*$, $k_2 = I^*$. Translating by traveling wave solution, we derive the corresponding wave system of (8) as follows

$$\begin{aligned} &D\phi''(t) - c\phi'(t) - \mu\phi(t) - \gamma e^{-\mu\tau}\varphi(t - c\tau) + (\gamma+\alpha)\varphi(t) = 0, \\ &D\varphi''(t) - c\varphi'(t) + [A/\mu - (\mu+\gamma+\alpha) - \beta\varphi(t)]\varphi(t) - \beta\phi(t)\varphi(t) = 0 \end{aligned} \tag{9}$$

with the following asymptotic boundary conditions

$$\lim_{t\to-\infty}(\phi(t),\varphi(t))=(0,0),\ \lim_{t\to\infty}(\phi(t),\varphi(t))=(k_1,k_2).$$

For $(\phi,\varphi)(t)\in C([-\tau,0],\mathbb{R}^2)$, denote $\boldsymbol{f}_c=(f_{c1},f_{c2})$ by

$$\begin{aligned}
&f_{c1}(\phi_t,\varphi_t)=-\mu\phi(0)-\gamma e^{-\mu\tau}\varphi(-\tau)+(\gamma+\alpha)\varphi(0),\\
&f_{c2}(\phi_t,\varphi_t)=[A/\mu-(\mu+\gamma+\alpha)-\beta\varphi(0)]\varphi(0)-\beta\phi(0)\varphi(0).
\end{aligned}$$

Obviously, $\boldsymbol{f}_c$ satisfies (H1)−(H3).

Let l_1, l_2 satisfy

$$\frac{A}{\mu}-(\gamma+\alpha+\mu)\le\frac{(\gamma+\alpha)l_2k_2}{l_1k_1}-\mu<2[\frac{A}{\mu}-(\gamma+\alpha+\mu)].\tag{10}$$

Denote $A_1=4D[\frac{(\gamma+\alpha)l_2k_2}{l_1k_1}-\mu]$, $A_2=4D[\frac{A}{\mu}-(\gamma+\alpha+\mu)]$. Let

$$c>c^*=\max\left\{\sqrt{A_1},\frac{4A_2-A_1}{2}\sqrt{\frac{1}{2(2A_2-A_1)}}\right\},\tag{11}$$

then we introduce the following positive numbers λ_i $(1\le i\le 4)$ satisfying

$$\lambda_{1,2}=\frac{c\mp\sqrt{c^2-A_1}}{2D},\ \lambda_{3,4}=\frac{c\mp\sqrt{c^2-A_2}}{2D}.\tag{12}$$

Noting that $\lambda_1<\min\{\lambda_2,2\lambda_3,\lambda_4\}$ by (10), (11) and (12), we can fix η_1,η_2 such that

$$\eta_1\in(1,\min\{\frac{\lambda_2}{\lambda_1},\frac{\lambda_4}{\lambda_1}\}),\ \eta_2\in(\frac{\lambda_1\eta_1}{\lambda_3},\min\{\frac{\lambda_2}{\lambda_3},2\}),$$

where η_1 is sufficiently close to 1 such that $\frac{\lambda_1\eta_1}{\lambda_3}<\min\{\frac{\lambda_2}{\lambda_3},2\}$.

For constant $q>1$, define

$$h_1(t)=l_1k_1(e^{\lambda_1t}-qe^{\eta_1\lambda_1t}),\ h_2(t)=l_2k_2(e^{\lambda_3t}-qe^{\eta_2\lambda_3t}).$$

It is easy to see that for $t\in R$, $h_i(t)$ is monotone increasing until it reach a global maximum, and then monotone decreasing to $-\infty$ $(i=1,2)$.

For $a_i<k_i$ $(i=1,2)$, denote

$$t_3=\max\{t:h_1(t)=a_1\},\ t_4=\max\{t:h_2(t)=a_2\}.$$

Since with the decreasing of $a_1(a_2)$, $t_3(t_4)$ is increasing, and for q sufficiently large, $t_i<0$ $(i=3,4)$, it is not difficult to verify that for given $\lambda>0$, we can make a_1,a_2 satisfy

$$(k_1-a_1)e^{\lambda(t_3-t_4)}>\frac{\gamma+\alpha}{\mu}(k_2-a_2),\ t_4<t_4+c\tau\le t_3<0.\tag{13}$$

Then there exist $\varepsilon_i>0$ $(i=3,4)$ such that

$$k_1-\varepsilon_3e^{-\lambda t_3}=h_1(t_3)=a_1,\ k_2-\varepsilon_4e^{-\lambda t_4}=h_2(t_4)=a_2.\tag{14}$$

We can deduce from (13) and (14) that $\mu\varepsilon_3>(\gamma+\alpha)\varepsilon_4$.

Suppose $\mu > \gamma e^{-\mu\tau}$. We can choose $\varepsilon_i > 0$ $(i = 1, 2)$ such that

$$\begin{aligned}
&\mu\varepsilon_1 - (\gamma+\alpha)\varepsilon_2 - \gamma e^{-\mu\tau}\varepsilon_4 > 0,\\
&\varepsilon_2 - \varepsilon_3 > 0,\\
&\mu\varepsilon_3 - (\gamma+\alpha)\varepsilon_4 - \gamma e^{-\mu\tau}\varepsilon_2 > 0,\\
&\varepsilon_4 - \varepsilon_1 > 0.
\end{aligned} \tag{15}$$

For the above constants and t_1, t_2 satisfying $t_3 + c\tau < t_1 < \min\{0, t_2\}$, define the continuous functions $\Phi(t) = (\phi_1(t), \varphi_1(t))$ and $\Psi(t) = (\phi_2(t), \varphi_2(t))$ as follows:

$$\phi_1(t) = \begin{cases} l_1k_1e^{\lambda_1 t}, & t \le t_1,\\ k_1 + \varepsilon_1 e^{-\lambda t}, & t > t_1, \end{cases}$$

$$\varphi_1(t) = \begin{cases} l_2k_2e^{\lambda_4 t}, & t \le t_2,\\ k_2 + \varepsilon_2 e^{-\lambda t}, & t > t_2, \end{cases}$$

$$\phi_2(t) = \begin{cases} l_1k_1(e^{\lambda_1 t} - qe^{\eta_1\lambda_1 t}), & t \le t_3,\\ k_1 - \varepsilon_3 e^{-\lambda t}, & t > t_3, \end{cases}$$

$$\varphi_2(t) = \begin{cases} l_2k_2(e^{\lambda_3 t} - qe^{\eta_2\lambda_3 t}), & t \le t_4,\\ k_2 - \varepsilon_4 e^{-\lambda t}, & t > t_4, \end{cases}$$

where $\lambda > 0$ is a constant to be chosen later. It is easy to know that $m_1 = \sup_{t\in\mathbb{R}}\phi_1(t) > k_1$, $m_2 = \sup_{t\in\mathbb{R}}\varphi_1(t) > k_2$, $\Phi(t)$ and $\Psi(t)$ satisfy (P1), (P2) and (P3).

Lemma 5. *$\Phi(t) = (\phi_1(t), \varphi_1(t))$ is an upper solution of system* (9).

Proof. Denote

$$\begin{aligned}
p_1(t) &:= D\phi_1''(t) - c\phi_1'(t) - \mu\phi_1(t) - \gamma e^{-\mu\tau}\varphi_2(t - c\tau) + (\gamma+\alpha)\varphi_1(t),\\
p_2(t) &:= D\varphi_1''(t) - c\varphi_1'(t) + [A/\mu - (\mu+\gamma+\alpha) - \beta\varphi_1(t)]\varphi_1(t) - \beta\phi_2(t)\varphi_1(t).
\end{aligned}$$

If $t < t_1$, $\phi_1(t) = l_1k_1e^{\lambda_1 t}$. It follows that

$$p_1(t) \le e^{\lambda_1 t}[Dl_1k_1\lambda_1^2 - cl_1k_1\lambda_1 - \mu l_1k_1 + (\gamma+\alpha)l_2k_2] = 0.$$

If $t_1 < t < t_2$, $\phi_1(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, $\varphi_1(t) = l_2k_2e^{\lambda_2 t}$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\lambda(t-c\tau)}$. We have that

$$p_1(t) \le e^{-\lambda t}[D\varepsilon_1\lambda^2 + c\varepsilon_1\lambda - \mu\varepsilon_1 + (\gamma+\alpha)\varepsilon_2 e^{-\lambda(t_2-t)} + \gamma e^{-\mu\tau}\varepsilon_4 e^{\lambda c\tau}].$$

For λ sufficiently small, $(\gamma+\alpha)\varepsilon_2 + \gamma e^{-\mu\tau}\varepsilon_4 - \mu\varepsilon_1 < 0$ indicates that $p_1(t) < 0$ and there exists a $\lambda_1^* > 0$ such that $p_1(t) < 0$ for all $\lambda \in (0, \lambda_1^*)$.

If $t > t_2$, $\phi_1(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, $\varphi_1(t) = k_2 + \varepsilon_2 e^{-\lambda t}$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\lambda(t-c\tau)}$. We derive that

$$p_1(t) = e^{-\lambda t}[D\varepsilon_1\lambda^2 + c\varepsilon_1\lambda - \mu\varepsilon_1 + (\gamma+\alpha)\varepsilon_2 + \gamma e^{-\mu\tau}\varepsilon_4 e^{\lambda c\tau}].$$

For λ sufficiently small, $(\gamma+\alpha)\varepsilon_2 + \gamma e^{-\mu\tau}\varepsilon_4 - \mu\varepsilon_1 < 0$ can deduce that $p_1(t) < 0$ and there exists a $\lambda_2^* > 0$ such that $p_1(t) < 0$ for all $\lambda \in (0, \lambda_2^*)$.

If $t < t_2$, $\varphi_1(t) = l_2k_2e^{\lambda_4 t}$. It follows that

$$p_2(t) \le l_2k_2e^{\lambda_4 t}[D\lambda_4^2 - c\lambda_4 + A/\mu - (\mu+\gamma+\alpha)] = 0.$$

If $t > t_2$, $\varphi_1(t) = k_2 + \varepsilon_2 e^{-\lambda t}$, $\phi_2(t) = k_1 - \varepsilon_3 e^{-\lambda t}$. We derive that

$$p_2(t) = e^{-\lambda t}[D\varepsilon_2\lambda^2 + c\varepsilon_2\lambda - \beta(\varepsilon_2 - \varepsilon_3)(k_2 + \varepsilon_2 e^{-\lambda t})].$$

For λ sufficiently small, $\varepsilon_2 > \varepsilon_3$ indicates that $p_2(t) < 0$ and there exists a $\lambda_3^* > 0$ such that $p_2(t) < 0$ for all $\lambda \in (0, \lambda_3^*)$.

Clearly, for all $\lambda \in (0, \min_{j=1,2,3}\{\lambda_j^*\})$, $p_i(t) \le 0$ $(i = 1, 2)$. This completes the proof.

Lemma 6. *$\Psi(t) = (\phi_2(t), \varphi_2(t))$ is a lower solution of system* (9).

Proof. Denote

$$q_1(t) := D\phi_2''(t) - c\phi_2'(t) - \mu\phi_2(t) - \gamma e^{-\mu\tau}\varphi_1(t - c\tau) + (\gamma + \alpha)\varphi_2(t),$$
$$q_2(t) := D\varphi_2''(t) - c\varphi_2'(t) + [A/\mu - (\mu + \gamma + \alpha) - \beta\varphi_2(t)]\varphi_2(t) - \beta\phi_1(t)\varphi_2(t).$$

If $t < t_4$, $\phi_2(t) = l_1k_1(e^{\lambda_1 t} - qe^{\eta_1\lambda_1 t})$, $\varphi_1(t - c\tau) = l_2k_2e^{\lambda_4(t-c\tau)}$, $\varphi_2(t) = l_2k_2(e^{\lambda_3 t} - qe^{\eta_2\lambda_3 t})$. We get

$$q_1(t) \ge -[Dl_1k_1(\eta_1\lambda_1)^2 - cl_1k_1\eta_1\lambda_1 + (\gamma+\alpha)l_2k_2 - \mu l_1k_1]qe^{\eta_1\lambda_1 t} - l_2k_2\gamma e^{-\mu\tau}e^{\lambda_4 t}.$$

Choosing q sufficiently large such that

$$q > -\frac{l_2k_2\gamma}{Dl_1k_1(\eta_1\lambda_1)^2 - cl_1k_1\eta_1\lambda_1 + (\gamma + \alpha)l_2k_2 - \mu l_1k_1} + 1.$$

Thus, we have $q_1(t) \ge 0$.

If $t_4 < t < t_3$, $\phi_2(t) = l_1k_1(e^{\lambda_1 t} - qe^{\eta_1\lambda_1 t})$, $\varphi_1(t - c\tau) = l_2k_2e^{\lambda_4(t-c\tau)}$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\lambda t}$. By the case above, it is obvious that $q_1(t) \ge 0$ since $k_2 - \varepsilon_4 e^{-\lambda t} \ge l_2k_2(e^{\lambda_3 t} - qe^{\eta_2\lambda_3 t})$ for $t_4 < t < t_3$.

If $t_3 < t < t_2$, $\phi_2(t) = k_1 - \varepsilon_3 e^{-\lambda t}$, $\varphi_1(t - c\tau) = l_2k_2e^{\lambda_4(t-c\tau)}$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\lambda t}$. We have that

$$q_1(t) \ge e^{-\lambda t}[-D\varepsilon_3\lambda^2 - c\varepsilon_3\lambda + \mu\varepsilon_3 - (\gamma + \alpha)\varepsilon_4 - \gamma e^{-\mu\tau}\varepsilon_2].$$

For λ sufficiently small, $\mu\varepsilon_3 - (\gamma + \alpha)\varepsilon_4 - \gamma e^{-\mu\tau}\varepsilon_2 > 0$ indicates that $q_1(t) > 0$ and there exists a $\lambda_4^* > 0$ such that $q_1(t) > 0$ for all $\lambda \in (0, \lambda_4^*)$.

If $t > t_2$, $\phi_2(t) = k_1 - \varepsilon_3 e^{-\lambda t}$, $\varphi_1(t - c\tau) = k_2 + \varepsilon_2 e^{-\lambda(t-c\tau)}$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\lambda t}$. We derive that

$$q_1(t) \ge e^{-\lambda t}[-D\varepsilon_3\lambda^2 - c\varepsilon_3\lambda + \mu\varepsilon_3 - (\gamma + \alpha)\varepsilon_4 - \gamma e^{-\mu\tau}\varepsilon_2 e^{\lambda c\tau}].$$

For λ sufficiently small, we can deduce from $\mu\varepsilon_3 - (\gamma + \alpha)\varepsilon_4 - \gamma e^{-\mu\tau}\varepsilon_2 > 0$ that $q_1(t) > 0$ and there exists a $\lambda_5^* > 0$ such that $q_1(t) > 0$ for all $\lambda \in (0, \lambda_5^*)$.

If $t < t_4$, $\varphi_2(t) = l_2k_2(e^{\lambda_3 t} - qe^{\eta_2\lambda_3 t})$, $\phi_1(t) = l_1k_1e^{\lambda_1 t}$. We get

$$\begin{aligned} q_2(t) > &- \{Dl_2k_2(\eta_2\lambda_3)^2 - cl_2k_2\eta_2\lambda_3 + [A/\mu - (\mu + \gamma + \alpha)l_2k_2]\}qe^{\eta_2\lambda_3 t} \\ &- \beta(l_2k_2)^2e^{2\lambda_3 t} - \beta l_1k_1l_2k_2e^{(\lambda_1+\lambda_3)t}. \end{aligned}$$

Choosing q sufficiently large satisfying

$$q > \frac{\beta l_2k_2(l_1k_1 + l_2k_2)}{-[Dl_2k_2(\eta_2\lambda_3)^2 - cl_2k_2\eta_2\lambda_3 + [A/\mu - (\mu + \gamma + \alpha)l_2k_2]} + 1.$$

Hence, $q_2(t) > 0$.

If $t_4 < t < t_1$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\lambda t}$, $\phi_1(t) = l_1 k_1 e^{\lambda_1 t}$. We have

$$\begin{aligned} q_2(t) &\geq e^{-\lambda t}[-D\varepsilon_4\lambda^2 - c\varepsilon_4\lambda + \beta(\varepsilon_4 - \varepsilon_1 e^{-\lambda(t_1-t)})(k_2 - \varepsilon_4 e^{-\lambda t})] \\ &\geq e^{-\lambda t}[-D\varepsilon_4\lambda^2 - c\varepsilon_4\lambda + \beta(\varepsilon_4 - \varepsilon_1)(k_2 - \varepsilon_4 e^{-\lambda t})]. \end{aligned}$$

For λ sufficiently small, $\varepsilon_4 > \varepsilon_1$ implies that $q_2(t) > 0$ and there exists a $\lambda_6^* > 0$ such that $q_2(t) > 0$ for all $\lambda \in (0, \lambda_6^*)$.

If $t > t_1$, $\varphi_2(t) = k_2 - \varepsilon_4 e^{-\lambda t}$, $\phi_1(t) = k_1 + \varepsilon_1 e^{-\lambda t}$. It follows that

$$q_2(t) = e^{-\lambda t}[-D\varepsilon_4\lambda^2 - c\varepsilon_4\lambda + \beta(\varepsilon_4 - \varepsilon_1)(k_2 - \varepsilon_4 e^{-\lambda t})].$$

For λ sufficiently small and the λ_6^* above, $\varepsilon_4 > \varepsilon_1$ guarantees that $q_2(t) > 0$ for all $\lambda \in (0, \lambda_6^*)$.

Obviously, for all $\lambda \in (0, \min_{j=4,5,6}\{\lambda_j^*\})$, $q_i(t) \geq 0$ $(i = 1, 2)$. The proof is completed. □

Applying Lemmas 5, 6 and Theorem 1, we know that if $R_0 > 1$ and $\mu > \gamma e^{-\mu\tau}$, system (8) has a traveling wave solution with speed $c > c^*$ connecting the steady states $(0, 0)$ and (k_1, k_2). Accordingly, we have the following conclusion.

Theorem 2. *Suppose $R_0 > 1$ and $\mu > \gamma e^{-\mu\tau}$. For every $c > c^*$, system* (2) *always has a traveling wave solution with speed c connecting the uninfected steady state $E_0(A/\mu, 0, 0)$ and the infected steady state $E^*(S^*, I^*, R^*)$.*

5. Concluding remark

In this paper, we have dealt with the existence of traveling wave solutions for an SIRS epidemic model with spatial diffusion and time delay describing the immunity period. The reaction terms satisfy the mixed quasi-monotonicity in which the monotonicity of the variable with delay is different from that without delay. By using the cross iteration scheme and Schauder's fixed point theorem, we reduced the existence of traveling wave solutions to the existence of a pair of upper-lower solutions which are easy to construct in practice. By constructing a new-style pair of upper-lower solutions, we derived the existence of a traveling wave solution connecting the uninfected steady state E_0 and the infected steady state E^*.

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