A Characterization of $M_1$-Spaces

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Abstract. In this paper, we prove the following theorem which gives a characterization of $M_1$-spaces by $g$-function.

**Theorem.** A space $(X, \mu)$ is an $M_1$-space if and only if there exists a function $g : \omega \times X \to \mu$ such that:

1. $\{x\} = \bigcap_n g(n, x)$.
2. $y \in g(n, x) \Rightarrow g(n, y) \subset g(n, x)$.
3. There exists a collection $\mathcal{U} = \{\bigcup_n U_n\}$ of open sets in $(X, \mu)$ such that:
   (a). $\bigcup\{g(n, x) : x \in U_n\}$ is regular open for each $U_n \in \mathcal{U}$.
   (b). If $H \subset X$ is closed with $y \notin H$, then $y \notin \text{Cl}_n(\bigcup\{g(n, x) : x \in U_n\})$ for some $U_n \in \mathcal{U}$ with $H \subset U_n$.

1. Introduction

Ceder [2] defined $M_i$-spaces, $i = 1, 2, 3$ and proved $M_1 \Rightarrow M_2 \Rightarrow M_3$. It is an interesting problem that whether or not these implications can be reversed. Recall that a space $X$ is an $M_1$-space if $X$ has a $\sigma$-closure preserving base $\mathcal{B}$. Recall that a collection $\mathcal{B}$ is a quasi-base for $X$ if for each open set $U$ of $X$ and a point $x \in U$, there is $B \in \mathcal{B}$ such that $x \in \text{Int}B \subset B \subset U$. A space $X$ is an $M_2$-space if $X$ has a $\sigma$-closure preserving quasi-base and an $M_3$-space if $X$ has a $\sigma$-cushioned pair-base.

Borges [1] gave some important results on $M_3$-spaces and renamed $M_3$-spaces as stratifiable spaces. Gruenhage [4] and Junnila [5] independently proved that stratifiable spaces are $M_2$-spaces. This is an important progress to the problem since stratifiable spaces have been shown to have many useful properties and are preserved by countable products, closed images, arbitrary subspaces; $M_1$-spaces have a simple and natural definition. In 1990, Rudin [7] suggested several conjectures on general topology for the 21st century. One of the conjectures is called “$M_3 \Rightarrow M_1$ problem”. According to her guess, the problem should have a positive answer. Also the problem is called “Problem on General Metric Spaces” in Wang [8].

In 2007, Lin [6] gave a survey of the most important results about $M_i$-spaces. It included discussions of the characteristics of $M_2$-spaces. In order to consider

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But, so far, $M_1$-spaces have only a simple and natural definition. It will be helpful to consider the $M_1 \Rightarrow M_3$ problem if we have a characterization of $M_1$-spaces. To do it, we construct a $g$-function which shows a characterization of $M_1$-spaces.

In this paper, the letter $N$ denotes the set of positive integers. $m$ and $n$ are used to denote members in $N$.

2. Section 2

In this section, we prove the following theorem which gives a characterization of $M_1$-spaces by $g$-function.

**Theorem 2.1.** A space $(X, \mu)$ is an $M_1$-space if and only if there exists a function $g : \omega \times X \rightarrow \mu$ such that:

1. $\{x\} = \cap_n g(n, x)$.
2. $y \in g(n, x) \Rightarrow g(n, y) \subset g(n, x)$.
3. There exists a collection $\mathcal{U} = \cup_n \mathcal{U}_n$ of open sets in $(X, \mu)$ such that:
   - (a) $\cup\{g(n, x) : x \in U_{na}\}$ is regular open for each $U_{na} \in \mathcal{U}$.
   - (b) If $H \subset X$ is closed with $y \notin H$, then $y \notin Cl_\mu(\cup\{g(n, x) : x \in U_{na}\})$ for some $U_{na} \in \mathcal{U}$ with $H \subset U_{na}$.

**Proof.** ($\Rightarrow$) Let $(X, \mu)$ be an $M_1$-space. Then $(X, \mu)$ has a $\sigma$-closure preserving base $\mathcal{B} = \cup_n \mathcal{B}_n$. Let $g(n, x) = X - \cup\{Cl_\mu B_{no} : B_{no} \in \mathcal{B}_n, x \notin Cl_\mu B_{no}\}$.

**Proof of (1).** By the definition of $g(n, x)$, we have $x \in g(n, x)$. So $x \in \cap_n g(n, x)$. On the other hand, if $y \in \cap_n g(n, x)$, then $y \in g(n, x)$ for each $n \in N$. Suppose $y \neq x$. Then there exist open sets $V_1, V_2$ such that $y \in V_1, x \in V_2$ and $V_1 \cap V_2 = \emptyset$. Then there exists a $B_{m3} \in \mathcal{B}_m$ such that $y \in B_{m3} \subset V_1$ for some $m \in N$. So $x \notin Cl_\mu B_{m3}$. Then $y \notin g(m, x)$, a contradiction to $y \in \cap_n g(n, x)$. So $y = x$. Then $\{x\} = \cap_n g(n, x)$.

**Proof of (2).** Take a $y \in g(n, x) = X - \cup\{Cl_\mu B_{no} : B_{no} \in \mathcal{B}_n, x \notin Cl_\mu B_{no}\}$. Then $x \notin Cl_\mu B_{no}$ implies $y \notin Cl_\mu B_{no}$. So $\cup\{Cl_\mu B_{no} : B_{no} \in \mathcal{B}_n, x \notin Cl_\mu B_{no}\} \subset \cup\{Cl_\mu B_{no} : B_{no} \in \mathcal{B}_n, y \notin Cl_\mu B_{no}\}$. So $g(n, y) \subset g(n, x)$.

**Proof of (3).** We prove it by the following Claims.

**Claim 2.2.** $Cl_\mu Int_\mu(Cl_\mu B_{no}) = Cl_\mu B_{no}$ for each $n \in N$ and each $B_{no} \in \mathcal{B}_n$.

**Proof.** $B_{no} \subset Int_\mu Cl_\mu B_{no}$ implies $Cl_\mu B_{no} \subset Cl_\mu Int_\mu(Cl_\mu B_{no})$. On the other hand, if $x \notin Cl_\mu B_{no}$, then there exist open sets $V_1, V_2$ such that $x \in V_1, Cl_\mu B_{no} \subset V_2$ and $V_1 \cap V_2 = \emptyset$. So $Int_\mu Cl_\mu B_{no} \subset V_2$. Then $x \notin Cl_\mu Int_\mu(Cl_\mu B_{no})$. So $Cl_\mu Int_\mu(Cl_\mu B_{no}) \subset Cl_\mu B_{no}$. Then $Cl_\mu Int_\mu(Cl_\mu B_{no}) = Cl_\mu B_{no}$. \hfill $\square$

**Claim 2.3.** $\cup\{g(n, x) : x \in X - Cl_\mu B_{no}\} = X - Cl_\mu B_{no}$ for each $n \in N$ and each.
Claim 2.4. \( X - Cl_{\mu}(X - Cl_{\mu}B_{\mu}) = Int_{\mu}Cl_{\mu}B_{\mu} \) for each \( n \in N \) and each \( B_{\mu} \in \mathcal{B}_n \).

Proof. Take an \( x \in X - Cl_{\mu}(X - Cl_{\mu}B_{\mu}) \). Then \( x \notin Cl_{\mu}(X - Cl_{\mu}B_{\mu}) \). Then there exists an open set \( V \) such that \( x \in V \) and \( V \cap (X - Cl_{\mu}B_{\mu}) = \emptyset \). So there exists a \( U \in Cl_{\mu}(X - Cl_{\mu}B_{\mu}) \) such that \( U \cap (X - Cl_{\mu}B_{\mu}) \neq \emptyset \) if \( U \) is a neighborhood of \( x \). So \( x \notin Int_{\mu}Cl_{\mu}B_{\mu} \). Then \( Int_{\mu}Cl_{\mu}B_{\mu} \subset X - Cl_{\mu}(X - Cl_{\mu}B_{\mu}) \). So \( X - Cl_{\mu}(X - Cl_{\mu}B_{\mu}) = Int_{\mu}Cl_{\mu}B_{\mu} \). □

Claim 2.5. \( \cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\} \) is regular open for each \( n \in N \) and each \( B_{\mu} \in \mathcal{B}_n \).

Proof. \( \cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\} \subset Cl_{\mu}(\cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\}) \) implies \( \cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\} \subset Int_{\mu}Cl_{\mu}(\cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\}) \).

On the other hand, take a \( y \in Int_{\mu}Cl_{\mu}(\cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\}) \), then there exists an open set \( U \) such that \( y \in U \subset Cl_{\mu}(\cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\}) \).

Then \( y \in U \subset X - Cl_{\mu}B_{\mu} \) by Claim 2.3 and Claim 2.4. So \( y \notin Cl_{\mu}(Cl_{\mu}B_{\mu}) = Cl_{\mu}B_{\mu} \) by Claim 2.2. Then \( y \notin Cl_{\mu}(Cl_{\mu}B_{\mu}) \) by Claim 2.3. So \( Int_{\mu}Cl_{\mu}(\cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\}) \subset \cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\} \). Then \( \cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\} = Int_{\mu}Cl_{\mu}(\cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\}) \). So \( \cup\{g(n, x) : x \in X - Cl_{\mu}B_{\mu}\} \) is regular open. □

Proof of (3)(continued). Let \( U_{\alpha} = X - Cl_{\mu}B_{\mu} \) for \( B_{\mu} \in \mathcal{B}_n \), \( U_{\alpha} = \{U_{\alpha} : \alpha \in \Lambda\} \), \( U = \cup U_{\alpha} \). Then \( \cup\{g(n, x) : x \in U_{\alpha}\} \) is regular open for each \( U_{\alpha} \in U \) by Claim 2.5. If \( y \notin H \), where \( H \) is closed, then there exist open sets \( V_1, V_2 \) such that \( y \in V_1 \), \( H \subset V_2 \) and \( V_1 \cap V_2 = \emptyset \). So there exists a \( B_{\mu} \in \mathcal{B}_n \) such that \( y \in B_{\mu} \subset V_1 \) for some \( n \in N \). Then \( Cl_{\mu}(B_{\mu}) \cap V_2 = \emptyset \). So \( H \subset U_{\alpha} \) and \( B_{\mu} \cap (\cup\{g(n, x) : x \in U_{\alpha}\}) = \emptyset \). Then \( y \notin Cl_{\mu}(\cup\{g(n, x) : x \in U_{\alpha}\}) \).

\((\Longleftrightarrow)\) Now we prove that \( (X, \mu) \) is an \( M_1 \)-space if \( (X, \mu) \) satisfies (1),(2) and (3). To do it, we prove the following claim at first.

Claim 2.6. \( Cl_{\mu}(X - Cl_{\mu}(\cup\{g(n, x) : x \in U_{\alpha}\})) = X - \cup\{g(n, x) : x \in U_{\alpha}\} \) if \( \cup\{g(n, x) : x \in U_{\alpha}\} \) is regular open for each \( U_{\alpha} \in U \).

Proof. \( \cup\{g(n, x) : x \in U_{\alpha}\} \subset Cl_{\mu}(\cup\{g(n, x) : x \in U_{\alpha}\}) \) implies \( X - \cup\{g(n, x) : x \in U_{\alpha}\} \subset Cl_{\mu}(\cup\{g(n, x) : x \in U_{\alpha}\}) \). Then \( X - \cup\{g(n, x) : x \in U_{\alpha}\} = Cl_{\mu}(X - Cl_{\mu}(\cup\{g(n, x) : x \in U_{\alpha}\})) \). On the other hand, \( \cup\{g(n, x) : x \in U_{\alpha}\} \) is a regular open set. Then \( \cup\{g(n, x) : x \in U_{\alpha}\} \).
x ∈ U_{na} = \text{Int}_\mu Cl\mu(\{g(n, x) : x ∈ U_{na}\}). Pick an x ∈ X - \bigcup\{g(n, x) : x ∈ U_{na}\}.

If x ∈ X - Cl\mu(\bigcup\{g(n, x) : x ∈ U_{na}\}), then x ∈ Cl\mu(X - Cl\mu(\bigcup\{g(n, x) : x ∈ U_{na}\})),

If x ∈ Cl\mu(\{g(n, x) : x ∈ U_{na}\}) - \bigcup\{g(n, x) : x ∈ U_{na}\}, then x ∉ \bigcup\{g(n, x) : x ∈ U_{na}\}.

Then there exists an U = \text{Int}_\mu Cl\mu(\{g(n, x) : x ∈ U_{na}\}) such that U ∩ (X - Cl\mu(\bigcup\{g(n, x) : x ∈ U_{na}\})) ≠ ∅ if U is a neighborhood of x. Then x ∈ Cl\mu(X - Cl\mu(\bigcup\{g(n, x) : x ∈ U_{na}\})).

This implies X - \bigcup\{g(n, x) : x ∈ U_{na}\} ⊂ Cl\mu(X - Cl\mu(\bigcup\{g(n, x) : x ∈ U_{na}\})). So Cl\mu(X - Cl\mu(\bigcup\{g(n, x) : x ∈ U_{na}\})) = X - \bigcup\{g(n, x) : x ∈ U_{na}\}. □

Next we continue to prove the sufficiency of Theorem 2.1.

Let B_{na} = X - Cl\mu(\{g(n, x) : x ∈ U_{na}\}) for each n ∈ N and each U_{na} ∈ \mathcal{U}_n.

Let \mathcal{B}_n = \{B_{na} : α ∈ \Lambda\} and \mathcal{B} = \bigcup_n \mathcal{B}_n. Let \Lambda_1 ⊆ \Lambda. Then, by Claim 2.6,

\begin{align*}
\mathcal{B}_n = & \bigcup\{Cl\mu B_{na} : α ∈ \Lambda_1\} \\
= & \bigcup\{Cl\mu(X - Cl\mu(\{g(n, x) : x ∈ U_{na}\})) : α ∈ \Lambda_1\} \\
= & \bigcup\{X - \bigcup\{g(n, x) : x ∈ U_{na}\} : α ∈ \Lambda_1\} \\
= & X - \bigcap\{\bigcup\{g(n, x) : x ∈ U_{na}\} : α ∈ \Lambda_1\}.
\end{align*}

Take a y ∈ \bigcap\{\bigcup\{g(n, x) : x ∈ U_{na}\} : α ∈ \Lambda_1\}. Then y ∈ \bigcup\{g(n, x) : x ∈ U_{na}\} for each α ∈ \Lambda_1. Then there exists an x_{na} ∈ U_{na} such that y ∈ g(n, x_{na}) for each α ∈ \Lambda_1. Then g(n, y) ∈ g(n, x_{na}) by (2). So g(n, y) ∈ \bigcap\{\bigcup\{g(n, x) : x ∈ U_{na}\} : α ∈ \Lambda_1\} is an open set. So \bigcup\{Cl\mu B_{na} : α ∈ \Lambda_1\} is closed. Then \mathcal{B}_n is a closure preserving collection.

Now we prove that \mathcal{B} = \bigcup_n \mathcal{B}_n is a base.

To do it let O be an open set with y ∈ O. Then y ∉ H = X - O and H is closed. Then there exists an U_{na} ∈ \mathcal{U}_n such that H ⊂ U_{na} and y ∉ Cl\mu(\bigcup\{g(n, x) : x ∈ U_{na}\}) for some n ∈ N by (3). So y ∈ X - Cl\mu(\bigcup\{g(n, x) : x ∈ U_{na}\}) ⊂ X - H = O.

Then y ∈ B_{na} ⊂ O. So \mathcal{B} = \bigcup_n \mathcal{B}_n is a σ-closure preserving base. Then (X, μ) is an M_1-space. □

References
