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Students' Understanding of the Derivative
- Literature Review of English and Korean Publications -

Jungeun Park[1)]

With a gradual increase in research on teaching and learning calculus, there have been various studies about students' thinking about the derivative. This paper reviews the results of the existing empirical studies published in Korean and English. These studies mainly have shown that how students think about the derivative is related to their understanding of the related concepts and the representations of the derivative. There are also recent studies that emphasize the importance of how students learn the derivative including different applications of the derivative in different disciplines. However, the current literature rarely addressed how students think about the derivative in terms of the language differences, e.g., in Korean and English. The different terms for the derivative at a point and the derivative of a function, which shows the relation between concepts, may be closely related to students' thinking of the derivative as a function. Future study on this topic may expand our understanding on the role language-specific terms play in students' learning of mathematical concepts.

Key Words: Derivative, Function, Rate of change, Limit

I. Introduction

This paper reviews what is known about students' understanding of the derivative in existing empirical studies published in Korea and English-speaking countries.[2)] With a gradual increase in learning and teaching collegiate mathematics, many studies explored students' thinking about calculus concepts. Some studies reported that even students who obtained good final grades in their calculus courses performed poorly in non-routine calculus problems due to the lack of relational understanding (e.g., Selden, Selden, Hauk & Mason, 2000). Of many calculus concepts, the derivative is known as a complex

1) University of Delaware(jungeunpark124@gmail.com)

2) To distinguish Korean studies and English-language publications, names of Korean authors are italicized in this paper.

concepts for students to understand because it contains many other concepts - ratio, limit, and function, and it can be represented in various ways - the slope of the tangent line, an instantaneous rate of change, and an expression using Leibniz's notation. Despite its complexity, sound understanding of the derivative is crucial to understanding advanced topics such as integral, Mean Value Theorem, and Fundamental Theorem of Calculus.

Most existing studies have explored students′ understanding about the derivative based on students' prior understandings and misconceptions of other concepts (e.g., Tall, 1986; Thompson, 1994) and their understanding of different representations of the derivative (e.g., Zandieh, 2000; Lim, 2005). In addition, recent studies have examined students' conceptions of the derivative based on departmental factors (e.g., different uses of the derivative in various disciplines, Bingolbali, 2008). What has not be investigated on this topic is language-specific terms. For example, the Korean terms, Mi-bun-gye-su (translated as a coefficient on an infinitesimal quantity) and Do-ham-su (translated as a path function), referring to derivative at a point and derivative function, respectively, are different from the corresponding English terms. These two Korean terms cannot be tied to a single word "derivative" and are not as illustrative or consistent as English terms (Han, 1998). This difference in terminology may explain possible differences in understanding of the derivative between Korean and English speaking countries as shown in existing international comparison studies on other mathematical topics such as whole numbers and fractions (e.g., Fuson and Kwon, 1992).

Because there has not been a comprehensive review on students' thinking about the derivative, this paper reviews studies on this topic, published in Korean and English to explore the factors that affect students' understanding of the derivative with three research questions:

(a) how are students' understanding of the limit, ratio, function, and tangency related to their understanding of the derivative?

(b) how are the different representations of the derivative related to students' understanding of the derivative? and

(c) how are the differences in learning environments (e.g., departments and languages) related to students′ understanding of the derivative?

II. Rationale

Since the definition of the derivative includes three concepts - function, ratio, and limit, these concepts become bases for learning the derivative. Since the derivative is interpreted as a slope of a tangent line to the graph of a function, students' concept of tangency is also related to their thinking about the derivative. Although there is a huge body of research for each of the limit, ratio, function, and tangency (e.g., Tall and

Vinner, 1981; Thompson, 1994), this paper will focus on students′ understanding of these concepts in the context of the derivative.

The derivative can be represented in symbolic, graphical, physical, and algebraic forms. Symbolic representations refer to its notations, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ and $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ if the limits exist. Graphical representations refer to the graph of tangent line to the graph of a function, whose slope is the derivative at a point, and the graph the derivative function. Physical representations mean the applications in physical situations such as velocity of a moving object. Algebraic representations are used for resulting products of applying the derivative formulas to equations of functions. It is possible that students have different levels of understanding about different representations (e.g., Zandieh, 2000), or prefer one representation over others (e.g., Asiala, Cottrill and Dubinsky, 1997). Students might have their own ways of making sense of the relationships among different representations of the derivative (e.g., Santos & Thomas, 2003).

The existing studies also reported how different emphasis on the concept would affect students' understanding of the derivative in different disciplines (Bingolbali, 2008). In an engineering department, the derivative is mostly interpreted as the rate of change to explain its meaning in physical situations whereas in a mathematics department, its mathematical meaning which describes behavior of a function over infinitesimal intervals tend to be emphasized to help students make sense of $\epsilon-\delta$ proofs. Emphasis on different aspects of the derivative might contribute to students' developing different dominant image of the derivative (Bingolbali, 2008).

Another topic that may help our understanding of students′ thinking about the derivative would be the role that language plays in students' learning of mathematical concepts. However, I was not able to find any studies about how language difference in derivative function and derivative at a point is related to students' understanding of these concepts. In this paper, I will review some studies about how language-related words for other mathematical concepts－fractions and whole numbers－are related to students' understanding of these concepts in order to gain insight into whether and how language matters in how students understand the derivative.

III. Method

To gather information about what have been known about students'understanding of the derivative in studies from English language publications, I started with the articles about this topic, which I have read in graduate mathematics education courses. From the references of these articles, I was able to find the names of researchers who have done many studies on students' understanding of the derivative. I searched their web pages (e.g., Tall) and downloaded their articles. I also used key word searches in the web site, Education Resources Information Center (ERIC) and Google Scholar with derivative, differentiability, rate of change, and understanding. To find Korean studies on this topic, I searched two web sites, the database of Korean Studies Information (www.papersearch.net) and Korea Education and Research Information Service, (www.riss4u.net). I also searched the website of National Assembly Library to find the theses and dissertations. I used the same key words that I used to search American literature.

There was a difference between the Korean studies and studies from English language publications regarding researchers and study participants. Compared to most studies from English language publications done by professional researchers with Ph.D. who mostly gathered the data from college calculus students (e.g., Selden, Mason, & Selden, 1994; Bingolbali & Monaghan, 2004 & 2008), the counterpart in Korea is mostly done by high school teachers as writing master's theses about high school students' thinking about the derivative (e.g., Kim, 2000; Lim, 2005). The types of research questions were different. Whereas studies from English language publications tried to explain students' cognitive structures regarding the derivative and related concepts, Korean studies focus on finding various types of misconceptions that students show and errors that they make while solving the problems involving the derivative. Therefore, I decided not to compare and constraint results of Korean and English language publications when I answer the three questions. Instead, I focused on a larger conceptual issue related to students' understanding of the derivative.

IV. Results

1. Students' understanding of mathematical concepts related to the derivative

1) Ratio: rate of change

Since the derivative can be interpreted as the instantaneous rate of change (IRC) of a function, it is important to explore (a) how students understand the average rate of change of a function (ARC), (b) how they make connections between ARC and IRC, and (c) how they use the concept of rate of change to understand the derivative. Studies have reported that students have trouble understanding ARC as a ratio of the difference between y-values to the difference between the corresponding x-values, and have a strong procedural understanding of ARC. In Orton's (1983) study, many (60 out of 110) students failed to find the average rate of change of a curve, $y=3x^2+1$ from $x=a$ to $x=a+h$ by mismatching between intervals of the independent and dependent variables.

They also failed to find the value of $f'(2)$ using the limit definition of the derivative on the interval from $(2+h)$ and $(2-h)$ as h approaches 0, in which the denominator should be $2h$ (Lee, 2005). Hauger (1998) also found that students' understanding of the rate of change is closely connected to the procedure of calculating ARC with x and y coordinates of the given points. According to Hauger (1998), this procedure-based concept of the rate of change cannot help students deepen their understanding of the connection between ARC and IRC because it is hard to relate this procedure to the IRC over infinitesimal intervals.

Studies have also found students' difficulties making connection between ARC and IRC. Many students in Orton's (1983) study did not see the connection between ARC and IRC: appreciating the limiting process of obtaining IRC at a point from the sequence of ARC values in graphical situations. Lim (2005) also reported students′ lack of understanding the relationship between ARC and IRC; the students, who correctly calculated ARC of $f(x)=3x^2+1$ on $[3, 3+h]$, found $f'(3)$ using the differentiation rules insead of finding the limit of ARC as h approaches 0.

Lastly, some studies have related ARC to students' thinking about the derivative and theorems in calculus. Thompson (1994) argued that the operational understanding ARC is crucial to make sense of the Fundamental Theorem of Calculus (FTC) and the Mean Value Theorem (MVT). He explained the operational understanding of ARC as knowing that "if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, we would end up with the same amount of change in the dependent quantity as actually occurred" (p. 165). For example, traveling at an average speed of 30km/hr means that one would trip exactly the same distance in the same amount of time if one repeats the trip as traveling at the constant rate of 30km/hr. Similarly, he interpreted MVT in a way that any differentiable function has its ARC over a given interval, and this ARC is equal to some instantaneous rate of change of the function within that interval. During the interviews, students did not show the operational understanding of ARC. However, after participating in his calculus lessons focusing on the operational understanding of ARC, the students showed improvement in explaining the derivative, MVT, and FTC, using the rate of change embedded in these concepts (Thompson, 1994).

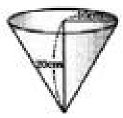
2) Limit

The derivative is defined as the limit of the difference quotient. Existing studies showed that students' misconceptions of the limit are related to their difficulties appreciating a tangent line at a point as a limit of sequent lines, whose slope is the derivative at the point. Based on the idea of the limit was that a sequence could approach its limit as close as possible but never get there (e.g., the sequence 0.9, 0.99, 0.999, 0.9999 approaches but never gets to 1, Tall & Vinner, 198; Han, 1997; Lim, 2005), students stated that the sequence of secant lines approaching to a tangent line could never get to the tangent line (e.g, "the tangent line can't be a limit because a part of the chord already past it," Tall, 1986, p. 2). Based on their thinking about chords of a circle; students answered that the limit was zero or a point (Tall, 1986) or did not exist because the length of the chord approaches zero as two points get closer, (e.g., "The lines get smaller," and "It disappears," Orton, 1983, p. 237). The authors pointed out that these misconceptions hamper students' understanding the limiting process to obtain the tangent line from several secant lines when the definition of derivative is explained graphically (Tall, 1986; Orton, 1983).

3) Function

Since taking the derivative of a function can be considered as an action operating on the function (Asiala, et al., 1997), students' understanding of a function should be taken into account. Thompson (1994) reported a common students' image that a function has two sides, a short expression on the left and a long expression on the right, which are separated by an equal sign (p. 268). He argued that this image can lead students to consider a function as one thing changing rather than as co-varying quantities, which might hamper them to interpret the behavior of a composite function appropriately. For example, in Santos and Thomas's (2003) research, a student who calculated the first and second derivatives of a given polynomial function correctly, but could not acknowledge the difference between $f''(x)$ and $f'(f'(x))$ can be an example of a lack of understanding in the covariance nature of a function. To him, taking the derivative of a function twice was the same as composing the derivative function with itself because he might have thought that a function vary only based on the independent variable x.

Students' weak understanding of what is varying in a function is also identified in their thinking about the independent variable of the derivative. For example, Carlson, Oehrtman, and Thompson (2008) observed that many calculus students could not explain the difference between the rates of change of the volumes of sphere and cube. In the case of a sphere, the surface area is the derivative of the volume when both the surface and volume are considered as functions of the radius of the sphere. However, in the case of a cube, one cannot obtain the surface area by taking the derivative of its volume. This difference comes from the difference in the nature of independent variables, the radius of a sphere and the side of a cube, which were not appreciated by many calculus students (Carlson et al., 2008). Lee (2005) also found students′ trouble finding the independent variable of the derivative of a function. Students did not pay attention to the variable which they were supposed to find the rate of change on. The dominant solution to a problem (Figure 1) was setting the variables x and y, finding the volume of the container, $V=\frac{2}{3}\pi x^3$, and taking the derivative of V. However, many students calculated the rate of change of the volume with respect to the radius, x, instead of time, t (Lee, 2005).

	In the container in the picture, one fills the water. The radius of the top of the water increases 0.5 cm/sec. When the radius is 6 *cm*, the rate of change of the velocity is $a\,\pi cm^3$. Find the integer *a*.	

[Figure 1] Problem 14 and variables in students' dominant solution (Lee, 2005, p. 45)

Some studies found that students'tendency to find an algebraic representation of a function is related to their understanding of the differentiability. Based on their misconception that a function should have one equation, students tend to think that a piecewise function is not differentiable at a point where it changes its equation because it has two derivatives at the point (Ferrini-Mundi & Graham, 1994; Kim, 2000). Student's preference of algebraic representations also plays an important role when they determine the differentiability of a function given by a graph. To determine if a curve represents a differentiable function, a student in Ferrini-Mundi and Graham (1994) first tried to find an equation for the curve, on which she could apply derivative formulas. If she could not find an equation or she found two equations (e.g., piece-wise function), she determined that the given graph did not represent a differentiable function. Even students, who know how to calculate the derivative of a piecewise function, showed a lack of understanding of the relationship between continuity and differentiability (Seldon et al., 1989, 1994, 2000; Viholanen, 2006). Viholanen (2006) found that students tended to calculate the derivative of equations on both sides at the point where the function changes, and to decide that the function was differentiable if the values were the same. This method leads to a mathematically incorrect conclusion that a discontinuous function is differentiable.

4) Tangency

Students' thinking about the derivative is also related to their thinking about tangent lines on graphical situations. Biza, Christou, and Zachariades's (2006) study, students answered that drawing more than one tangent line at a point is possible, and that a tangent line exists at a cusp. These misconceptions can affect students' understanding of differentiability because some students consider drawing a tangent line as equivalent to differentiating (Choi, 2001; Park, 2007). To these students, existence of more than one tangent line at one point and a tangent line at a cusp might imply that its derivative function has two values at one point and, that y value of the derivative function exists at the point the original function is not differentiable.

Another misconception that a tangent line to the curve should intersect the curve only once at the tangency point (Biza et al., 2006; Choi, 2001; Kim, 2005; Park, 2007) was found when students tried to find the equation of a tangent line of a curve. Even after calculating the equation of the tangent line algebraically using the derivative. Students did not make the connection between the equation and the graph of the tangent line because it intersects the curve at other points besides a tangency point when they extend the line (Kim, 2005; Park, 2007).

2. Students'understanding of the representations of the derivative

This section reviews studies about students′ thinking about symbolic, graphical, physical, and algebraic representations of the derivative, and relationships among these representations.

1) Symbolic representations

There have been studies about how students understand the symbols dx,dx, dy, and dy/dx and the relationship between the symbols $\triangle y/\triangle x$, and dy/dx. Studies found that students, who correctly interpret $\triangle x$ and $\triangle y$ correctly as the changes in x and y, respectively, have difficulties making sense of dx and dy. They tended to interpret dx as the derivative of x without mentioning an independent variable such as the differential of x and the rate of change of x, or as an amount of x or x-increment (Orton, 1983). In Yoon and Eun's (2001) study, only 66% of the students, who correctly calculated dy/dx of $y=x^2$, calculated dy for the same function correctly.

Many studies showed that students recognized the symbol dy/dx as a signal of taking the derivative of a function without distinguishing the independent and dependent variable. A student calculated $\frac{dm}{dq}=\frac{6m-7}{3m^2}$ when the given function was $q=\frac{3m^2-7}{m^3}$ by treating m as the independent variable (Santos & Thomas, 2003). Also, student did not see the equivalence between $\frac{d^2y}{dx^2}$ and $\frac{d(\frac{dy}{dx})}{dx}$. For $\frac{d^2y}{dx^2}$, they applied the differentiation twice, but did not see $\frac{d(\frac{dy}{dx})}{dx}$ this way. Students' weak understanding of these symbols was also found when they explain the relationship between $\triangle y/\triangle x$ and dy/dx. In Lim's (2005) study, most of students, who correctly interpreted $\triangle y/\triangle x$ as an average rate of change (ARC) or the slope, and dy/dx as the derivative, an instantaneous rate of change (IRC) or the slope of tangent line, did not correctly explain the limit process between $\triangle y/\triangle x$ and dy/dx. These findings are aligned with the results from the previous section about the ratio that students consider ARC and IRC as separate quantities.

2) Graphical representation

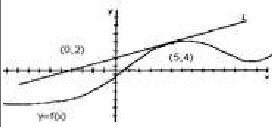
Studies have identified two students' misconceptions about the graphical representations of the derivative: assumptions that the graphs of a function and its derivative resemble each other, and the derivative of an increasing (decreasing) function is always positive (negative). Choi (2001) found that students are likely to assume that the graphs of a function and its derivative take the same direction, and thus the same sign of the slope resulting in the same shapes of graphs. In Nemirovsky and Rubin's (1992) study, when asked to graph the derivative function, many students drew a similar graph to the original function by matching overall characteristics of the two graphs. Nemirovsky and Rubin (1992) argued that this tendency was likely due to the plausibility of the physical contexts, if an object has positive increasing velocity, its distance also increases as time passes, and thus using examples from various situations can help understand this relationship better. Studies also found that students assume that an increasing (decreasing) function has a non-zero positive (negative) slope when they identify the interval where a function increases. They did not consider including the equal sign in the inequality, $f'(x) > 0$ due to their image that a function f does not change near the points where $f'(x) = 0$ (Choi, 2001).

3) Physical representations

The first and second derivatives can be interpreted as the velocity and acceleration of a moving object, respectively. Existing studies found students' weak understanding of the derivative in physical contexts. For example, in Bezuidenhout's study, for a given distance function, $S(t)$, students interpreted $S'(80) = 1.15$ incorrectly, e.g., "for the velocity of 80 km/h, the deceleration is 1.15m/s," "rate of change of the distance of a car at 80km/h is 11.15km/h," and "at 80km/h the change in time is 1.15 seconds" (Bezuidenhout,1998, p. 395). During interviews, students did not provide correct units or interpretations of these quantities, which is aligned with the results of their weak understanding of functions and the rate of change with respect to which quantity is varying. This lack of understanding of the rate of change in physical contexts also seems related to their confusion of the velocity and acceleration. In Lim's (2005) study, when asked to graph the distance-time function based on the verbal data, "the object is moving with a positive constant velocity," many students drew a concave-up curve instead of a straight line due to their confusion between "the positive constant velocity" with a constantly increasing velocity (p. 32).

4) Algebraic representations

Studies have found that students prefer algebraic representations of the derivative over other representations, and their dominant image of the derivative is "applying the derivative formulas" (e.g., Asiala et al., 1997). When asked what the derivative is, many students recall derivative formulas or the equations obtained by applying derivative formulas to functions (Choi, 2001; Chung, 2003). However, students showed various misconceptions and errors when they applied derivative formulas (e.g., Chung 2003; Hirst, 2002). Students prefer algebraic representations even when an algebraic approach is not explicitly mentioned in a problem, and not helpful to solve the problem (Asiala et al., 1997; Kim, 2005; Lim, 2005). Asiala et al. (1997) conducted interviews to explore students' mental constructions of the graphical representation of the derivative. When a curve and its tangent line are given (Figure 2), to calculate f'(5), many students found the equation of the tangent line and differentiated it instead of finding the slope based on the given two points. Then, when asked to explain the relationship between the tangent line and curve, they tried to integrate the slope (a constant), to find the equation of the curve.



Suppose that the line L tangent to the graph of the function f at the point $(5, 4)$ as indicated in the figure. Find $f(5)$, $f'(5)$.

[Figure 2] Problem 6 (Asiala et. al., 1997, p. 404)

Similarly, in Lim's (2005) study, when asked to graph the derivative of a given concave-up curve, only one third of the students tried to use the slopes of tangent lines or the pattern of increasing or decreasing of the curve whereas half of the students tried to use even-degree polynomials in general, a quadratic function $y = ax^2 + bx + c$ in particular.

Despite of their preference of algebraic representations, students showed incorrect use of algebraic representations of the derivative when they applied derivative formulas. Studies have showed that most prominent procedural errors come from extrapolation - applying derivative rules based on external similarity of functions (Hirst, 2002; Yoon & Eun, 2001). When asked to differentiate $y = f(g(x))$, many students obtained $y = f'(g(x))$ ignoring the chain rule. Similarly, when differentiating a product of functions, $y = f(x)g(x)$, or a rational function, $y = f(x)/g(x)$, they obtained $y = f'(x)g'(x)$, and $y = f'(x)/g'(x)$, respectively (Lim, 2005; Chung, 2003). Many students applied the power rule $(x^n)' = nx^{n-1}$, to exponential or trigonometric functions, and obtained $(e^x)' = xe^{x-1}$, $(x^x)' = xx^{x-1}$, and $[(\sin x)^x]' = x(\sin x)^{x-1}$ (Hirst, 2002; Yoon & Eun, 2001).

5) Relationships among representations

Studies on students' understanding of representations of the derivative found that students showed different levels or dimensions of understanding depending on how the derivative is represented. These studies employed hierarchical frameworks presented as matrices with columns of several levels of understanding and rows of different representations. Zandieh (2000) used a matrix-form framework for various representations of the derivative: graphical (slope), verbal (ratio and rate), physical (velocity), and symbolic (difference quotient). Zandieh (2000) specified three hierarchical concept layers in the definition of the derivative, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$: the different quotient, the limit of the quotient, and the derivative function as process of taking limit at infinitely many x values in the domain of a function (Table 1).

<Table 1> Zandieh's (2000) framework for the derivative (p. 106)

	Contexts				
	Graphical	Verbal	Paradigmatic/Physical	Symbolic	Other
Process-Object layer	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

With this framework, Zandieh (2000) analyzed nine high school calculus students' responses to the question, "What is a derivative?" Each of the students had different level of understanding of the derivative in different contexts. One students showed strong understanding in symbolic representations with detailed explanations of all three concept layers, but did not show conceptual understanding in graphical representations in any layer; he just mentioned that the derivative is the slope of a tangent line of a function. Another subject explained graphical and symbolic representations and their relationship, but did not mention IRC (verbal) or any physical examples.

Lim (2005) employed Zandieh's framework to analyze students' answer patterns on problems involving the derivative. Lim (2005) classified her survey problems into slots in Zandieh's framework, and calculated percentages of the correct answer for each slot. Students' response patterns in symbolic, graphical, verbal representations were all similar: high performance in function layer but relatively low success in limit and ratio layers. Because the function layer is defined by applying limit layer on the ratio layer, this pattern implies that students' understanding of the function layer is based on the pseudo-structures (superficial knowledge of a concept without appreciating underlying meaning) of ratio and limit (Lim, 2005).

Studies also explored students' understanding of the derivative based on how they explain the relationships among different representations of the derivative. For example, Hahkioniemi (2005) discussed two levels of understanding of the derivative based on how strong connections among different representations students make are. Students, who have a weak understanding of connections between representations, are able to "predict, identify, or produce the counter part of a given external representation" (p. 2). However, when an action is given to a representation, they cannot "predict, identify, or produce" the result of the action in its counterpart because this metal activity involves strong understanding of the link between the representations (p. 2). In his study, three out of five high school students showed weak understanding; they explained the meaning of each representation of the derivative separately, but could not describe how phases of taking derivative in symbolic representations are related to their counterparts in graphical representations. However, one student who showed strong understanding of this connection, explained the equivalence between $\lim_{x \to 1}\frac{f(x)-f(1)}{x-1}$ and $\lim_{h \to 0}\frac{f(1+h)-f(1)}{h}$ using the rate of change in graphical representations. Santos and Thomas (2002) also compared different levels of understanding with two students: Steven with procedure-oriented understanding and James with object-oriented understanding. During the interview, Steven, who consistently mentioned and used differentiation rules, did not make any connection to other representations of the derivative when prompted by the interviewer. However, James explained how the function, ratio, and limit layers in the limit definition of the derivative are related to the gradients of the graph of the function.

As discussed in this section, students understanding of the derivative is related to how they think about the ratio, limit, function and tangency, and the representations of the derivative. Besides these mathematical factors, exiting studies have also identified non-mathematical factors which affects the way that students make sense of the concept of the derivative.

2. Students' thinking about the derivative in different learning environment

Students' thinking about the derivative is also related to how they learn the derivative. For example, it could be an emphasis on different aspects of the derivative in different disciplines. It could also be what language they used to learn the derivative due to the language-specific mathematical terms. There have been studies about how language-specific terms for numbers and fractions are related to children's understanding of these concepts (e.g., Fuson & Kwon, 1992). However, studies about language issues and students' thinking about the derivative are rare.

1) Departmental difference

The derivative is used differently in different disciplines; it is mostly used as the rage of change in mechanical engineering department (ME) and as the slope of tangent line in mathematics (M) department. Bingolbali and Monaghan (2004 & 2008) employed a socio-cultural perspective to investigate the difference between ME and M students' concept images of the derivative. They found that ME students tended to understand the derivative as the rate of change whereas mathematics students showed a tendency interpreting the derivative as a slope of the tangent line. Where as there was no significant difference between M and ME students' performance in the pretest which was held in the beginning of the semester of the first calculus course, there was significant difference between there achievement in the posttest which was administered at the end of the semester. In the posttest, ME students outperformed M students on problems involving the rate of change while the M students outperformed on problems involving the slope of tangent line. Based on the interviews with students, the authors concluded that this difference came from students' affiliation and positional identity related to their major including the difference between calculus courses they took.

2) Difference in language-specific terms

Terms of a mathematical concept very across different languages, which may be related to how students understand the concepts. For example, Fuson and Kwon (1992) showed the importance of number words for children to understand multi-digit numbers. In Korean, when they read a number, each digit in a number is said, and its value is named. For example, "11" is read as Ten one, in contrast to, Eleven. Fuson and Kwon (1992) specified the difference between English and Korean number words as a possible linguistic reason why Korean children outperformed the U.S. children in multi-digit addition and subtraction. Similarly, Korean terms for fraction contains the part-whole concept, which is different from the corresponding English terms (Miura et al., 1999). For example, 1/2 is read as sam-bun-ui-il, which is translated as "of three parts, one" (p. 3). It includes the concept that the whole is divided into three parts, unlike the term, "one third." Based on this observation, the authors conducted experiments with Korean and American first-grade students who had not learned fraction. During the experimental lessons, teachers explained how to read fractions and what they mean without emphasizing shaded and unshaded parts in visual representations of fractions. OF three test administered in the middle and end of the first year, and the beginning of the second year, Korean students significantly outperformed American students in the last two tests, although their performance were not significantly different in the first test. Miura et al. (1999) argued that Korean terms, which explicitly state the part-whole concept, may help children understand the simple fractions easier.

This language-related approach, which explains the relation between language-specific terms for mathematical concepts and students' understanding of the concepts, may explain other aspects of students' understanding of the derivative than mathematical concepts in the derivative and its representations. Exploring how similarity and dissimilarity of the terms for "the derivative at a point" and "the derivative function" are related to students' understanding of the derivative, may reveal a new factor, Word use, forming their cognitive structures related to the derivative.

V. Conclusion and Discussions

This paper reviewed existing studies from Korean and English language publications on students' thinking about the derivative in terms of (a) students' understanding of related concepts such as functions, ratios, limits, and tangency, (b) students' understanding of various representations of the derivative, and (c) different mathematical learning environments. As discussed earlier, there have been rich body of research on the first two topics that address mathematical aspects of the derivative. However, the last topic that address how students learn the derivative, have not been investigated in details. The only aspect identified in existing studies on this topic was differences between disciplines: Mathematics and Mechanical Engineering (Bingolbali and Monaghan, 2004 & 2008). Another aspect of how students learn the derivative may be the language-specific terms for the derivative, which provided the initial motivation for this literature review. However, due to the lack of the studies about the derivative, I reviewed two international comparison studies on numbers and fractions. Similar to these studies, the difference between Korean and English terms for "derivative function" and "the derivative at a point" might affect students' understanding of the derivative. Unlike these English terms, Korean terms, "Mi-Bun-Gye-Su" and "Do-Ham-Su" do not have common words, "derivative." Thus, these two terms do not show the relation between these terms. However, in English, one word, "derivative" is colloquially used for both "derivative at a point" and "derivative of a function," which may confuse students about whether the word "derivative" refers to (a) a point-specific value or (b) a function. Students' understanding about these two different aspects of the derivative may be closely related to the language-specific terms for "the derivative at a point" and "the derivative function"; the English terms show the consistency of the concepts, and the Korean terms shows the difference of the concepts. Use of these two terms especially in the beguiling stage is important because it addresses an important aspect of the concept, the derivative as a function. Considering the fact that the concept of a function is developed from a point-specific concept to the concept on a continuum, the similar transition from the derivative at a point to the derivative of a function should be addressed a great care in word use. For example, in English, the difference between point-specific and continuous concepts should be emphasized due to the same use of the word, "derivative." In Korean, the connection between 미분계수 and 도함수 need to be addressed because of the two totally different terms. Future research on this topic in Korean and English would help expand our understanding of how students think about the derivative as a function in relation to these two mathematical terms.

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학생들의 미분에 대한 이해에 관한 문헌 연구

박정은[3)]

초 록

최근 미적분학에 관한 수학 교육연구가 점차 많아지고 있다. 많은 미적분 개념 중 미분은 학생들이 이해하기 어려운 개념으로 알려져 있다. 본 논문은 영어나 한국어로 출판된 연구들 중 학생의 미분에 대한 이해를 다룬 연구들의 결과를 요약, 정리 하고 있다. 연구들은 학생들의 미분에 관한 이해는 학생들이 어떻게 미분에 관련된 다른 개념들, 즉 극한, 함수, 변화율, 접선, 그리고 여러 가지 미분을 표현하는 방법을 이해하는 가와 관련이 있음을 보여주고 있다. 또한 최근 연구들은 학생이 어떠한 환경에서 미분을 학습하였는지가 학생의 미분에 관한 이해와 밀접한 관련이 있음도 보여 주고 있다. 예를 들면, 미분의 개념은 다른 전공에서 다르게 이용 된다. 현존하는 연구에서 다루어지지 않은 주제로 미분계수와 도함수가 다른 언어에서 어떻게 다르게 표현하는 지와 학생들이 가지고 있는 미분의 개념의 연계성을 들 수 있으며, 이 주제에 관한 후행 연구는 이런 언어 차이와 학생들이 함수로서의 미분을 어떻게 이해하는 지에 관한 우리의 이해를 증진 시킬 수 있을 것이다.

주요 용어: 미분, 함수, 변화율, 극한

3) 델라웨어 대학교(jungeunpark124@gmail.com)