EXISTENCE OF WEAK NON-NEGATIVE SOLUTIONS FOR A CLASS OF NONUNIFORMLY BOUNDARY VALUE PROBLEM

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Abstract. The goal of this paper is to study the existence of non-trivial non-negative weak solution for the nonlinear elliptic equation:

$$-\text{div}(h(x)\nabla u) = f(x, u) \text{ in } \Omega$$

with Dirichlet boundary condition in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, where $h(x) \in L^1_{\text{loc}}(\Omega)$, $f(x, s)$ has asymptotically linear behavior. The solutions will be obtained in a subspace of the space $H^1_0(\Omega)$ and the proofs rely essentially on a variation of the mountain pass theorem in [12].

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ with smooth boundary $\partial \Omega$. We study the existence of non-trivial weak solution of the following Dirichlet problem

$$\begin{cases} -\text{div}(h(x)\nabla u) = f(x, u) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial \Omega, \end{cases}$$

where $h(x) \in L^1_{\text{loc}}(\Omega), h(x) \geq 1$ a.e. $x \in \Omega$.

Due to the presence of $h(x) \in L^1_{\text{loc}}(\Omega)$, the problem now may be non-uniform in sense that the functional associated to the problem may be infinity for some $u$ in $H^1_0(\Omega)$. In what follow, we deduce the problem (1.1) to a uniform one by using an appropriate weighted Sobolev space. Then applying a variation of the mountain pass theorem in [12], we prove that the problem (1.1) admits a non-trivial non-negative weak solution in a subspace of the $H^1_0(\Omega)$. Let us introduce some hypotheses:

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F1) \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Caratheodory function satisfying
\[
f(x, s) = 0 \quad \text{for all } s \leq 0, \text{ a.e. } x \in \Omega.
\]

F2) There exists a constant \( C > 0 \) such that \( \frac{|f(x, s)|}{s} \leq C \) a.e. \( x \in \Omega \),
\( \forall s \in (0, +\infty) \) and \( f \) is “asymptotically linear” in the sense that there
exists \( \beta \in C(\overline{\Omega}) \) such that \( \beta(x) = \lim_{s \to +\infty} \frac{f(x, s)}{s} \) uniformly a.e. \( x \in \Omega \).

Firstly, we introduce some following remark:

Remark 1.1. There is a rich literature dealing with asymptotically linear prob-
lem and existence results on bounded domain or unbounded domain in t he
case that \( h(x) = 1 \) which have been obtained via variational methods (see
\([1, 4, 18, 19, 20]\) and the reference therein). However, to the be st of our knowl-
edge, there has never been any study on the existence results of asymptotically
linear of the problem (1.1) in the case \( h(x) \in L^1_{\text{loc}}(\Omega) \). This case will be appro-
priate in our paper.

Remark 1.2. The problem (1.1) when the nonlinearity satisfies the condition
\( 0 < \mu F(x, s) \leq f(x, s) s \quad \text{for } \mu > 2, |s| \geq M, \)
where \( F(x, s) = \int_0^s f(x, t) dt \) has been studied either when \( h(x) \in L^\infty(\Omega) \) or
\( h(x) \in L^1_{\text{loc}}(\Omega) \) (see \([9, 21, 22]\)). We point out that the condition (1.2) implies
that \( f \) has to be superlinear at infinity. So this kind of assumption is not
appropriate in our situation.

Let \( H^1_0(\Omega) \) be the usual Sobolev space under the norm
\[
||u|| = \left( \int_{\Omega} (|\nabla u|^2 + |u|^2) dx \right)^{\frac{1}{2}}.
\]
We now consider following subspaces \( H \) of \( H^1_0(\Omega) \)
\[
H = \left\{ u \in H^1_0(\Omega) : \int_{\Omega} h(x)|\nabla u|^2 dx < +\infty \right\}.
\]
Then \( H \) is a Hilbert space with the norm
\[
||u||_H^2 = \int_{\Omega} h(x)|\nabla u|^2 dx
\]
and the scalar product (see \([9, 22]\))
\[
\langle u, v \rangle_H = \int_{\Omega} h(x)\nabla u \cdot \nabla v dx, \quad u, v \in H.
\]
Furthermore we have \( ||u||_{H^1_0(\Omega)} \leq ||u||_H, \ u \in H \) and the continuous embedding
\( H \hookrightarrow H^1_0(\Omega) \hookrightarrow L^q(\Omega), 2 \leq q \leq 2^* = \frac{2N}{N-2} \) hold true. Moreover, the embedding
\( H \hookrightarrow L^2(\Omega) \) is compact.

Definition 1.1. We say that \( u \in H \) is a weak solution of the problem (1.1) if
\[
\int_{\Omega} h(x)\nabla u \cdot \nabla \varphi dx - \int_{\Omega} f(x, u) \varphi dx = 0
\]
for all $\varphi \in H$.

2. Auxiliary results

We define the functional $J : H \rightarrow \mathbb{R}$ given by

$$J(u) = \frac{1}{2} \int_{\Omega} h(x)|\nabla u|^2 dx - \int_{\Omega} F(x, u) dx$$

$$= T(u) - P(u), \quad u \in H,$$

where

$$F(x, t) = \int_{0}^{t} f(x, s) ds, \quad T(u) = \frac{1}{2} \int_{\Omega} h(x)|\nabla u|^2 dx,$$

$$P(u) = \int_{\Omega} F(x, u) dx, \quad u \in H.$$  \hfill (2.5)

Firstly we remark that the critical points of the functional $J$ correspond to the weak solution of the problem (1.1). Moreover, due to the presence of $h(x) \in L^{1}_{loc}(\Omega)$, in general, the functional $T$ (and thus $J$) does not belong to $C^{1}(H)$. This means that we cannot apply the classical mountain pass theorem by Ambrossetti-Rabinowitz. In order to overcome this difficulty, we shall apply a weak version of the mountain pass theorem introduced by D. M. Duc [12]. But we first recall the following useful concept of weak continuous differentiability:

**Definition 2.1.** Let $J$ be a functional from a Banach space $Y$ into $\mathbb{R}$. We say that $J$ is weakly continuously differentiable on $Y$ if and only if three following conditions are satisfied:

i) $J$ is continuous on $Y$.

ii) For any $u \in Y$ there exists a linear map $DJ(u)$ from $Y$ into $\mathbb{R}$ such that

$$\lim_{t \rightarrow 0} \frac{J(u + t\varphi) - J(u)}{t} = \langle DJ(u), \varphi \rangle, \forall \varphi \in Y.$$  

iii) For any $\varphi \in Y$, the map $u \mapsto \langle DJ(u), \varphi \rangle$ is continuous on $Y$.

We denote by $C^{1}_{w}(Y)$ the set of weakly continuously differentiable functionals on $Y$. It is clear that $C^{1}(Y) \subset C^{1}_{w}(Y)$, where $C^{1}(Y)$ is the set of all continuously Fréchet differentiable functionals on $Y$. With similar arguments as those used in the proof of Proposition 2.2 in [22], we conclude the following proposition which concerns the smoothness of the functional $J$.

**Proposition 2.1.** The functional $J$ given by (2.4) is weakly continuously differentiable on $H$ and we have

$$\langle DJ(u), \varphi \rangle = \int_{\Omega} h(x) \nabla u \nabla \varphi dx - \int_{\Omega} f(x, u) \varphi dx \quad \text{for all} \ u, \varphi \in H.$$  

By Proposition 2.1, the critical points of the functional $J$ correspond to the weak solutions of the problem (1.1).
Proposition 2.2 (see Lemma 2.3 in [9]). The functional $T$ given by (2.5) is weakly lower semicontinuous on the space $H$.

Proposition 2.3. Let $v \in L^\infty(\Omega)$ such that $\Omega^+ = \{ x \in \Omega : v(x) > 0 \}$ is an open set in $\mathbb{R}^N$. Set

$$\Lambda := \inf_{u \in H} \left\{ \int h(x)|\nabla u|^2 dx : \int_\Omega v(x)u^2 dx = 1 \right\}.$$ 

Then

i) $S = \{ u \in H : \int_\Omega v(x)u^2 dx = 1 \} \neq \emptyset,$

ii) there exists $u_0 \in S : \int_\Omega h(x)|\nabla u_0|^2 dx = \Lambda$ and $u_0 \geq 0$, $u_0 \neq 0$ in $\Omega$.

Proof. i) Let $u \in C^\infty_0(\Omega^+)$, $u \neq 0$ and $u \in H$, then $\int_{\Omega^+} v(x)u^2 dx > 0$.

Choose $\overline{\tau} \in H$ as $\overline{\tau} = \frac{u(x)}{(\int_{\Omega^+} v(x)u^2 dx)^{1/2}}$ as $x \in \Omega^+$ and $\overline{\tau} = 0$ as $x \in \Omega \setminus \Omega^+$. Then

$$\int_\Omega v(x)\overline{\tau}^2 dx = \int_{\Omega^+} v(x)\frac{u^2}{\int_{\Omega^+} v(x)u^2 dx} dx = 1.$$ 

Hence $S \neq \emptyset$.

ii) Let $\{ u_m \} \subset H$ be a minimizing sequence, i.e.,

$$\int_\Omega h(x)|\nabla u_m|^2 dx \rightarrow \Lambda \quad \text{and} \quad \int_\Omega v(x)u_m^2 dx = 1.$$ 

So $\{ u_m \}$ is bounded in $H$. Then there exists a subsequence of $\{ u_m \}$ still denoted by $\{ u_m \}$ such that $u_m \rightharpoonup \hat{u}$ in $H$ and $u_m \rightarrow \hat{u}$ in $L^2(\Omega)$. We have $\hat{u} \in S$. Indeed,

$$1 = \lim_{m \rightarrow +\infty} \int_\Omega v(x)u_m^2 dx = \int_\Omega v(x)\hat{u}^2 dx$$

(from $v \in L^\infty(\Omega)$ we deduce $\int_\Omega v(x)(u_m^2 - \hat{u}^2) dx \rightarrow 0$). Then by the minimizing properties of $\{ u_m \}$ and by the weakly lower semicontinuity of the functional $\int_\Omega h(x)|\nabla u|^2 dx$ (see Proposition 2.2) we have

$$\Lambda = \lim_{m \rightarrow +\infty} \inf \int_\Omega h(x)|\nabla u_m|^2 dx \geq \int_\Omega h(x)|\nabla \hat{u}|^2 dx \geq \Lambda.$$ 

So we get $\Lambda = \int_\Omega h(x)|\nabla \hat{u}|^2 dx$.

We have $\hat{u} \in H$ and $\hat{u}$ is a minimizer of

$$\inf \left\{ \int_\Omega h(x)|\nabla u|^2 dx : \int_\Omega v(x)u^2 dx = 1 \right\}.$$ 

We show that $|\hat{u}|$ is a minimizer too. Since $\hat{u} \in H \subset H^1_0(\Omega)$ then $|\hat{u}| \in H^1_0(\Omega)$ (see Lemma 7.6, p. 145 in [14]).

Moreover

$$\int_\Omega v(x)\hat{u}^2 dx = \int_\Omega v(x)|\hat{u}|^2 dx,$$ 

so $|\hat{u}| \in S$. 


Finally

$$\Lambda = \int_{\Omega} h(x)|\nabla \hat{u}|^2 = \int_{\Omega} h(x)|\hat{u}|^2 \, dx.$$ 

So $|\hat{u}| \in H$ and $|\hat{u}|$ is a minimizer then $\hat{u} \geq 0$. We suppose that $\hat{u} = 0$ then we deduce that $\int_{\Omega} v(x)\hat{u}^2 \, dx = 0$, a contradiction. Set $u_0 = |\hat{u}|$, $u_0 \geq 0$ and $u_0 \neq 0$ in $\Omega$. The proof of Proposition 2.3 is complete. □

3. Main results

Let us introduce following hypotheses.

F3) There exists $x_0 \in \Omega$ such that $\beta(x_0) > 0$ where $\beta$ is defined by F2).

Denoted by

$$\Omega_\beta = \{ x \in \Omega : \beta(x) > 0 \}$$

and assume that

$$\Lambda_\beta = \inf_{u \in H(\Omega_\beta)} \frac{\int_{\Omega_\beta} h(x)|\nabla u|^2 \, dx}{\int_{\Omega_\beta} \beta(x)u^2 \, dx} > 1.$$ 

F4) There exist two positive constants $\tau_1, \tau_2$ such that

$$\lim_{s \to +\infty} \frac{2F(x,s)}{s^2} \leq \tau_1 < \lambda_1 \leq \lim_{s \to +\infty} \frac{2F(x,s)}{s^2}$$ 

uniformly a.e. $x \in \Omega$,

where $\lambda_1 = \inf_{u \in H} \frac{\int_{\Omega} h(x)|\nabla u|^2 \, dx}{\int_{\Omega} u^2 \, dx}$.

Our main result is given by the following theorem.

**Theorem 3.1.** Assuming hypotheses F1)-F4) are fulfilled. Then the problem (1.1) has at least one non-negative non-trivial weak solution in space $H$.

In order to prove Theorem 3.1, we need some following propositions.

**Proposition 3.1.** Assuming F1), F2), F4) are fulfilled. Then there exist $\alpha, \rho > 0$ such that $J(u) \geq \alpha$ if $\|u\|_H = \rho$. Moreover, there exists $\varphi_0 \in H$ such that $J(t\varphi_0) \to -\infty$ as $t \to +\infty$.

**Proof.** By F4) $\lim_{s \to +\infty} \frac{2F(x,s)}{s^2} \geq \tau_2$ uniformly a.e. $x \in \Omega$, we deduce that there exists $s_0 > 0$ such that $\frac{2F(x,s)}{s^2} \geq \tau_2$ for all $s > s_0$ or $F(x,s) \geq \frac{1}{2}\tau_2 s^2$ for all $s > s_0$ uniformly a.e. $x \in \Omega$. We choose $t_0 \in (0,s_0]$ such that $F(x,t_0) < \frac{1}{2}\tau_2 t_0^2$ a.e. $x \in \Omega$.

Fix $\varepsilon > 0$. There exists $B(\varepsilon, t_0)$ such that $F(x,t_0) \geq \frac{1}{2}(\tau_2 - \varepsilon)t_0^2 - B(\varepsilon, t_0)$. Denote $B(\varepsilon) = \sup_{u \leq s_0} B(\varepsilon, t_0)$. We obtain for any given $\varepsilon > 0$ there exists $B = B(\varepsilon)$ such that

$$F(x,s) \geq \frac{1}{2}(\tau_2 - \varepsilon)s^2 - B$$ 

for all $s \in (0, +\infty)$ a.e. $x \in \Omega$. 
Remark that by F2) we deduce \( \lim_{\varepsilon \to +\infty} \frac{F(x,s)}{\varepsilon^2} = 0 \) a.e. \( x \in \Omega \) and \( q > 2 \). Fix arbitrarily \( \varepsilon > 0 \). In the same way, using the second inequality of F4) and F2), it follows that there exists \( A = A(\varepsilon) > 0 \) such that
\[
2F(x,s) \leq (\tau_1 + \varepsilon)s^2 + 2A(\varepsilon)s^q \quad \text{for all } s > 0 \text{ a.e. } x \in \Omega.
\]
For any given \( \varepsilon > 0 \) there exists \( A = A(\varepsilon) > 0, B = B(\varepsilon) > 0 \) such that
\[
\frac{1}{2}(\tau_2 - \varepsilon)s^2 - B \leq F(x,s) \leq \frac{1}{2}(\tau_1 + \varepsilon)s^2 + As^q \quad \text{for all } s \in (0, +\infty) \text{ a.e. } x \in \Omega.
\]
Now we choose \( \varepsilon > 0 \) so that \( \tau_1 + \varepsilon < \lambda_1 < \tau_2 - \varepsilon \), we have
\[
J(u) = \frac{1}{2}||u||^2_H - \int_{\Omega} F(x,u)dx \\
\geq \frac{1}{2}||u||^2_H - \frac{1}{2} \int_{\Omega} (\varepsilon + \tau_1)u^2 dx - \int_{\Omega} A|u|^q dx \\
\geq \frac{1}{2}(1 - \varepsilon + \tau_1)||u||^2_H - Ak||u||^q_H.
\]
With \( q > 2 \), choose \( \rho = ||u||_H \) small enough then we have
\[
\alpha = \frac{1}{2}(1 - \varepsilon + \tau_1)||u||_H^2 - Ak||u||^q_H > 0.
\]
Moreover,
\[
J(u) \leq \frac{1}{2}||u||^2_H - \frac{1}{2} \int_{\Omega} (\tau_2 - \varepsilon)|u|^2 dx + B|\Omega|.
\]
Choose \( \varphi_0 \in C_0^\infty(\Omega), \varphi_0 > 0 \) such that \( \varphi_0 \) is a \( \lambda_1 \)-eigen-function, that is, it satisfies \( \lambda_1 \int_{\Omega} \varphi_0^2 dx = \int_{\Omega} b(x)|\nabla \varphi|^2 dx \). Denote \( u_0 = t\varphi_0 \) then
\[
J(u_0) \leq \frac{1}{2}(1 - \frac{\tau_2 - \varepsilon}{\lambda_1})t^2 ||\varphi_0||^2 + B|\Omega| \to -\infty \text{ as } t \to +\infty.
\]

**Proposition 3.2.** Assuming hypotheses F1)-F4) are fulfilled. Let \( \{u_m\} \) be a Palais-Smale sequence in \( H \), i.e.,
\[
\lim_{m \to \infty} J(u_m) = c, \quad \lim_{m \to +\infty} ||DJ(u_m)||_{H^*} = 0.
\]
Suppose that \( \{u_m\} \) is not bounded in \( H \). Then there exists a subsequence of \( \{u_m\} \) until denoted \( \{u_m\} \) such that \( ||u_m||_H \to +\infty \) as \( m \to +\infty \). Putting \( w_m = \frac{\lim_{m \to \infty} u_m}{||u_m||_H} \). Then there exists a subsequence \( \{w_{m_k}\} \) of \( \{w_m\} \) such that \( \{w_{m_k}\} \to w \) in \( H \) satisfying
i) \( w \neq 0 \) in \( \Omega \),
ii) \( w > 0 \) in \( \Omega \),
iii) \(-\text{div}(h(x)\nabla w) = \beta(x)w \) in \( \Omega \).

**Proof.** We have \( ||w_m||_H = 1 \), so \( \{w_m\} \) is bounded in \( H \) then there exists a subsequence \( \{u_{m_k}\} \) such that
\[
w_{m_k} \to w \text{ in } H,
\]
\[
w_{m_k} \to w \text{ in } L^2(\Omega),
\]
$w_{m_k} \to w$ a.e. in $\Omega$.

i) Arguing by contradiction, if $w = 0$, then $w_{m_k} \to 0$ in $L^2(\Omega)$ and

$$\frac{DJ(u_{m_k})(u_{m_k})}{\|u_{m_k}\|_H} \to 0$$

(from definition of PS sequence).

So, a fortiori

$$\frac{DJ(u_{m_k})(u_{m_k})}{\|u_{m_k}\|_H} \to 0.$$

This yields

$$\int_{\Omega} h(x)|\nabla w_{m_k}|^2 dx - \int_{\Omega} f(x,u_{m_k})w_{m_k}^2 dx \to 0,$$

$$\|w_{m_k}\|_H^2 - \int_{\Omega} f(x,u_{m_k})w_{m_k}^2 dx \to 0,$$

$$1 - \int_{\Omega} f(x,u_{m_k})w_{m_k}^2 dx \to 0.$$

Since $\frac{f(x,u_{m_k})}{u_{m_k}}$ is bounded and $w_{m_k} \to 0$ in $L^2(\Omega)$, we get relation $1 = 0$. Hence we must have $w \neq 0$.

ii) Knowing that $\frac{DJ(u_{m_k})(\varphi)}{\|\varphi\|\|u_{m_k}\|} \to 0, \forall \varphi \in H, \varphi \neq 0$. We deduce

$$\int_{\Omega} h(x)\nabla u_{m_k} \cdot \nabla \varphi dx - \int_{\Omega} f(x,u_{m_k})\varphi dx \to 0, \forall \varphi \in H.$$

Since $f(x,s) = 0$ for $s \leq 0$

$$\int_{\Omega} h(x)\nabla u_{m_k} \cdot \nabla \varphi dx - \int_{\Omega} f(x,u_{m_k}^+)\varphi dx \to 0,$$

$$\int_{\Omega} h(x)\nabla w_{m_k} \cdot \nabla \varphi dx - \int_{\Omega} f(x,u_{m_k})w_{m_k}^+ \varphi dx \to 0.$$

Since $\frac{f(x,u_{m_k})}{u_{m_k}}$ is bounded, it has a subsequence still denoted by $\frac{f(x,u_{m_k})}{u_{m_k}}$, it converges weakly in $L^2$ to some function $\theta \in L^\infty$. Then,

$$\int_{\Omega} h(x)\nabla w_{m_k} \cdot \nabla \varphi dx - \int_{\Omega} f(x,u_{m_k})w_{m_k}^+ \varphi dx \to 0,$$

$$\int_{\Omega} h(x)\nabla w \cdot \nabla \varphi dx - \int_{\Omega} \theta(x)w^+ \varphi dx = 0, \forall \varphi \in H.$$

Choosing $\varphi = w^-$ we have

$$\int_{\Omega} h(x)\nabla w \cdot \nabla w^- dx - \int_{\Omega} \theta(x)w^+ w^- dx = 0,$$

$$\int_{\Omega} h(x)|\nabla w^-|^2 dx = 0$$

which implies $w^- = 0$ then $w \geq 0$. 
So \( w \geq 0 \) satisfies the equation

\[
-\text{div}(h(x)\nabla w) = \theta(x)w \quad \text{in } \Omega.
\]

Moreover, for any \( \Omega' \subset \subset \Omega \), we have \( h \in L^1_{\text{loc}}(\Omega') \), \( w(x) \neq 0 \), \( w(x) \geq 0 \) in \( \Omega' \) and

\[
-\text{div}(h(x)\nabla w) = \theta(x)w \quad \text{in } \Omega'.
\]

By the Hanark inequality (see [14] Theorem 8.20 and Corollary 8.21), it follows that \( w(x) > 0 \) in \( \Omega' \). This implies that \( w(x) > 0 \) in \( \Omega \).

iii) Since \( w > 0 \), \( u_m \to +\infty \) a.e. in \( \Omega \). So

\[
\frac{f(x, u_{m_k})}{u_{m_k}} \to \beta(x) \quad \text{a.e. } x \in \Omega,
\]

\[
\frac{f(x, u_{m_k})}{u_{m_k}} \to \theta(x) \quad \text{in } L^2(\Omega)
\]

this yields \( \beta(x) = \theta(x) \). Then \( w \) verifies the equation

\[
\int_{\Omega} h(x)\nabla w \nabla \varphi dx = \int_{\Omega} \beta(x)w\varphi dx \quad \text{for all } \varphi \in H
\]

so

\[
-\text{div}(h(x)\nabla w) = \beta(x)w \quad \text{in } \Omega.
\]

The proof of Proposition 3.2 is complete.

**Proposition 3.3.** Assuming hypotheses F1)-F4) are fulfilled. Then the functional \( J: H \to \mathbb{R} \) is defined by (2.4) satisfies the Palais-Smale condition on \( H \).

**Proof.** Let \( \{u_m\} \) be a sequence in \( H \) such that

\[
\lim_{m \to \infty} J(u_m) = c, \quad \lim_{m \to +\infty} \|DJ(u_m)\|_{H^*} = 0.
\]

First, we shall prove that \( \{u_m\} \) is bounded in \( H \). We suppose by contradiction that \( \{u_m\} \) is not bounded in \( H \). Then there exists a subsequence still denoted \( \{u_m\} \) such that \( \|u_m\|_H \to +\infty \) as \( m \to +\infty \). Then putting \( w_m = \frac{u_m}{\|u_m\|} \), by Proposition 3.2, we have the subsequence \( \{w_{m_k}\} \) of \( \{w_m\} \) satisfying \( w_{m_k} \to w \) in \( H \) and

i) \( w \neq 0 \) in \( \Omega \),

ii) \( w > 0 \) in \( \Omega \),

iii) \( -\text{div}(h(x)\nabla w) = \beta(x)w \) in \( \Omega \).

Hence

\[
\int_{\Omega} h(x)|\nabla w|^2 dx = \int_{\Omega} \beta(x)w^2 dx.
\]

So we have

\[
1 = \frac{\int_{\Omega} h(x)|\nabla w|^2 dx}{\int_{\Omega} \beta(x)w^2 dx}.
\]
Recall \( \Omega_\beta = \{ x \in \Omega : \beta(x) > 0 \} \subset \Omega \), we deduce

\[
1 = \frac{\int_{\Omega} h(x)|\nabla w|^2 dx}{\int_{\Omega} \beta(x)w^2 dx} \geq \frac{\int_{\Omega} h(x)|\nabla w|^2 dx}{\int_{\Omega_\beta} \beta(x)w^2 dx} \geq \frac{\int_{\Omega} h(x)|\nabla w|^2 dx}{\int_{\Omega_\beta} \beta(x)w^2 dx}
\]

\[
\geq \inf_{u \in H(\Omega_\beta)} \frac{\int_{\Omega} h(x)|\nabla u|^2 dx}{\int_{\Omega_\beta} \beta(x)w^2 dx} = \Lambda_\beta.
\]

By F3) we have a contradiction. So we deduce that all Palais Smale sequences of the functional \( J \) are bounded in \( H \).

Next we prove that \( \{ u_m \} \) has a subsequence converging strongly in \( H \). Since \( \{ u_m \} \) is bounded in \( H \), \( H \) is a Hilbert space, there exists a subsequence \( \{ u_{m_k} \} \) such that it converges weakly to some \( u \) in \( H \) and \( \{ u_{m_k} \} \) converge strongly in \( L^2(\Omega) \). Then by Proposition 2.2 we find that

\[
(3.9) \quad T(u) \leq \lim_{k \to \infty} \inf T(u_{m_k}).
\]

Now we prove \( \lim_{k \to +\infty} T(u_{m_k}) = T(u) \). Indeed,

\[
\langle DT(u_{m_k}), u_{m_k} - u \rangle = \langle DJ(u_{m_k}), u_{m_k} - u \rangle + \langle DP(u_{m_k}), u_{m_k} - u \rangle.
\]

By the definition of \( \text{(PS)} \) sequence we have

\[
\lim_{k \to +\infty} \langle DJ(u_{m_k}), u_{m_k} - u \rangle = 0.
\]

From F2)

\[
|\langle DP(u_{m_k}), u_{m_k} - u \rangle| = \left| \int_{\Omega} f(x, u_{m_k})(u_{m_k} - u) dx \right|
\leq \int_{\Omega} |f(x, u_{m_k})| |u_{m_k} - u| dx
\leq C \left( \int_{\Omega} |u_{m_k}|^2 dx \right)^{\frac{1}{2}} \cdot \left( \int_{\Omega} |u_{m_k} - u|^2 dx \right)^{\frac{1}{2}}.
\]

Since \( u_{m_k} \to u \) in \( L^2(\Omega) \) we have

\[
\lim_{k \to +\infty} \langle DP(u_{m_k}), u_{m_k} - u \rangle = 0.
\]

Hence

\[
\lim_{k \to +\infty} \langle DT(u_{m_k}), u_{m_k} - u \rangle = 0.
\]

On the other hand, since \( T \) is convex the following inequality holds true

\[
T(u) - T(u_{m_k}) \geq \langle DT(u_{m_k}), u - u_{m_k} \rangle.
\]

Letting \( k \to +\infty \) we have

\[
T(u) - \lim_{k \to +\infty} T(u_{m_k}) = \lim_{k \to +\infty} \left[ T(u) - T(u_{m_k}) \right]
\geq \lim_{k \to +\infty} \langle DT(u_{m_k}), u - u_{m_k} \rangle = 0.
\]
This implies that

\[ T(u) \geq \lim_{k \to +\infty} T(u_{m_k}). \]  

From (3.9), (3.10) we get \( \lim_{k \to +\infty} T(u_{m_k}) = T(u) \).

Now we prove that the sequence \( \{u_{m_k}\} \) converges strongly to \( u \) in \( H \). Indeed, we suppose by contradiction that \( \{u_{m_k}\} \) is not converges strongly to \( u \) in \( H \).

Then there exist a constant \( \epsilon_0 > 0 \) and a subsequence \( \{u_{m_{kj}}\} \) of \( \{u_{m_k}\} \) such that \( \|u_{m_{kj}} - u\|_H \geq \epsilon_0 \) for \( j = 1, 2, \ldots \).

Recalling inequality

\[ \left| \frac{\alpha + \beta}{2} \right|^2 + \left| \frac{\alpha - \beta}{2} \right|^2 = \frac{1}{2}(|\alpha|^2 + |\beta|^2), \quad \forall \alpha, \beta \in \mathbb{R}. \]

We deduce that for any \( j = 1, 2, \ldots \)

\[ \frac{1}{2}T(u_{m_{kj}}) + \frac{1}{2}T(u) - T\left(\frac{u_{m_{kj}} + u}{2}\right) \geq \frac{1}{4}\|u_{m_{kj}} - u\|^2_H = \left(\frac{\epsilon_0}{2}\right)^2. \]  

(3.11)

Again instead of the remark that since \( \{u_{m_{kj}} + u/2\} \) converges weakly to \( u \) in \( H \), applying Proposition 2.2 we have

\[ T(u) \leq \lim_{j \to +\infty} \inf T\left(\frac{u_{m_{kj}} + u}{2}\right). \]

Then from (3.11), letting \( j \to \infty \) we obtain

\[ T(u) - \lim_{j \to +\infty} \inf T\left(\frac{u_{m_{kj}} + u}{2}\right) \geq \left(\frac{\epsilon_0}{2}\right)^2 > 0 \]

which is a contradiction. Therefore, \( \{u_{m_k}\} \) converges strongly to \( u \) in \( H \). Thus the functional \( J \) satisfies the Palais-Smale condition on \( H \). The proof of Proposition 3.3 is complete.

Proposition 3.4.

i) \( J(0) = 0 \).

ii) The acceptable set \( G = \{ \gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = u_0 \} \) is not empty (with \( u_0 \) in Proposition 3.1).

Proof. i) Follows from F1) and the definition of \( J \) we have \( J(0) = 0 \).

ii) Let \( \gamma(t) = tu_0 \), so \( \gamma(0) = 0, \gamma(1) = u_0 \), then \( \gamma(t) \in G \) and \( G \neq \emptyset \).

Proof of Theorem 3.1. By Propositions 3.1-3.4, all assumptions of the variations of the mountain pass theorem introduced in [12] are satisfied. Therefore there exists \( \tilde{w} \in H \) such that

\[ 0 < \alpha \leq J(\tilde{w}) = \inf \{ \max J(\gamma([0,1])) : \gamma \in G \} \]
and \( \langle DJ(\tilde{w}), v \rangle = 0 \) for all \( v \in H \), i.e., \( \tilde{w} \) is a weak solution of the problem (1.1). Moreover since \( J(\tilde{w}) > J(0) \), \( \tilde{w} \) is a nontrivial weak solution of the problem (1.1). We have
\[
\int_{\Omega} h(x) \nabla \tilde{w} \nabla \varphi dx - \int_{\Omega} f(x, \tilde{w}) \varphi dx = 0, \quad \forall \varphi \in H.
\]
Choose \( \varphi = \tilde{w}, f(x, w) = 0 \) as \( w \leq 0 \). So we obtain
\[
\int_{\Omega} h(x) \nabla \tilde{w} \nabla \tilde{w} dx = 0 \quad \text{or} \quad ||\tilde{w}||_H = 0.
\]
Then \( \tilde{w} \geq 0 \), is a weak solution non-negative non-trivial of the problem (1.1). Theorem 3.1 is completely proved. □

References


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