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GENERALIZED WEYL'S THEOREM FOR FUNCTIONS OF OPERATORS AND COMPACT PERTURBATIONS

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Abstract. Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space. In this paper, a necessary and sufficient condition is given for an operator T on $\mathcal{H}$ to satisfy that $f(T)$ obeys generalized Weyl's theorem for each function f analytic on some neighborhood of $\sigma(T)$. Also we investigate the stability of generalized Weyl's theorem under (small) compact perturbations.

1. Introduction

This paper is a continuation of a previous paper of the authors and Feng [22], where the stability of Weyl's theorem under analytic functional calculus is studied. This paper is also inspired by [1, 3, 4], where the stability of property (w) under some perturbations is studied. The aim of this paper is to study the stability of generalized Weyl's theorem under analytic functional calculus and (small) compact perturbations. Our results provide some concise spectral characterizations of the stability of generalized Weyl's theorem under the above transformations. To proceed, we first introduce some necessary notations and terminology.

Throughout this paper, $\mathcal{H}$ will always denote a complex separable infinite dimensional Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and by $\mathcal{K}(\mathcal{H})$ the ideal of compact operators in $\mathcal{B}(\mathcal{H})$.

Let $T \in \mathcal{B}(\mathcal{H})$. We denote by $\sigma(T)$ and $\sigma_p(T)$ the spectrum of T and the point spectrum of T respectively. Denote by $\ker T$ and $\operatorname{ran} T$ the kernel of T and the range of T respectively. T is called a *semi-Fredholm* operator, if $\operatorname{ran} T$ is closed and either $\dim\ker T$ or $\dim\ker T^*$ is finite; in this case, $\operatorname{ind} T := \dim\ker T - \dim\ker T^*$ is called the *index* of T. In particular, if $-\infty < \operatorname{ind} T <$

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∞, then T is called a *Fredholm operator*. T is called a *Weyl operator* if it is Fredholm of index 0. The *Wolf spectrum* $\sigma_{lre}(T)$, the *essential spectrum* $\sigma_e(T)$ and the *Weyl spectrum* $\sigma_w(T)$ of T are defined as

$$\sigma_{lre}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\},$$

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

respectively. $\rho_{s-F}(T) := \mathbb{C} \setminus \sigma_{lre}(T)$ is called the *semi-Fredholm domain* of T.

We denote

$$\rho^0_{s-F}(T) := \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(T - \lambda) = 0\},$$

$$\rho^+_{s-F}(T) := \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(T - \lambda) > 0\}$$

and

$$\rho^-_{s-F}(T) := \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(T - \lambda) < 0\}.$$

Let $T \in \mathcal{B}(\mathcal{H})$. If σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subset \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$. We let $E(\sigma; T)$ denote the *Riesz idempotent* of T corresponding to σ, that is,

$$E(\sigma; T) = \frac{1}{2\pi i} \int_\Gamma (\lambda - T)^{-1} \mathrm{d}\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω. In this case, we denote $\mathcal{H}(\sigma; T) = \operatorname{ran} E(\sigma; T)$. If $\lambda \in \operatorname{iso}\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$ and we simply write $\mathcal{H}(\lambda; T)$ instead of $\mathcal{H}(\{\lambda\}; T)$; if, in addition, $\dim \mathcal{H}(\lambda; T) < \infty$, then λ is called a *normal eigenvalue* of T. The set of all normal eigenvalues of T will be denoted by $\sigma_0(T)$.

For $T \in \mathcal{B}(\mathcal{H})$ and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $\operatorname{ran} T^n$ viewed as a map from $\operatorname{ran} T^n$ into $\operatorname{ran} T^n$. If for some n the range space $\operatorname{ran} T^n$ is closed and $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm operator*. In this case, from [12, Proposition 2.1], $T_{[m]}$ is Fredholm and $\operatorname{ind}(T_{[m]}) = \operatorname{ind}(T_{[n]})$ for all $m \geq n$. This enables us to define the index of a B-Fredholm operator T as the index of the Fredholm operator $T_{[n]}$, where n is any nonnegative integer such that $\operatorname{ran} T^n$ is closed and $T_{[n]}$ is Fredholm. T is called a *B-Weyl operator* if it is a B-Fredholm operator of index 0. The *B-Weyl spectrum* of T, denoted by $\sigma_{BW}(T)$, is defined as $\{\lambda \in \mathbb{C} : T - \lambda \text{ is not B-Weyl}\}$. For details, the reader is referred to [12].

Following Berkani and Koliha [11], we say that generalized Weyl's theorem holds for $T \in \mathcal{B}(\mathcal{H})$, denoted by $T \in$ (gW), if there is the equality

$$\sigma_{BW}(T) = \sigma(T) \setminus E(T),$$

where $E(T) := \sigma_p(T) \cap \operatorname{iso}\sigma(T)$ (here and in what follows, $\operatorname{iso}\sigma(T)$ denotes the set of all isolated points of $\sigma(T)$). This is a generalization of the classical Weyl's theorem.

Following Coburn [15], we say that Weyl's theorem holds for $T \in \mathcal{B}(\mathcal{H})$, denoted by $T \in (\mathrm{W})$, if there is the equality

$$\sigma_w(T) = \sigma(T) \setminus \Pi_{00}(T),$$

where $\Pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \dim\ker(T-\lambda) < \infty\}$. Operators satisfying generalized Weyl's theorem always satisfy Weyl's theorem (for details, see [11]).

The study of Weyl's theorem for bounded linear operators has a long history. In 1909, Weyl [24] proved that Weyl's theorem holds for self-adjoint operators. In 1966, Coburn [15] extended Weyl's theorem for several class operators including hyponormal operators. Since then, Weyl's theorem has been extended to various operators both on Hilbert spaces and Banach spaces. In 2003, Berkani and Koliha [11] generalized the notion of Weyl's theorem and initiated the study for generalized Weyl's theorem. Berkani [8] proved that normal operators satisfy generalized Weyl's theorem. Generalized Weyl's theorem has been extended to hyponormal operators [10]. Meanwhile many publications on Weyl's theorem and generalized Weyl's theorem have appeared (see, e.g., [2, 5, 7, 6, 13, 14, 17, 22, 25]).

Generalized Weyl's theorem has also been investigated for functions of operators. For $T \in \mathcal{B}(\mathcal{H})$ and $f \in \mathrm{Hol}(\sigma(T))$, let $f(T)$ denote the analytic functional calculus of T with respect to f. In this paper, we denote by $\mathrm{Hol}(\sigma(T))$ the set of all functions f which are analytic on some neighborhood of $\sigma(T)$(the neighborhood depends on f). The reader is referred to ([19], Chapter VII) for more results on analytic functional calculus. Cao, Guo and Meng [14] proved that if T or T^* is p-hyponormal or M-hyponormal, then $f(T) \in (\mathrm{gW})$ for all $f \in \mathrm{Hol}(\sigma(T))$. Zguitti [25] proved that if T is algebraically paranormal, then $f(T) \in (\mathrm{gW})$ for all $f \in \mathrm{Hol}(\sigma(T))$. Curto and Han [17] proved that if T is algebraically M-hyponormal, then $f(T) \in (\mathrm{gW})$ for all $f \in \mathrm{Hol}(\sigma(T))$. For more results, the reader is referred to [5, 13].

In [10], Berkani and Arroud proved that if T is hyponormal, then $f(T) \in (\mathrm{gW})$ for all $f \in \mathrm{Hol}(\sigma(T))$. In particular, they obtained the following result.

Theorem 1.1 ([10], Theorems 2.4 and 2.10)**.** *Let T be a Banach space operator and suppose that $\operatorname{iso}\sigma(T) \subset \sigma_p(T)$. Then $f(T) \in (\mathrm{gW})$ for all $f \in \mathrm{Hol}(\sigma(T))$ if and only if the following conditions hold.*

(i) $T \in (\mathrm{gW})$.
(ii) $\mathrm{ind}(T-\lambda) \cdot \mathrm{ind}(T-\mu) \geq 0$ *for all* $\lambda, \mu \notin \sigma_e(T)$.

The main result of this paper is the following theorem which extends Theorem 1.1.

Theorem 1.2 (Main Theorem)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T) \in (\mathrm{gW})$ for all $f \in \mathrm{Hol}(\sigma(T))$ if and only if the following conditions hold.*

(i) $T \in (\mathrm{gW})$.
(ii) $\mathrm{ind}(T-\lambda) \cdot \mathrm{ind}(T-\mu) \geq 0$ *for all* $\lambda, \mu \notin \sigma_e(T)$.

(iii) *If* $E(T) \neq \emptyset$*, then* $\operatorname{iso}\sigma(T) \subset \sigma_p(T)$*.*

Also, there exists a lot of work dealing with the stability of generalized Weyl's theorem under commuting finite rank perturbations and quasinilpotent perturbations (see, for example, [9, 10, 18]). In this paper, we shall investigate the stability of generalized Weyl's theorem under (small) compact perturbations. Now we are going to list our results in this aspect.

First, we obtain the following result which implies that each operator in $\mathcal{B}(\mathcal{H})$ has an arbitrarily small compact perturbation obeying generalized Weyl's theorem.

Theorem 1.3. *Given* $T \in \mathcal{B}(\mathcal{H})$ *and* $\varepsilon > 0$*, there exists* $K \in \mathcal{K}(\mathcal{H})$ *with* $\|K\| < \varepsilon$ *such that* $T + K \in (\mathrm{gW})$*.*

The following result characterizes those operators for which generalized Weyl's theorem is stable under small compact perturbations.

Theorem 1.4. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then there exists* $\delta > 0$ *such that* $T+K \in (\mathrm{gW})$ *for all* $K \in \mathcal{K}(\mathcal{H})$ *with* $\|K\| < \delta$ *if and only if the following conditions hold*;

(i) $T \in (\mathrm{gW})$*.*
(ii) $\mathbb{C} \setminus \sigma_w(T)$ *consists of finitely many connected components.*
(iii) $\operatorname{iso}\sigma_w(T) = \emptyset$*.*

The following result characterizes those operators for which generalized Weyl's theorem is stable under compact perturbations.

Theorem 1.5. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then* $T + K \in (\mathrm{gW})$ *for all* $K \in \mathcal{K}(\mathcal{H})$ *if and only if the following conditions hold*;

(i) $T \in (\mathrm{gW})$*.*
(ii) $\mathbb{C} \setminus \sigma_w(T)$ *is connected.*
(iii) $\operatorname{iso}\sigma_w(T) = \emptyset$*.*

Note that if $N \in \mathcal{B}(\mathcal{H})$ is normal, then $\sigma(N) = \sigma_w(N) \cup \sigma_0(N)$ and $\sigma_w(N) = \sigma_e(N)$. Also, we note that $\operatorname{iso}\sigma_0(N) = \sigma_0(N)$. Applying Theorems 1.4 and 1.5 to normal operators, we obtain the following corollary.

Corollary 1.6. *Let* $N \in \mathcal{B}(\mathcal{H})$ *be normal. Then*

(i) *there exists* $\delta > 0$ *such that* $N + K \in (\mathrm{gW})$ *for all* $K \in \mathcal{K}(\mathcal{H})$ *with* $\|K\| < \delta$ *if and only if* $\mathbb{C} \setminus \sigma(N)$ *consists of finitely many connected components and* $\operatorname{iso}\sigma_e(N) = \emptyset$;
(ii) $N + K \in (\mathrm{gW})$ *for all* $K \in \mathcal{K}(\mathcal{H})$ *if and only if* $\mathbb{C} \setminus \sigma(N)$ *is connected and* $\operatorname{iso}\sigma_e(N) = \emptyset$*.*

The rest part of this paper is organized as follows. In Section 2, we shall make some preparation for the proofs of main results. Section 3 is devoted to the proof of Theorem 1.2. In Section 4, we shall give the proofs of Theorems 1.3, 1.4 and 1.5.

2. Preparation

We first give some useful lemmas.

Lemma 2.1 ([23], Theorem 2.10)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that $\sigma(T) = \sigma_1 \cup \sigma_2$, where $\sigma_i (i = 1, 2)$ are clopen subsets of $\sigma(T)$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Then $\mathcal{H}(\sigma_1; T) + \mathcal{H}(\sigma_2; T) = \mathcal{H}$, $\mathcal{H}(\sigma_1; T) \cap \mathcal{H}(\sigma_2; T) = \{0\}$ and T admits the following matrix representation*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{array}{l} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_2; T), \end{array}$$

where $\sigma(T_i) = \sigma_i (i = 1, 2)$.

Using [21, Corollary 3.22] and the above lemma, we can obtain the following result whose proof is left to the reader.

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that σ is a clopen subset of $\sigma(T)$. Then*

$$T = \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \begin{array}{c} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_1; T)^{\perp} \end{array} \sim \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{array}{c} \mathcal{H}(\sigma_1; T) \\ \mathcal{H}(\sigma_1; T)^{\perp}, \end{array}$$

where $\sigma(A) = \sigma_1$ and $\sigma(B) = \sigma(T) \setminus \sigma_1$.

Therein and throughout the paper $S \sim T$ denotes that S and T are similar.

Lemma 2.3 ([16], Proposition 6.9)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $\lambda_0 \in \operatorname{iso} \sigma(T)$. Then the following statements are equivalent.*

(i) $\lambda_0 \in \sigma_0(T)$.
(ii) $\lambda_0 \in \rho^0_{s-F}(T)$.
(iii) $\lambda_0 \in \rho_{s-F}(T)$.

Let $T \in \mathcal{B}(\mathcal{H})$. For $\lambda \in \rho_{s-F}(T)$, the *minimal index* of $\lambda - T$ is defined by

$$\min \operatorname{ind}(\lambda - T) := \min\{\dim \ker(\lambda - T), \dim \ker(\lambda - T)^*\}.$$

Lemma 2.4 ([21], Corollary 1.14)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then the function $\lambda \mapsto \min \operatorname{ind}(\lambda - T)$ is constant on every component of $\rho_{s-F}(T)$ except for an at most denumerable subset $\rho^s_{s-F}(T)$ of $\rho_{s-F}(T)$ without limit points in $\rho_{s-F}(T)$. Furthermore, if $\mu \in \rho^s_{s-F}(T)$ and λ is a point of $\rho_{s-F}(T)$ in the same component as μ but $\lambda \notin \rho^s_{s-F}(T)$, then*

$$\min \operatorname{ind}(\lambda - T) < \min \operatorname{ind}(\mu - T).$$

Lemma 2.5 ([22], Lemma 2.7)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $f \in \operatorname{Hol}(\sigma(T))$. If $0 \in \sigma(f(T))$ and $\dim \ker f(T) < \infty$, then there exists $g \in \operatorname{Hol}(\sigma(T))$ such that $f(T) = g(T)$ and*

$$g(z) = (z - \lambda_1)^{k_1} (z - \lambda_2)^{k_2} \cdots (z - \lambda_n)^{k_n} g_0(z),$$

where $\lambda_i \in \sigma(T)(1 \leq i \leq n)$, $g_0 \in \operatorname{Hol}(\sigma(T))$ and $g_0(z) \neq 0$ for all $z \in \sigma(T)$.

Lemma 2.6 ([26], Corollary 2.9)**.** *Given $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with*

$$\|K\| < \varepsilon + \max\{\operatorname{dist}[\lambda, \partial\sigma_e(T)] : \lambda \in \sigma_0(T)\},$$

such that $\sigma_p(T+K) = \rho^{+}_{s-F}(T)$.

The following result is a direct consequence of Theorem 2.4 in [10].

Lemma 2.7 ([10], Theorem 2.4)**.** *Let T be a Banach space operator and $f \in \mathrm{Hol}(\sigma(T))$. If $\operatorname{ind}(T-\lambda)\cdot\operatorname{ind}(T-\mu) \geq 0$ for all $\lambda, \mu \notin \sigma_e(T)$, then $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$.*

Lemma 2.8 ([8], Lemma 4.1)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is B-Weyl if and only if there exists an idempotent P which commutes with T such that $T|_{\ker P}$ is Weyl and $T|_{\operatorname{ran} P}$ is nilpotent.*

Lemma 2.9 ([8], Theorem 4.2)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. If $\lambda \in \operatorname{iso}\sigma(T)$, then the following are equivalent.*

(i) *$\lambda \notin \sigma_{BW}(T)$.*
(ii) *There exists an idempotent P commuting with T such that $(T-\lambda)|_{\ker P}$ is invertible and $(T-\lambda)|_{\operatorname{ran} P}$ is nilpotent.*

3. Proof of Theorem 1.2

In this paper, for $\lambda \in \mathbb{C}$ and $\delta > 0$ we denote $B_\delta(\lambda) = \{z \in \mathbb{C} : |z-\lambda| < \delta\}$. We first give a useful lemma.

Lemma 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in$ (gW) if and only if the following conditions hold*;

(i) *$[\sigma(T) \setminus \sigma_w(T)] \subset \sigma_0(T)$.*
(ii) *$E(T) \subset [\sigma(T) \setminus \sigma_{BW}(T)]$.*

Proof. "$\Longrightarrow$" By definition, it is trivial to see that $T \in$ (gW) implies (ii).

Since $\sigma_{BW}(T) \subset \sigma_w(T)$, it follows from $T \in$ (gW) that

$$[\sigma(T) \setminus \sigma_w(T)] \subset [\sigma(T) \setminus \sigma_{BW}(T)] = E(T).$$

Hence it follows from Lemma 2.3 that

$$[\sigma(T) \setminus \sigma_w(T)] \subset [\rho^0_{s-F}(T) \cap E(T)] = \sigma_0(T).$$

"$\Longleftarrow$" Since (ii) holds for T, it remains to show that $[\sigma(T)\setminus\sigma_{BW}(T)] \subset E(T)$. Choose an arbitrary $\lambda \in [\sigma(T) \setminus \sigma_{BW}(T)]$. Then, by Lemma 2.8, there exists an idempotent such that

$$T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \operatorname{ran} P \\ \ker P, \end{matrix}$$

where $A - \lambda$ is nilpotent and $\operatorname{ind}(B-\lambda) = 0$. It is obvious that $\lambda \in \sigma_p(T)$. In fact, if not, then it follows that A is absent and $T - \lambda = B - \lambda$ is invertible, contradicting the fact that $\lambda \in \sigma(T)$.

Now it remains to prove that $\lambda \in \operatorname{iso}\sigma(T)$. For a proof by contradiction, we assume that $\lambda \notin \operatorname{iso}\sigma(T)$. Then there exists $\{\lambda_n\}_{n=1}^{\infty} \subset \sigma(T) \setminus \{\lambda\}$ such that $\lambda_n \to \lambda$. Since $\operatorname{ind}(B-\lambda)=0$, by the continuity of the index function, there exists $\delta > 0$ such that $\operatorname{ind}(B-\mu)=0$ for all $\mu \in B_\delta(\lambda)$. Note that $\sigma(A)=\{\lambda\}$. It follows that $\operatorname{ind}(T-\mu)=0$ for all $\mu \in B_\delta(\lambda)\setminus\{\lambda\}$. Since $\lambda_n \to \lambda$, we may assume that $\{\lambda_n\}_{n=1}^{\infty} \subset B_\delta(\lambda)$. It follows that $\{\lambda_n\}_{n=1}^{\infty} \subset [\sigma(T)\setminus\sigma_w(T)]$ and hence $\{\lambda_n\}_{n=1}^{\infty} \subset \sigma_p(T)$. Noting that $\sigma(A)=\{\lambda\}$, we obtain $\{\lambda_n\}_{n=1}^{\infty} \subset \sigma_p(B)$.

Since (i) holds, by Lemma 2.4, we deduce that $T-\mu$ is invertible for all $\mu \in B_\delta(\lambda)\setminus\{\lambda\}$ except for an at most denumerable subset. By $\sigma(A)=\{\lambda\}$, it follows that $B-\mu$ is invertible for all $\mu \in B_\delta(\lambda)\setminus\{\lambda\}$ except for an at most denumerable subset. Then, by Lemma 2.4, $\{\lambda_n\}_{n=1}^{\infty} \subset \rho_{s-F}^{s}(B)$, and hence $\lambda \in \sigma_{lre}(B)$, contradicting the fact that $\operatorname{ind}(B-\lambda)=0$. This completes the proof. □

Corollary 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If $[\sigma(T)\setminus\sigma_w(T)] \subset \sigma_0(T)$ and $E(T) \subset \sigma_0(T)$, then $T \in$ (gW).*

Lemma 3.3 ([22], Theorem 1.2)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T) \in$ (W) for all $f \in \operatorname{Hol}(\sigma(T))$ if and only if the following conditions hold*;

(i) *$T \in$ (W).*
(ii) *$\operatorname{ind}(T-\lambda)\cdot\operatorname{ind}(T-\mu) \geq 0$ for all $\lambda, \mu \notin \sigma_e(T)$.*
(iii) *If $\sigma_0(T) \neq \emptyset$, then $\operatorname{iso}\sigma(T) \subset \sigma_p(T)$.*

Now, we are going to give the proof of Theorem 1.2.

Proof of Theorem 1.2. "$\Longrightarrow$" Assume that $f(T) \in$ (gW) for all $f \in \operatorname{Hol}(\sigma(T))$.

(i) Set $f_1(\lambda)=\lambda$. Then, evidently, $T=f_1(T) \in$ (gW).

(ii) If (ii) does not hold, then, by Lemma 3.3, there exists a polynomial f_2 such that $f_2(T) \notin$ (W) and, moreover, $f_2(T) \notin$ (gW), a contradiction.

(iii) For a proof by contradiction, we assume that (iii) does not hold. Then we can choose $\lambda \in E(T)$ and $\mu \in \operatorname{iso}\sigma(T)$ satisfying $\mu \notin \sigma_p(T)$. Set $g(z) = (z-\lambda)(z-\mu)$. It is easy to see that $0 \in E(g(T))$. Since $g(T) \in$ (gW), there exists $n_0 \in \mathbb{N}$ such that $\operatorname{ind}(g(T)_{[n]})=0$ for $n \geq n_0$. Without loss of generality, assume that (i) holds for T, hence $\lambda \notin \sigma_{BW}(T)$. By Lemma 2.9, there exists an idempotent P commuting with T such that $T-\lambda$ admits the following representation

$$T-\lambda = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{matrix} \operatorname{ran} P \\ \ker P, \end{matrix}$$

where A is nilpotent and B is invertible. Noting that

$$g(T) = \begin{bmatrix} A(A+\lambda-\mu) & 0 \\ 0 & B(B+\lambda-\mu) \end{bmatrix} \begin{matrix} \operatorname{ran} P \\ \ker P, \end{matrix}$$

there exists some large enough $k \in \mathbb{N}$ such that $A^k=0$ and $\operatorname{ind}(g(T)_{[k]})=0$. Noting that

$$g(T)^k = \begin{bmatrix} 0 & 0 \\ 0 & B^k(B+\lambda-\mu)^k \end{bmatrix} \begin{matrix} \operatorname{ran} P \\ \ker P, \end{matrix}$$

it follows that $\operatorname{ran} g(T)^k = \operatorname{ran} B^k(B+\lambda-\mu)^k$ and hence $g(T)_{[k]} = [B(B+\lambda-\mu)]_{[k]}$. Thus $\operatorname{ind}([B(B+\lambda-\mu)]_{[k]}) = 0$. By Lemma 2.8, there exists an idempotent Q on $\ker P$ commuting with $B(B+\lambda-\mu)$ such that $B(B+\lambda-\mu)$ can be represented as

$$B(B+\lambda-\mu) = \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix} \begin{matrix} \operatorname{ran} Q \\ \ker Q, \end{matrix}$$

where E is nilpotent and $\operatorname{ind} F = 0$. Note that $\mu \notin \sigma_p(T)$ and hence $B(B+\lambda-\mu)$ is injective. It follows that E is absent and $B(B+\lambda-\mu) = F$ is invertible. Hence $B+\lambda-\mu$ is invertible.

It is easy to see that

$$T-\mu = \begin{bmatrix} A+\lambda-\mu & 0 \\ 0 & B+\lambda-\mu \end{bmatrix} \begin{matrix} \operatorname{ran} P \\ \ker P. \end{matrix}$$

Because $\lambda \neq \mu$, $A+\lambda-\mu$ is invertible. Hence $T-\mu$ is invertible, a contradiction.

"$\Longleftarrow$" Choose an arbitrary $f \in \operatorname{Hol}(\sigma(T))$. It suffices to prove that $f(T) \in$ (gW).

Step 1. $[\sigma(f(T)) \setminus \sigma_w(f(T))] \subset \sigma_0(f(T))$.

If $\lambda \in [\sigma(f(T)) \setminus \sigma_w(f(T))]$, then $\operatorname{ind}(f(T)-\lambda) = 0$. Now we are going to show that $\lambda \in \sigma_0(f(T))$. It is easy to see that $0 < \dim\ker(f(T)-\lambda) < \infty$. Then, by Lemma 2.5, we may directly assume that $\{\lambda_i\}_{i=1}^n$ is an enumeration of $\{z \in \sigma(T) : f(z)-\lambda = 0\}$ and

$$f(z)-\lambda = (z-\lambda_1)^{k_1}\cdots(z-\lambda_n)^{k_n} g(z),$$

where $g(z) \neq 0$ for all $z \in \sigma(T)$. Then

$$f(T)-\lambda = (T-\lambda_1)^{k_1}\cdots(T-\lambda_n)^{k_n} g(T),$$

where $g(T)$ is invertible.

Since $\lambda \notin \sigma_w(f(T))$, we have $\lambda_i \notin \sigma_e(T)$ and $\sum_{i=1}^n k_i \cdot \operatorname{ind}(T-\lambda_i) = 0$. It follows from (ii) that $\operatorname{ind}(T-\lambda_i) = 0$ for $1 \le i \le n$. Since (i) holds for T, using Lemma 3.1, we obtain that $\lambda_i \in \sigma_0(T)$. Now it is easy to check that $\lambda \in \sigma_0(f(T))$.

Step 2. $E(f(T)) \subset [\sigma(f(T)) \setminus \sigma_{BW}(f(T))]$.

Let $\lambda \in E(f(T))$ be fixed. We first assume that $f(\cdot)$ is not constant on any connected component of its domain and

$$f(T)-\lambda = (T-\lambda_1)^{k_1}\cdots(T-\lambda_n)^{k_n} g(T),$$

where $\lambda_i \in \sigma(T)$ for $1 \le i \le n$ and $g(T)$ is invertible. From $\lambda \in E(f(T))$, we have $\lambda_i \in \operatorname{iso}\sigma(T)$ for all i and there exists an i_0 such that $\lambda_{i_0} \in \sigma_p(T)$, hence $\lambda_{i_0} \in E(T)$. By (iii), we can deduce that $\lambda_i \in \sigma_p(T)$ for all i and hence $\lambda_i \in E(T)$ for all i. As (i) holds for T, consequently we obtain $\lambda_i \notin \sigma_{BW}(T)$ for all i. Note that (ii) holds for T. By Lemma 2.7, we have $\lambda \notin \sigma_{BW}(f(T))$.

If $f(z) \equiv \lambda$ on some clopen subset σ_1 of $\sigma(T)$, then, by Lemma 2.1, $f(T)-\lambda$ is zero on $\mathcal{H}(\sigma_1;T)$. In this case, one can deal with $T|_{\mathcal{H}(\sigma_1;T)}$ and $T|_{\mathcal{H}(\sigma(T)\setminus\sigma_1;T)}$

respectively. From Lemma 2.9, it is easy to see that the above argument also applies. □

If one checks the proof of Theorem 1.2, then one can easily obtain the following result.

Corollary 3.4. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then* $f(T) \in (\mathrm{gW})$ *for all* $f \in \mathrm{Hol}(\sigma(T))$ *if and only if* $p(T) \in (\mathrm{gW})$ *for each polynomial* $p(\lambda)$*.*

Remark 3.5. Since Lemmas 2.8 and 2.9 also hold for Banach space operators, one can easily check that the result of Theorem 1.2 can be extended to Banach space operators.

4. Proofs of Theorems 1.3, 1.4 and 1.5

We first give the proof of Theorem 1.3.

Proof of Theorem 1.3. For given $\varepsilon > 0$, set $\sigma_1 = \{\lambda \in \sigma_0(T) : \mathrm{dist}[\lambda, \partial\sigma_e(T)] \geq \varepsilon\}$. Then σ_1 is a finite, clopen subset of $\sigma(T)$. Set $\sigma_2 = \sigma(T) \setminus \sigma_1$. By Corollary 2.2, T admits the following representation

$$T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1;T) \\ \mathcal{H}(\sigma_1;T)^{\perp}, \end{matrix}$$

where $\sigma(T_i) = \sigma_i (i = 1, 2)$. Then one can verify that

$$\max\{\mathrm{dist}[\lambda, \partial\sigma_e(T_2)] : \lambda \in \sigma_0(T_2)\} < \varepsilon.$$

By Lemma 2.6, there exists a compact operator $\overline{K}$ on $\mathcal{H}(\sigma_1;T)^{\perp}$ with $\|\overline{K}\| < \varepsilon$ such that $\sigma_p(T_2 + \overline{K}) = \rho^{+}_{s-F}(T_2)$. Denote $\overline{T_2} = T_2 + \overline{K}$ and set

$$K = \begin{bmatrix} 0 & 0 \\ 0 & \overline{K} \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1;T) \\ \mathcal{H}(\sigma_1;T)^{\perp}. \end{matrix}$$

Then $K \in \mathcal{K}(\mathcal{H})$, $\|K\| < \varepsilon$ and

$$T + K = \begin{bmatrix} T_1 & * \\ 0 & \overline{T_2} \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma_1;T) \\ \mathcal{H}(\sigma_1;T)^{\perp}. \end{matrix}$$

Now it remains to show that $T + K \in (\mathrm{gW})$.

$\sigma_p(\overline{T_2}) = \rho^{+}_{s-F}(\overline{T_2})$ implies that $\sigma(\overline{T_2}) = \sigma_w(\overline{T_2})$ and hence $\sigma(\overline{T_2}) \cap \sigma(T_1) = \emptyset$. Since $\sigma_p(\overline{T_2}) = \rho^{+}_{s-F}(\overline{T_2}) = \rho^{+}_{s-F}(T + K)$ and $\dim \mathcal{H}(\sigma_1;T) < \infty$, we can deduce that $\sigma_0(T + K) = \sigma_1 = \sigma(T_1)$ and $\sigma_p(T + K) = \rho^{+}_{s-F}(T + K) \cup \sigma_0(T + K)$. It follows that $E(T + K) = \sigma_0(T + K)$. On the other hand, if $\lambda \in [\sigma(T + K) \setminus \sigma_w(T + K)]$, then it is easy to see that $\lambda \in \sigma_p(T + K)$ and hence $\lambda \in [\sigma_p(T+K) \setminus \rho^{+}_{s-F}(T+K)] = \sigma_0(T+K)$. By Corollary 3.2, we obtain $T + K \in (\mathrm{gW})$. □

Given a subset σ of $\mathbb{C}$, we denote $\sigma^c = \mathbb{C} \setminus \sigma$.

Lemma 4.1 ([20], Theorem 3.1)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and let Φ be the union of a collection of bounded components of $(\sigma(T) \setminus \sigma_0(T))^c$. Then there exists $K \in \mathcal{K}(\mathcal{H})$ such that $\sigma(T+K) = \sigma(T) \cup \Phi$.*

Lemma 4.2 ([22], Proposition 4.10)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. If $\sigma(T) = \sigma_w(T) \cup \sigma_0(T)$ and $\mathbb{C} \setminus \sigma_w(T)$ consists of at most finitely many connected components, then there exists $\delta > 0$ such that $\sigma(T+K) = \sigma_w(T+K) \cup \sigma_0(T+K)$ for all $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \delta$.*

Lemma 4.3 ([22], Theorem 1.4)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then there exists $\delta > 0$ such that $T + K \in$ (W) for all $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \delta$ if and only if the following conditions hold*;

(i) $T \in$ (W).
(ii) $\mathbb{C} \setminus \sigma_w(T)$ *consists of finitely many connected components.*
(iii) $\mathrm{iso}[\sigma_w(T)] = \emptyset$.

Now we are going to give the proofs of Theorems 1.4 and 1.5.

Proof of Theorem 1.4. "$\Longrightarrow$" Since there exists $\delta > 0$ such that $T + K \in$ (gW) for all $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \delta$, it follows that $T \in$ (gW). If (ii) or (iii) does not hold, then, for arbitrarily given $\varepsilon > 0$, by Lemma 4.3, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T + K \notin$ (W), hence $T + K \notin$ (gW).

"$\Longleftarrow$" Assume that (i), (ii), and (iii) hold for T. By Lemma 3.1, it follows from $T \in$ (gW) that $\sigma(T) = \sigma_w(T) \cup \sigma_0(T)$. Furthermore, since (ii) holds, by Lemma 4.2, there exists $\delta > 0$ such that $\sigma(T + K) = \sigma_w(T + K) \cup \sigma_0(T + K)$ for all $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \delta$. To complete the proof for the sufficiency, by Lemma 3.1, we need only prove that $E(T + K) \subset [\sigma(T + K) \setminus \sigma_{BW}(T + K)]$ for all $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \delta$.

In fact, if not, there exist $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \delta$ and $\lambda_0 \in E(T + K) \cap \sigma_{BW}(T + K)$. Since $\sigma_{BW}(T + K) \subset \sigma_w(T + K)$, we have $\lambda_0 \in E(T + K) \cap \sigma_w(T + K)$ and hence $\lambda_0 \in \mathrm{iso}\,\sigma_w(T + K) = \mathrm{iso}\,\sigma_w(T) = \emptyset$, a contradiction. Thus we have proved that $T + K \in$ (gW) for all $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \delta$. $\square$

Proof of Theorem 1.5. "$\Longrightarrow$" By Theorem 1.4, if (i) or (iii) does not hold, for arbitrarily given $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \varepsilon$ such that $T+K \notin$ (gW). If $\mathbb{C}\setminus\sigma_w(T)$ is not connected, denote by Ω a bounded component of $\mathbb{C} \setminus \sigma_w(T)$. Without loss of generality, assume that (i) holds for T. Then by Lemma 3.1, $\sigma(T) = \sigma_w(T) \cup \sigma_0(T)$ and hence Ω is a bounded component of $[\sigma(T) \setminus \sigma_0(T)]^c$. By Lemma 4.1, there exists $K \in \mathcal{K}(\mathcal{H})$ such that $\sigma(T + K) = \sigma(T)\cup\Omega$. Obviously, $\Omega \subset [\sigma(T+K)\setminus\sigma_w(T+K)]$ and hence $\sigma(T+K)\setminus\sigma_w(T+K)$ is not a subset of $\sigma_0(T + K)$. By Lemma 3.1, we conclude that $T + K \notin$ (gW).

"$\Longleftarrow$" Choose an arbitrary compact operator K on $\mathcal{H}$, we shall prove that $T + K \in$ (gW). Note that $\mathbb{C} \setminus \sigma_w(T + K) = \mathbb{C} \setminus \sigma_w(T)$ is connected, which contains $\sigma(T + K)^c$. Using Lemma 2.4, we have $\min \mathrm{ind}(T + K - \lambda) = 0$ on $\mathbb{C} \setminus \sigma_w(T+K)$ except an at most denumerable set $\sigma_0(T+K)$. Hence $T+K-\lambda$

is invertible for all $\lambda \in [(\sigma_w(T+K) \cup \sigma_0(T+K))]^c$. Now we conclude that $\sigma(T+K) = \sigma_w(T+K) \cup \sigma_0(T+K)$.

On the other hand, if $\lambda \in E(T+K)$, then, using a similar argument as in proof for the sufficiency of Theorem 1.4, one can prove that $\lambda \in [\sigma(T+K) \setminus \sigma_{BW}(T+K)]$. This completes the proof. $\square$

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References

[1] P. Aiena, *Property* (w) *and perturbations. II*, J. Math. Anal. Appl. **342** (2008), no. 2, 830–837.

[2] P. Aiena and M. Berkani, *Generalized Weyl's theorem and quasi-affinity*, Studia Math. **198** (2010), no. 2, 105–120.

[3] P. Aiena and M. T. Biondi, *Property* (w) *and perturbations*, J. Math. Anal. Appl. **336** (2007), no. 1, 683–692.

[4] P. Aiena, M. T. Biondi, and F. Villafañe, *Property* (w) *and perturbations. III*, J. Math. Anal. Appl. **353** (2009), no. 1, 205–214.

[5] M. Amouch, *Weyl type theorems for operators satisfying the single-valued extension property*, J. Math. Anal. Appl. **326** (2007), no. 2, 1476–1484.

[6] I. J. An and Y. M. Han, *Weyl's theorem for algebraically quasi-class A operators*, Integral Equations Operator Theory **62** (2008), no. 1, 1–10.

[7] S. K. Berberian, *An extension of Weyl's theorem to a class of not necessarily normal operators*, Michigan Math. J. **16** (1969), 273–279.

[8] M. Berkani, *Index of B-Fredholm operators and generalization of a Weyl theorem*, Proc. Amer. Math. Soc. **130** (2002), no. 6, 1717–1723 (electronic).

[9] ———, *On the equivalence of Weyl theorem and generalized Weyl theorem*, Acta Math. Sin. (Engl. Ser.) **23** (2007), no. 1, 103–110.

[10] M. Berkani and A. Arroud, *Generalized Weyl's theorem and hyponormal operators*, J. Aust. Math. Soc. **76** (2004), no. 2, 291–302.

[11] M. Berkani and J. J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69** (2003), no. 1-2, 359–376.

[12] M. Berkani and M. Sarih, *On semi B-Fredholm operators*, Glasg. Math. J. **43** (2001), no. 3, 457–465.

[13] X. H. Cao, *Topological uniform descent and Weyl type theorem*, Linear Algebra Appl. **420** (2007), no. 1, 175–182.

[14] X. H. Cao, M. Z. Guo, and B. Meng, *Weyl type theorems for p-hyponormal and M-hyponormal operators*, Studia Math. **163** (2004), no. 2, 177–188.

[15] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.

[16] J. B. Conway, *A course in Functional Analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.

[17] R. E. Curto and Y. M. Han, *Generalized Browder's and Weyl's theorems for Banach space operators*, J. Math. Anal. Appl. **336** (2007), no. 2, 1424–1442.

[18] B. P. Duggal, *Hereditarily polaroid operators, SVEP and Weyl's theorem*, J. Math. Anal. Appl. **340** (2008), no. 1, 366–373.

[19] N. Dunford and J. T. Schwartz, *Linear Operators. Part I*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1988.

[20] D. A. Herrero, *Economical compact perturbations. II. Filling in the holes*, J. Operator Theory **19** (1988), no. 1, 25–42.

[21] ______, *Approximation of Hilbert Space Operators. Vol. 1*, second ed., Pitman Research Notes in Mathematics Series, vol. 224, Longman Scientific & Technical, Harlow, 1989.

[22] C. G. Li, S. Zhu, and Y. L. Feng, *Weyl's theorem for functions of operators and approximation*, Integral Equations Operator Theory **67** (2010), no. 4, 481–497.

[23] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, second ed., Dover Publications Inc., Mineola, NY, 2003.

[24] H. Weyl, *Uber beschrankte quadratische formen, deren differenz, vollsteig ist*, Rend. Circ. Mat. Palermo **27** (1909), 373–392.

[25] H. Zguitti, *A note on generalized Weyl's theorem*, J. Math. Anal. Appl. **316** (2006), no. 1, 373–381.

[26] S. Zhu and C. G. Li, *SVEP and compact perturbations*, J. Math. Anal. Appl. **380** (2011), no. 1, 69–75.

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