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ON FUZZY n-FOLD STRONG IDEALS OF BH-ALGEBRAS

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Abstract. Fuzzifications of the notion of n-fold strong ideals are considered. Characterizations of fuzzy n-fold strong ideals are given.

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1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3,4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have some connections with other areas: D. Mundici [7] proved MV-algebras are categorically equivalent to bounded commutative algebra, and J. Meng [8] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim [5] introduced the notion of a BH-algebra, which is a generalization of BCK/BCI-algebras. They defined the notions of ideal, maximal ideal and translation ideal and investigated some properties. E. H. Roh and S. Y. Kim [10] estimated the number of BH^*-subalgebras of order i in a transitive BH^*-algebras by using Hao's method. S. S. Ahn and J. H. Lee ([2]) defined the notion of strong ideals in BH-algebra and studied some properties of it. They considered the notion of a rough set in BH-algebras. S. S. Ahn and E. M. Kim ([1]) introduced the notion of n-fold strong ideal in BH-algebra and gave some related properties of it. We also described the role of initial segments in BH-algebras.

In this paper, we consider the fuzzifications of the notion of n-fold strong ideals. We investigate some of their properties, and consider characterizations of fuzzy n-fold strong ideals. Using a family of n-fold strong ideals, we establish fuzzy n-fold strong ideals.

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2. Preliminaries

By a *BH-algebra* ([5]), we mean an algebra $(X;*,0)$ of type (2,0) satisfying the following conditions:

(I) $x*x=0$,
(II) $x*0=x$,
(III) $x*y=0$ and $y*x=0$ imply $x=y$, for all $x,y\in X$.

For brevity, we also call X a BH-algebra. In X we can define an order relation "$\leq$" by $x\leq y$ if and only if $x*y=0$. A non-empty subset S of a BH-algebra X is called a *subalgebra* of X if, for any $x,y\in S$, $x*y\in S$, i.e., S is a closed under binary operation.

Definition 2.1 ([5])**.** A non-empty subset A of a BH-algebra X is called an *ideal* of X if it satisfies:

(I1) $0\in A$,
(I2) $x*y\in A$ and $y\in A$ imply $x\in A$, $\forall x,y\in X$.

An ideal A of a BH-algebra X is said to be a *translation ideal* of X if it satisfies:

(I3) $x*y\in A$ and $y*x\in A$ imply $(x*z)*(y*z)\in A$ and $(z*x)*(z*y)\in A$, $\forall x,y,z\in X$.

Obviously, $\{0\}$ and X are ideals of X. For any elements x and y of a BH-algebra X, $x*y^n$ denotes $(\cdots((x*y)*y)*\cdots)*y$ in which y occurs n times.

Definition 2.2. A non-empty subset A of a BH-algebra X is called a *strong ideal* ([2]) of X if it satisfies (I1) and

(I4) $(x*y)*z\in A$ and $y\in A$ imply $x*z\in A$ for all $x,y,z\in X$.

An ideal A of a BH-algebra X is called an *n-fold strong ideal* ([1]) of X if it satisfies (I1) and

(I5) for every $x,y,z\in X$ there exists a natural number n such that $x*z^n\in A$ whenever $(x*y)*z^n\in A$ and $y\in A$.

A mapping $f:X\to Y$ of BH-algebras is called a *homomorphism* if $f(x*y)=f(x)*f(y)$ for all $x,y\in X$. For a homomorphism $f:X\to Y$ of BH-algebras, the *kernel* of f, denoted by ker$f$, defined to be the set

$$\ker f=\{x\in X|f(x)=0\}.$$

Definition 2.3 ([10])**.** A BH-algebra X is called a *BH^*-algebra* if it satisfies the identity $(x*y)*x=0$ for all $x,y\in X$.

Lemma 2.4. *Let X be a BH^*-algebra. Then the following identity holds:*

$$0*x=0,\ \forall x\in X.$$

Definition 2.5. A BH-algebra $(X;*,0)$ is said to be *transitive* if $x*y=0$ and $y*z=0$ imply $x*z=0$ for all $x,y,z\in X$.

We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu : X \to [0,1]$. For a fuzzy set μ in X and $t \in [0,1]$, define $U(\mu;t)$ to be the set $U(\mu;t) = \{x \in X | \mu(x) \geq t\}$, which is called a *level subset* of μ.

Definition 2.6 ([6]). A fuzzy set μ in a BH-algebra X is called a *fuzzy BH-ideal* (here call it a *fuzzy ideal*) of X if

(FI1) $\mu(0) \geq \mu(x), \forall x \in X$,
(FI2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}, \forall x, y \in X$.

A fuzzy set μ in a BH-algebra X is called a *fuzzy translation BH-ideal* of X if it satisfies (FI1), (FI2) and

(FI3) $\min\{\mu((x*z)*(y*z)), \mu((z*x)*(z*y))\} \geq \min\{\mu(x*y), \mu(y*x)\}, \forall x, y, z \in X$.

Definition 2.7. Let A and B be any two sets, μ be any fuzzy set in A and $f : A \to B$ be any function. Set $f^{-1}(y) = \{x \in A | f(x) = y\}$ for $y \in B$. The fuzzy set ν in B defined by

$$\nu(y) = \begin{cases} \vee\{\mu(x) |\ x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in B$, is called the *image* of μ under f and is denoted by $f(\mu)$.

Definition 2.8. Let A and B be any two sets, $f : A \to B$ be any function and ν be any fuzzy set in $f(A)$. The fuzzy set μ in A defined by

$$\mu(x) = \nu(f(x)) \text{ for all } x \in X$$

is called the *preimage* of ν under f and is denoted by $f^{-1}(\nu)$.

3. Fuzzy ideals

Proposition 3.1. *Let μ be a fuzzy ideal of a BH-algebra X. If $x \leq y$ for any $x, y \in X$, then $\mu(x) \leq \mu(y)$.*

Proof. If $x \leq y$ for any $x, y \in X$, then $x * y = 0$. Hence, by (FI1) and (FI2), we have $\mu(x) \leq \min\{\mu(x * y), \mu(y)\} = \min\{\mu(0), \mu(y)\} = \mu(y)$. □

Proposition 3.2. *A fuzzy set μ in a BH-algebra X is a fuzzy ideal of X if and only if it satisfies* (FI1) and

(FI2′) $(\forall x, y, z \in X)((x * y) * z = 0 \Rightarrow \mu(x) \geq \min\{\mu(y), \mu(z)\})$.

Proof. Let μ be a fuzzy ideal of X and let $x, y, z \in X$. Suppose that $(x*y)*z = 0$. Since μ is a fuzzy ideal, we have

$$\begin{aligned} \mu(x * y) &\geq \min\{\mu((x * y) * z), \mu(z)\} \\ &= \min\{\mu(0), \mu(z)\} \\ &= \mu(z). \end{aligned}$$

Hence $\mu(x) \geq \min\{\mu(x*y), \mu(y)\} \geq \min\{\mu(y), \mu(z)\}$. Therefore $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.

Conversely, assume μ satisfies (FI1) and (FI2′). Since $(x*y)*(x*y) = 0$ for any $x, y \in X$, we have $\mu(x) \geq \min\{\mu(y), \mu(x*y)\}$ by (FI2′). Thus μ is a fuzzy ideal of X. □

It is easy to prove by induction the following.

Proposition 3.3 *Let μ be a fuzzy set satisfying* (FI1) *in a BH-algebra X. Then μ is a fuzzy ideal of X if and only if for any $x_1, \cdots, x_n \in X (n \geq 2)$,*

$$(\cdots(x*x_1)*\cdots)*x_n = 0 \text{ implies } \mu(x) \geq \min\{\mu(x_1), \cdots, \mu(x_n)\}.$$

The following two theorems give the homomorphic properties of fuzzy ideals.

Theorem 3.4. *Let X and Y be BH-algebras and $f : X \to Y$ be a homomorphism and ν be a fuzzy ideal of Y. Then $f^{-1}(\nu)$ is a fuzzy ideal of X.*

Proof. Let $x \in X$. Since $f(x) \in Y$ and ν is a fuzzy ideal of Y, $\nu(0) \geq \nu(f(x)) = (f^{-1}(\nu))(x)$ for any $x \in X$, but $\nu(0) = \nu(f(0)) = (f^{-1}(\nu))(0)$. Thus we get $(f^{-1}(\nu))(0) \geq \nu(f(x)) = (f^{-1}(\nu))(x)$ for any $x \in X$. Thus $f^{-1}(\nu)$ satisfies (FI1). Now let $x, y \in X$. Since ν is a fuzzy ideal of Y, we have

$$\begin{aligned}\nu(f(x)) &\geq \min\{\nu(f(x)*f(y)), \nu(f(y))\}\\ &= \min\{\nu(f(x*y)), \nu(f(y))\}\end{aligned}$$

and hence $f^{-1}(\nu)(x) \geq \min\{f^{-1}(\nu)(x*y), f^{-1}(\nu)(y)\}$. Thus $f^{-1}(\nu)$ is a fuzzy ideal of X. □

Lemma 3.5. *Let X and Y be BH-algebras and let $f : X \to Y$ be a homomorphism and μ be a fuzzy ideal of X. Then, if μ is constant on $kerf = f^{-1}(0)$, then $f^{-1}(f(\mu)) = \mu$.*

Proof. Let $x \in X$ and $f(x) = y$. Hence we have

$$\begin{aligned}f^{-1}(f(\mu))(x) =&(f(\mu))(f(x))\\ =&(f(\mu))(y) = \vee\{\mu(a) | a \in f^{-1}(y)\}.\end{aligned}$$

For all $a \in f^{-1}(y)$, we obtain $f(x) = f(a)$. Hence $f(x*a) = 0$, i.e., $x*a \in kerf$. Thus $\mu(x*a) = \mu(0)$. Therefore $\mu(x) \geq \min\{\mu(x*a), \mu(a)\} = \mu(a)$. Similarly, we have $\mu(a) \geq \mu(x)$. Hence $\mu(x) = \mu(a)$. Thus $f^{-1}(f(\mu))(x) = \vee\{\mu(a) | a \in f^{-1}(y)\} = \mu(x)$, i.e., $f^{-1}(f(\mu)) = \mu$. □

Theorem 3.6 *Let X and Y be BH-algebras and let $f : X \to Y$ be a surjective homomorphism and μ be a fuzzy ideal of X be such that* $\ker f \subseteq A_\mu$*, where $A_\mu := \{x \in X | \mu(x) = \mu(0)\}$. Then $f(\mu)$ is a fuzzy ideal of Y.*

Proof. Since μ is a fuzzy ideal of X and $0 \in f^{-1}(0)$, we have

$$\begin{aligned}(f(\mu))(0) =& \vee \{\mu(a) | a \in f^{-1}(0)\}\\ =&\mu(0) \geq \mu(x) \text{ for any } x \in X.\end{aligned}$$

Hence

$$(f(\mu))(0) \geq \vee \{\mu(x)|x \in f^{-1}(y)\} \\ =(f(\mu))(y) \text{ for any } y \in Y.$$

Thus $f(\mu)$ satisfies (FI1). Suppose that $(f(\mu))(x_B) < \min\{f(\mu)(y_B * x_B), f(\mu)(y_B)\}$ for some $x_B, y_B \in Y$. Since f is surjective, there exist $x_A, y_A \in X$ such that $f(x_A) = x_B$ and $f(y_A) = y_B$. Hence $f(\mu)(f(x_A)) < \min\{f(\mu)(f(y_A * x_A)), f(\mu)(f(y_A))\}$. Therefore

$$f^{-1}(f(\mu))(x_A) < \min\{f^{-1}(f(\mu))(y_A * x_A), f^{-1}(f(\mu))(y_A)\}.$$

Since $\ker f \subseteq A_\mu$, μ is constant on $ker f$. Hence, by Lemma 3.5, we get $\mu(x_A) < \min\{\mu(x_A * y_A), \mu(y_A)\}$ which is a contradiction. Thus $f(\mu)$ is a fuzzy ideal of Y. □

4. Fuzzy n-fold strong ideals

Definition 4.1. A fuzzy set μ in X is called a *fuzzy n-fold strong ideal* of X if it satisfies (FI1) and

(FI4) $\mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y)\}$ for all $x, y, z \in X$, where n is a natural number.

Example 4.2. Let $X := \{0, 1, 2, 3\}$ be a BH-algebra([1]) with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	2	0

Define a fuzzy set μ in X by $\mu(3) = 0.3$ and $\mu(x) = 0.7$ for all $x \neq 3$. Then μ is a fuzzy n-fold strong ideal of X.

A fuzzy set μ in a BH-algebra X is called a *fuzzy subalgebra* of X if $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Lemma 4.3. *In a BH^*-algebra, every fuzzy ideal is a fuzzy subalgebra.*

Proof. Since X is a BH^*-algebra, we have

$$\mu(x * y) \geq \min\{\mu((x * y) * x), \mu(x)\} \\ = \min\{\mu(0), \mu(x)\} \\ = \mu(x),$$

for all $x, y \in X$. Hence $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Thus μ is a fuzzy subalgebra of X. □

Theorem 4.4. *In a BH-algebra, every fuzzy n-fold strong ideal is a fuzzy ideal.*

Proof. Let μ be a fuzzy n-fold strong ideal of a BH-algebra X. Taking $x := x,\ y := y$ and $z := 0$ in (FI4) and using (II), we get

$$\begin{aligned}\mu(x) =& \mu(x * 0^n) \\ &\geq \min\{\mu((x * y) * 0^n), \mu(y)\} \\ &= \min\{\mu(x * y), \mu(y)\}\end{aligned}$$

for all $x, y \in X$. Hence μ is a fuzzy ideal of X. □

Corollary 4.5. *In a BH^*-algebra, every fuzzy n-fold strong ideal is a fuzzy subalgebra.*

The converse of Corollary 4.5 is not be true as seen in the following example.

Example 4.6. Let $X = \{0, 1, 2, 3\}$ be a BH^*-algebra as in Example 4.2 and let μ be a fuzzy set in X given by

$$\mu(x) := \begin{cases} \alpha_1 & \text{if } x \in \{0, 2\} \\ \alpha_2 & \text{otherwise} \end{cases}$$

where $\alpha_1 > \alpha_2$ in $[0, 1]$. It is easy to show that μ is a fuzzy subalgebra of X. But μ is not a fuzzy n-fold strong ideal of X for every positive integer n, because $\mu(3 * 0^n) = \mu(3) = \alpha_2 < \alpha_1 = \min\{\mu((3 * 2) * 0^n), \mu(2)\}$.

Theorem 4.7. *Let X and Y be BH-algebras and $f : X \to Y$ be a homomorphism and ν be a fuzzy n-fold strong ideal of Y. Then $f^{-1}(\nu)$ is a fuzzy n-fold strong ideal of X.*

Proof. Let $x \in X$. Since $f(x) \in Y$ and ν is a fuzzy n-fold strong ideal of Y, $\nu(0) \geq \nu(f(x)) = (f^{-1}(\nu))(x)$, but $\nu(0) = \nu(f(0)) = (f^{-1}(\nu))(0)$ for any $x \in X$. Thus we get $(f^{-1}(\nu))(0) \geq \nu(f(x)) = (f^{-1}(\nu))(x)$ for any $x \in X$. Thus $f^{-1}(\nu)$ satisfies (FI1). Now let $x, y, z \in X$. Since ν is a fuzzy n-fold strong ideal of Y, we have

$$\begin{aligned}\nu(f(x) * f(z)^n) &\geq \min\{\nu((f(x) * f(y)) * f(z)^n), \nu(f(y))\} \\ &= \min\{\nu(f((x * y) * z^n)), \nu(f(y))\}\end{aligned}$$

and so $\nu(f(x * z^n)) \geq \min\{\nu(f((x * y) * z^n)), \nu(f(y))\}$. Hence we get $f^{-1}(\nu)(x * z^n) \geq \min\{f^{-1}(\nu)((x * y) * z^n), f^{-1}(\nu)(y)\}$. Thus $f^{-1}(\nu)$ is a fuzzy n-fold strong ideal of X. □

Proposition 4.8. *Let A be a non-empty subset of a BH-algebra X, n be a positive integer and μ be a fuzzy set in X defined by*

$$\mu(x) := \begin{cases} \alpha_1 & \text{if } x \in A \\ \alpha_2 & \text{otherwise} \end{cases}$$

where $\alpha_1 > \alpha_2$ in $[0, 1]$. Then μ is a fuzzy n-fold strong ideal of X if and only if A is an n-fold strong ideal of X. Moreover, $X_\mu = A$, where $X_\mu := \{x \in X | \mu(x) = \mu(0)\}$.

Proof. Assume that μ is a fuzzy n-fold strong ideal of X. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, we have $\mu(0) = \alpha_1$ and so $0 \in A$. Let $x, y, z \in X$ be such that $(x * y) * z^n \in A$ and $y \in A$. Using (FI4), we know that

$$\mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y)\} = \alpha_1$$

and thus $\mu(x * z^n) = \alpha_1$. Hence $x * z^n \in A$, and A is an n-fold strong ideal of X.

Conversely, suppose that A is an n-fold strong ideal of X. Since $0 \in A$, it follows that $\mu(0) = \alpha_1 \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. If $y \notin A$ or $x * z^n \in A$, then we have

$$\mu(x * z^n) \geq \min\{\mu((x * y) * z^n), \mu(y)\}.$$

Assume that $y \in A$ and $x * z^n \notin A$. Then by (I5), we have $(x * y) * z^n \notin A$. Therefore $\mu(x * z^n) = \alpha_2 = \min\{\mu((x * y) * z^n), \mu(y)\}$. Thus μ is a fuzzy n-fold strong ideal of X. Finally, we have

$$X_\mu = \{x \in X | \mu(x) = \mu(0)\} = \{x \in X | \mu(x) = \alpha_1\} = A.$$

This completes the proof. □

Theorem 4.9. *Let μ be a fuzzy set in a BH-algebra X and let n be a positive integer. Then μ is a fuzzy n-fold strong ideal of X if and only if the non-empty level set $U(\mu; \alpha)$ of μ is an n-fold strong ideal of X.*

Proof. Suppose that μ is a fuzzy n-fold strong ideal of X and $U(\mu; \alpha) \neq \emptyset$ for any $\alpha \in [0, 1]$. Then there exists $x \in U(\mu; \alpha)$ and so $\mu(x) \geq \alpha$. It follows from (FI1) that $\mu(0) \geq \mu(x) \geq \alpha$ so that $0 \in U(\mu; \alpha)$. Let $x, y, z \in X$ be such that $(x * y) * z^n \in U(\mu; \alpha)$ and $y \in U(\mu; \alpha)$. Then by (FI4), we have

$$\begin{aligned}\mu(x * z^n) &\geq \min\{\mu((x * y) * z^n), \mu(y)\}\\ &\geq \min\{\alpha, \alpha\} = \alpha,\end{aligned}$$

and thus $x * z^n \in U(\mu; \alpha)$. Hence $U(\mu; \alpha)$ is an n-fold strong ideal of X.

Conversely, assume that $U(\mu; \alpha) \neq \emptyset$ is an n-fold strong ideal of X for every $\alpha \in [0, 1]$. For any $x \in X$, let $\mu(x) = \alpha$. Then $x \in U(\mu; \alpha)$. Since $0 \in U(\mu; \alpha)$, it follows that $\mu(0) \geq \alpha = \mu(x)$ so that $\mu(0) \geq \mu(x)$ for all $x \in X$. Now we only to show that μ satisfies (FI4). If not, then there exist $a, b, c \in X$ such that

$$\mu(a * c^n) < \min\{\mu((a * b) * c^n), \mu(b)\}.$$

Taking $\alpha_0 = \frac{1}{2}(\mu(a * c^n) + \min\{\mu((a * b) * c^n), \mu(b)\})$, then we have

$$\mu(a * c^n) < \alpha_0 < \min\{\mu((a * b) * c^n), \mu(b)\}.$$

Hence $(a * b) * c^n \in U(\mu; \alpha_0)$ and $b \in U(\mu; \alpha_0)$, but $a * c^n \notin U(\mu; \alpha_0)$, which means that $U(\mu; \alpha_0)$ is not an n-fold strong ideal of X. This is a contradiction. Therefore μ is a fuzzy n-fold strong ideal of X. □

Corollary 4.10. *Let μ be a fuzzy set in a BH-algebra X and let n be a positive integer. Then μ is a fuzzy n-fold strong ideal of X if and only if $X_b := \{x \in X | \ \mu(x) \geq \mu(b)\}$ is an n-fold strong ideal of X for any $b \in X$.*

Proof. Assume that X_b is an n-fold strong ideal of X for any $b \in X$. It is enough to show that $U(\mu;\alpha)$ is an n-fold strong ideal of X for any $\alpha \in [0,1]$. Choose y's so that $\alpha \geq \mu(y)$. Then $\{x \in X|\mu(x) \geq \alpha\} = \cap_y\{x \in X|\mu(x) \geq \mu(y)\}$. By assumption, $U(\mu;\alpha)$ is a fuzzy n-fold strong ideal of X.

Conversely, by Theorem 4.9, it is easy to prove. □

Corollary 4.11. *If μ is a fuzzy n-fold strong ideal of X, then $X_\mu = \{x \in X|\ \mu(x) = \mu(0)\}$ is an n-fold strong ideal of X.*

Theorem 4.12. *Let μ be a fuzzy set in X satisfying* (FI1). *Then μ is a fuzzy n-fold strong ideal of X if and only if for every $t \in [0,1]$ and every $x,y,z \in X$ such that $\mu(y * z^n) < t \leq \mu(x)$, we have $\mu((y*x)*z^n) < t$.*

Proof. Assume that μ is a fuzzy n-fold strong ideal of X. Let $t \in [0,1]$ and $x,y,z \in X$ be such that $\mu(y * z^n) < t \leq \mu(x)$. Then $y * z^n \notin U(\mu;t)$ and $x \in U(\mu;t)$. If $(y*x)*z^n \in U(\mu;t)$, then $y*z^n \in U(\mu;t)$ because $U(\mu;t)$ is an n-fold strong ideal of X by Theorem 4.9. This is impossible, and so $(y*x)*z^n \notin U(\mu;t)$ which shows that $\mu((y*x)*z^n) < t$.

Conversely, suppose that $\mu((y*x)*z^n) < t$ for all $t \in [0,1]$ and $x,y,z \in X$ satisfying $\mu(y*z^n) < t \leq \mu(x)$. Consider any level subset $U(\mu;s)$ of μ and let $a,b,c \in X$ be such that $(a*b)*c^n \in U(\mu;s)$ and $b \in U(\mu;s)$. Obviously, $0 \in U(\mu;s)$. If $a*c^n \notin U(\mu;s)$, then $\mu(a*c^n) < s \leq \mu(b)$, and so $\mu((a*b)*c^n) < s$ by assumption. This implies that $(a*b)*c^n \notin U(\mu;s)$, a contradiction. Hence $a*c^n \in U(\mu;s)$, and thus $U(\mu;s)$ is an n-fold strong ideal of X. By Theorem 4.9, μ is a fuzzy n-fold strong ideal of X. □

Theorem 4.13. *Let $\{A_t|t \in T\}$ be a family of n-fold strong ideals of X such that*

(i) $G = \cup_{t\in T}A_t$

(ii) *$s > t$ if and only if $A_s \subset A_t$ for all $s,t \in T$,*

where T is a non-empty subset of $[0,1]$, *Define a fuzzy set μ in X by*

$$\mu := \sup\{t \in T|x \in A_t\}.$$

Then μ is a fuzzy n-fold strong ideal of X.

Proof. Using Theorem 4.9, it is sufficient to show that the non-empty level subset of μ is an n-fold strong ideal of X. Assume that $U(\mu;s)$ is non-empty, where $s \in [0,1]$. Then either $s = \sup\{t \in T|t < s\}$ or $s \neq \sup\{t \in T|t < s\}$, i.e., either $s = \sup\{t \in T|A_s \subseteq A_t\}$ or $s \neq \sup\{t \in T|A_s \subseteq A_t\}$. In the first case, since $x \in U(\mu;s)$ if and only if $x \in A_t$ for all $t < s$ if and only if $x \in \cup_{t<s}A_t$, we have $U(\mu;s) = \cup_{t<s}A_t$, which is an n-fold strong ideal of X.

The second case implies that there exists $\epsilon > 0$ such that $(s-\epsilon,s)\cap T$ is empty. If $x \in \cup_{t\geq s}A_t$, then $x \in A_t$ for some $t \geq s$, and hence $\mu(x) \geq t \geq s$. It follows that $x \in U(\mu;s)$ so that $\cup_{t\geq s}A_t \subseteq U(\mu;s)$. Now, if $x \notin \cup_{t\geq s}A_t$, then $x \notin A_t$ for all $t \geq s$. Hence $x \notin A_t$ for all $t > s-\epsilon$, i.e., if $x \in A_t$ then $t \leq s-\epsilon$. Thus $\mu(x) \leq s-\epsilon$, and therefore $x \notin U(\mu;s)$. This shows that $U(\mu;s) = \cup_{t\geq s}A_t$, which is an n-fold strong ideal of X. This completes the proof. □

Using a chain of n-fold strong ideals, we establish a fuzzy n-fold strong ideal.

Theorem 4.14. *Let μ be a fuzzy set in X with $Im(\mu) = \{t_0, t_1, \cdots, t_m\}$, where $t_0 > t_1 > \cdots > t_m$ in $[0,1]$. If $A_0 \subset A_1 \subset \cdots \subset A_m = X$ is a chain of n-fold strong ideals of X such that $\mu(A_k - A_{k-1}) = t_k$ for $k = 0, 1, \cdots, m$, where $A_{-1} := \emptyset$, then μ is a fuzzy n-fold strong ideal of X.*

Proof. Obviously, $\mu(0) \geq \mu(x)$ for all $x \in X$. To prove μ satisfies (FI4), we consider the following four cases:

$$(x * y) * z^n \in A_k - A_{k-1}, y \in A_k - A_{k-1},$$
$$(x * y) * z^n \in A_k - A_{k-1}, y \notin A_k - A_{k-1},$$
$$(x * y) * z^n \notin A_k - A_{k-1}, y \in A_k - A_{k-1},$$
$$(x * y) * z^n \notin A_k - A_{k-1}, y \notin A_k - A_{k-1}.$$

The first case implies that $x * z^n \in A_k$, because A_k is an n-fold strong ideal. Hence we have

$$\begin{aligned}\mu(x * z^n) \geq & t_k = \mu((x * y) * z^n) = \mu(y) \\ & = \min\{\mu((x * y) * z^n), \mu(y)\}.\end{aligned}$$

In the second case, we know that either $y \in A_{k-1}$ or $y \in A_m - A_{m-1} \subset A_m - A_k \subset A_m$ for some $m > k$. Since $(x * y) * z^n \in A_k - A_{k-1} \subset A_k$, it follows that either $x * z^n \in A_k$ or $x * z^n \in A_m - A_k \subset A_m$. Therefore we have

$$\begin{aligned}\mu(x * z^n) \geq & t_k = \mu((x * y) * z^n) \\ & \geq \min\{\mu((x * y) * z^n), \mu(y)\}\end{aligned}$$

for $y \in A_{k-1}, x * z^n \in A_k$. Similarly,

$$\mu(x * z^n) \geq t_m = \mu(y) = \min\{\mu((x * y) * z^n), \mu(y)\}$$

for $y \in A_m - A_{m-1}$ and $x * z^n \in A_m - A_k$. In the last two cases the process of verification is analogous. □

Corollary 4.15. *Let μ be a fuzzy set in X with $Im(\mu) = \{t_0, \cdots, t_m\}$, where $t_0 > t_1 > \cdots > t_m$ in $[0,1]$. If $A_0 \subset A_1 \subset \cdots \subset A_m = X$ is a chain of n-fold strong ideals of X such that $\mu(A_k) \geq t_k$ for $k = 0, 1, \cdots, m$, then μ is a fuzzy n-fold strong ideal of X.*

Proof. Straightforward. □

Corollary 4.16. *Let μ be a fuzzy n-fold strong ideal of X. If $Im(\mu) = \{t_0, \cdots, t_m\}$, where $t_0 > t_1 > \cdots > t_m$ in $[0,1]$, then $U(\mu; t_k), k = 0, 1, \cdots, m$, are n-fold strong ideals of X such that*

$$\mu(U(\mu; t_0)) = t_0 \text{ and } \mu(U(\mu; t_k) - U(\mu; t_{k-1})) = t_k \text{ for } k = 0, 1, \cdots, m.$$

Proof. By Theorem 4.9, $U(\mu; t_k)$ are strong ideals of X. Obviously, $\mu(U(\mu; t_0)) = t_0$. Since $\mu(U(\mu; t_1)) \geq t_1$, we have $\mu(x) = t_0$ for $x \in U(\mu; t_0)$, $\mu(x) \geq t_1$ for $x \in U(\mu; t_0) - U(\mu : t_1)$. Repeating this process, we conclude that $\mu(U(\mu; t_k) - U(\mu; t_{k-1})) = t_k$ for $k = 1, \cdots, m$. □

Lemma 4.17. *Every fuzzy set μ in X is represented by $\mu(x) = \sup\{t \in [0,1] | x \in U(\mu;t)\}$ for all $x \in X$.*

Proof. Let $s := \sup\{t \in [0,1] | x \in U(\mu;t)\}$ and let $\epsilon > 0$ be given. Then there exists $t \in [0,1]$ such that $s < s + \epsilon$ and $x \in U(\mu;t)$, and hence $s < \mu(x) + \epsilon$. Since ϵ is arbitrary, it follows that $s \leq \mu(x)$. Let $\mu(x) = w$. Then $x \in U(\mu;w)$, i.e., $w \in \{t \in [0,1] | \mu(x) \in U(\mu;t)\}$. Thus $w \leq \sup\{t \in [0,1] | x \in U(\mu;t)\} = s$, and therefore $\mu(x) = s$, completing the proof. □

Let A be a subset of X. The least n-fold strong ideal of X containing A is called an *n-fold strong ideal generated by A*, written $\langle A \rangle$.

For any fuzzy set μ in X, the least fuzzy n-fold strong ideal of X containing μ is called a *fuzzy n-fold strong ideal of X induced by μ*, denoted by $\langle \mu \rangle$.

Theorem 4.18. *Let μ be a fuzzy set in X. Then the fuzzy set μ^* in X defined by*

$$\mu^* := \sup\{t \in [0,1] | x \in \langle U(\mu;t) \rangle\}$$

for all $x \in X$ is the fuzzy n-fold strong ideal $\langle \mu \rangle$ induced by μ.

Proof. For any $r \in Im(\mu^*)$, let $r_k = r - \frac{1}{k}$ for any $k \in \mathbb{N}$. If $x \in U(\mu^*;r)$, then $\mu^*(x) \geq r$ and so

$$\sup\{t \in [0,1] | x \in \langle U(\mu;t) \rangle\} \geq r > r - \frac{1}{k} = r_k$$

for any $k \in \mathbb{N}$. Hence there exists $s \in \{t \in [0,1] | x \in \langle U(\mu;t) \rangle\}$ such that $s > r_k$. Thus $U(\mu;s) \subseteq U(\mu;r_k)$ and so $x \in \langle U(\mu;s) \rangle \subseteq \langle U(\mu;r_k) \rangle$ for all $k \in \mathbb{N}$. Consequently, $x \in \cap_{k \in \mathbb{N}} \langle U(\mu;r_k) \rangle$.

On the other hand, if $x \in \cap_{k \in \mathbb{N}} \langle U(\mu;r_k) \rangle$, then $r_k \in \{t \in [0,1] | x \in \langle U(\mu;t) \rangle\}$ for any $k \in \mathbb{N}$. Therefore

$$r - \frac{1}{k} = r_k \leq \sup\{t \in [0,1] | x \in \langle U(\mu;t) \rangle\} = \mu^*(x)$$

for all $k \in \mathbb{N}$. Since k is arbitrary, it follows that $r \leq \mu^*(x)$ so that $x \in U(\mu^*;r)$. Hence $U(\mu^*;r) = \cap_{k \in \mathbb{N}} \langle U(\mu;r_k) \rangle$, which is an n-fold strong ideal of X. Using Theorem 4.9, we know that μ^* is a fuzzy n-fold strong ideal of X.

Now we prove that μ^* contains μ. For any $x \in X$, let $s \in \{t \in [0,1] | x \in U(\mu;t)\}$. Then $x \in U(\mu;s)$ and so $x \in \langle U(\mu;s) \rangle$. Thus $s \in \{t \in [0,1] | x \in \langle U(\mu;t) \rangle\}$, which implies that

$$\{t \in [0,1] | x \in U(\mu;t)\} \subseteq \{t \in [0,1] | x \in \langle U(\mu;t) \rangle\}.$$

Using Lemma 4.17, we have

$$\begin{aligned}\mu(x) &= \sup\{t \in [0,1] | x \in U(\mu;t)\} \\ &\leq \sup\{t \in [0,1] | x \in \langle U(\mu;t) \rangle\} \\ &= \mu^*(x).\end{aligned}$$

Hence $\mu \subseteq \mu^*$.

Finally, let ν be a fuzzy n-fold strong ideal of X containing μ. Let $x \in X$. If $\mu^*(x) = 0$, then obviously $\mu^*(x) \leq \nu(x)$. Assume that $\mu^*(x) = r \neq 0$. Then $x \in U(\mu^*; r) = \cap_{k\in\mathbb{N}}\langle U(\mu; r_k)\rangle$, i.e., $x \in U(\mu; r_k)$ for all $k \in \mathbb{N}$. It follows that $\nu(x) \geq \mu(x) \geq r_k - \frac{1}{k}$ for all $k \in \mathbb{N}$ so that $\nu(x) \geq r = \mu^*(x)$ since k is arbitrary. This shows that $\mu^* \subseteq \nu$, completing the proof. □

Theorem 4.19. *Let $\{A_k | k \in \mathbb{N}\}$ be a family of n-fold strong ideals of a BH-algebra X which is nested, i.e., $A_0 \supset A_1 \supset A_2 \supset \cdots$. Let μ be a family set in X defined by*

$$\mu(x) = \begin{cases} \frac{k}{k+1} & \text{if } x \in A_k - A_{k+1},\ k = 0, 1, 2, \cdots \\ 1 & \text{if } x \in \cap_{k=0}^{\infty} A_K \end{cases}$$

for all $x \in X$, where A_0 stands for X. Then μ is a fuzzy n-fold strong ideal of X.

Proof. Clearly $\mu(0) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. Suppose that $(x * y) * z^n \in A_k - A_{k+1}, y \in A_r - A_{r+1}$ for $k = 0, 1, 2, \cdots, r = 0, 1, 2, \cdots$. Without loss of generality, we may assume that $k \leq r$. Then obviously $y \in A_k$. Since A_k is an n-fold strong ideal, it follows that $x * z^n \in A_k$ so that

$$\mu(x * z^n) \geq \frac{k}{k+1} = \min\{\mu((x * y) * z^n), \mu(y)\}.$$

If $(x * y) * z^n \in \cap_{k=0}^{\infty} A_k$ and $y \in \cap_{k=0}^{\infty} A_k$, then there exists $i \in \mathbb{N}$ such that $(x * y) * z^n \in A_i - A_{i+1}$. It follows that $x * z^n \in A_i$, so that

$$\mu(x * z^n) \geq \frac{i}{i+1} = \min\{\mu((x * y) * z^n), \mu(y)\}.$$

Finally, assume that $(x * y) * z^n \in \cap_{k=0}^{\infty} A_k$ and $y \notin \cap_{k=0}^{\infty} A_k$. Then $y \in A_j - A_{j+1}$ for some $j \in \mathbb{N}$. Hence $x * z^n \in A_j$, and thus

$$\mu(x * z^n) \geq \frac{j}{j+1} = \min\{\mu((x * y) * z^n), \mu(y)\}.$$

Consequently, μ is a fuzzy n-fold strong ideal of X. □

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