TETRAVALENT SYMMETRIC GRAPHS OF ORDER $9p$

SONG-TAO GUO AND YAN-QUAN FENG

Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify tetravalent symmetric graphs of order $9p$ for each prime $p$.

1. Introduction

Let $G$ be a permutation group on a set $\Omega$ and $\alpha \in \Omega$. Denote by $G_\alpha$ the stabilizer of $\alpha$ in $G$, that is, the subgroup of $G$ fixing the point $\alpha$. We say that $G$ is semiregular on $\Omega$ if $G_\alpha = 1$ for every $\alpha \in \Omega$ and regular if $G$ is transitive and semiregular. Throughout this paper, we consider undirected finite connected graphs without loops or multiple edges. For a graph $X$ we use $V(X)$, $E(X)$ and $\text{Aut}(X)$ to denote its vertex set, edge set, and automorphism group, respectively. For $u, v \in V(X)$, denote by $\{u, v\}$ the edge incident to $u$ and $v$ in $X$.

A graph $X$ is said to be vertex-transitive if $\text{Aut}(X)$ acts transitively on $V(X)$. An $s$-arc in a graph is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of the graph $X$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$. In particular, a 1-arc is called an arc for short and a 0-arc is a vertex. For a subgroup $G \leq \text{Aut}(X)$, a graph $X$ is said to be $(G, s)$-arc-transitive and $(G, s)$-regular if $G$ is transitive and regular on the set of $s$-arcs in $X$, respectively. A $(G, s)$-arc-transitive graph is said to be $(G, s)$-transitive if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is simply called $G$-symmetric. A graph $X$ is simply called $s$-arc-transitive, $s$-regular and $s$-transitive if it is $(\text{Aut}(X), s)$-arc-transitive, $(\text{Aut}(X), s)$-regular and $(\text{Aut}(X), s)$-transitive, respectively.

Arc-transitive or $s$-transitive graphs have received considerable attention in the literature. For example, $s$-transitive graphs of order $np$ was classified in [3, 4, 23] depending on $n=1, 2$ or 3, where $p$ is a prime. Li [13] showed that there exists an $s$-transitive graph of odd order if and only if $s \leq 3$. For the case of valency 4, Gardiner and Praeger [8, 9] characterized tetravalent

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symmetric graphs, and Li et al. [14] classified vertex-primitive tetravalent \( s \)-transitive graphs. The classification of tetravalent \( s \)-transitive Cayley graphs on abelian groups was given by Xu and Xu [25]. We may deduce a classification of tetravalent 1-regular Cayley graphs on dihedral groups from [12, 18, 21, 22]. Zhou [31] gave a classification of tetravalent 1-regular graphs of order \( 2pq \) for \( p, q \) primes. Recently, Zhou [29] classified tetravalent \( s \)-transitive graphs of order \( 4p \), and Zhou and Feng [30] classified tetravalent \( s \)-transitive graphs of order \( 2p^2 \). In this paper we classify tetravalent \( s \)-transitive graphs of order \( 9p \).

Throughout the paper we denote by \( C_n \) and \( K_n \) the cycle and the complete graph of order \( n \), respectively. Denote by \( Z_n \) the cyclic group of order \( n \), by \( Z_n^\ast \) the multiplicative group of \( Z_n \) consisting of numbers coprime to \( n \), by \( D_{2n} \) the dihedral group of order \( 2n \), and by \( F_n \) the Frobenius group of order \( n \).

2. Preliminary results

For a subgroup \( H \) of a group \( G \), denote by \( C_G(H) \) the centralizer of \( H \) in \( G \) and by \( N_G(H) \) the normalizer of \( H \) in \( G \).

**Proposition 2.1** ([11, Chapter I, Theorem 4.5]). The quotient group \( N_G(H)/C_G(H) \) is isomorphic to a subgroup of the automorphism group \( \text{Aut}(H) \) of \( H \).

The following proposition is due to Burnside.

**Proposition 2.2** ([19, Theorem 8.5.3]). Let \( p \) and \( q \) be primes, and let \( m \) and \( n \) be non-negative integers. Then every group of order \( p^m q^n \) is solvable.

Let \( G \) be a permutation group on a set \( \Omega \). The size of \( \Omega \) is called the degree of \( G \) acting on \( \Omega \).

**Proposition 2.3** ([6, Corollary 3.5B]). Every transitive permutation group of prime degree \( p \) is either 2-transitive or solvable with a regular normal Sylow \( p \)-subgroup.

The following proposition is about the permutation group of degree \( p^2 \) for \( p \) a prime.

**Proposition 2.4** ([28, Proposition 1]). Any transitive group of degree \( p^2 \) has a regular subgroup.

For a finite group \( G \) and a subset \( S \) of \( G \) such that \( 1 \not\in S \) and \( S = S^{-1} \), the Cayley graph \( \text{Cay}(G, S) \) on \( G \) with respect to \( S \) is defined to have vertex set \( V(\text{Cay}(G, S)) = G \) and edge set \( E(\text{Cay}(G, S)) = \{ \{g, sg\} \mid g \in G, s \in S \} \). Clearly, a Cayley graph \( \text{Cay}(G, S) \) is connected if and only if \( S \) generates \( G \). Furthermore, \( \text{Aut}(G, S) = \{ \alpha \in \text{Aut}(G) \mid S^\alpha = S \} \) is a subgroup of the automorphism group \( \text{Aut}(\text{Cay}(G, S)) \). Given a \( g \in G \), define the permutation \( R(g) \) on \( G \) by \( x \mapsto xg, x \in G \). Then \( R(G) = \{ R(g) \mid g \in G \} \), called the right regular representation of \( G \), is a permutation group isomorphic to \( G \). The
Cayley graph is vertex-transitive because it admits the right regular representation \( R(G) \) of \( G \) as a regular group of automorphisms of \( \text{Cay}(G, S) \). A Cayley graph \( \text{Cay}(G, S) \) is said to be normal if \( R(G) \) is normal in \( \text{Aut}(\text{Cay}(G, S)) \).

A graph \( X \) is isomorphic to a Cayley graph on \( G \) if and only if \( \text{Aut}(X) \) has a subgroup isomorphic to \( G \), acting regularly on vertices (see [20]). For two subsets \( S \) and \( T \) of \( G \) not containing the identity 1, if there is an \( \alpha \in \text{Aut}(G) \) such that \( S\alpha = T \), then \( S \) and \( T \) are said to be equivalent, denoted by \( S \equiv T \).

We may easily show that if \( S \equiv T \), then \( \text{Cay}(G, S) \cong \text{Cay}(G, T) \) and \( \text{Cay}(G, S) \) is normal if and only if \( \text{Cay}(G, T) \) is normal.

**Proposition 2.5** ([26, Proposition 1.5]). A Cayley graph \( \text{Cay}(G, S) \) is normal if and only if \( \text{Aut}(\text{Cay}(G, S)) \) is isomorphic to \( G \), acting regularly on vertices (see [20]).

From [1, Corollary 1.3], we have the following proposition.

**Proposition 2.6.** Let \( X = \text{Cay}(G, S) \) be a connected tetravalent Cayley graph on a finite abelian group \( G \) of odd order. Then \( X \) is normal except for \( G = \mathbb{Z}_5 \) and \( X = \mathbb{K}_5 \).

For two subgroups \( M \) and \( N \) of a group \( G \), \( M \rtimes N \) stands for the semidirect product of \( M \) by \( N \). The next proposition characterizes the vertex stabilizers of connected tetravalent \( s \)-transitive graphs (see [14, Lemma 2.5] and [13, Theorem 1.1]).

**Proposition 2.7.** Let \( X \) be a connected tetravalent \((G, s)\)-transitive graph of odd order. Then \( s \leq 3 \) and the stabilizer \( G_v \) of a vertex \( v \in V(X) \) in \( G \) is as follows:

1. \( G_v \) is a 2-group for \( s = 1 \);
2. \( G_v \cong A_4 \) or \( S_4 \) for \( s = 2 \);
3. \( G_v \cong \mathbb{Z}_3 \times A_4 \), \( \mathbb{Z}_4 \times S_4 \), or \( S_3 \times S_4 \) for \( s = 3 \).

To introduce tetravalent symmetric graphs of order \( 3p \) for \( p \) a prime, we define some graphs. Let \( p > 3 \) be a prime and let \( \mathbb{Z}_{3p} = \mathbb{Z}_3 \times \mathbb{Z}_p = \langle a \rangle \times \langle b \rangle \) be the cyclic group of order \( 3p \). Define \( \mathcal{C}A_{3p} = \text{Cay}(\mathbb{Z}_{3p}, \{ab, a^{-1}b, ab^{-1}, a^{-1}b^{-1}\}) \).

By the definition of \( G(3p, 2) \) given in [23, Example 3.4], it is easy to see that \( \mathcal{C}A_{3p} \cong G(3p, 2) \) and \( \text{Aut}(\mathcal{C}A_{3p}) = \mathbb{Z}_{3p} \rtimes \mathbb{Z}_2^2 \). The next proposition is about the classification of connected tetravalent symmetric graphs of order \( 3p \) (see [23, Theorem]).

**Proposition 2.8.** Let \( p > 7 \) be a prime and \( X \) a connected tetravalent symmetric graph of order \( 3p \). Then \( X \cong \mathcal{C}A_{3p} \).

### 3. Graph constructions and isomorphisms

In this section we introduce connected tetravalent symmetric graphs of order \( 9p \) for \( p \) a prime. The first example is the lexicographic product of \( C_9 \) and \( 2K_1 \).
Example 3.1. The lexicographic product $C_9[2K_1]$ is defined as the graph with vertex set $V(C_9) \times V(2K_1)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(C_9[2K_1])$, $u$ is adjacent to $v$ in $C_9[2K_1]$ if and only if $\{x_1, x_2\} \in E(C_9)$. Then $C_9[2K_1]$ is a connected tetravalent 1-transitive Cayley graph on the group $Z_9 \times Z_2$ and $\text{Aut}(C_9[2K_1]) = Z_2^3 \rtimes D_{18}$.

From [25, Example 3.2], we have the following example.

Example 3.2. Let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong Z_3 \times Z_3 \times Z_2$. The Cayley graph $G_{18} = \text{Cay}(G, \{ca, ca^{-1}, cb, cb^{-1}\})$ is 1-transitive and $\text{Aut}(G_{18}) = G \rtimes D_8$.

Xu and Xu [25] gave a classification of tetravalent arc-transitive Cayley graphs on finite abelian groups. The following example is extracted from [25, Example 3.2 and Theorem 3.5].

Example 3.3. Let $p \geq 3$ be a prime and $G = \langle a \rangle \times \langle b \rangle \cong Z_3 \times Z_{3p}$. Then the Cayley graph $\mathcal{C}A^1_{(3,3p)} = \text{Cay}(G, \{b, b^{-1}, ab, a^{-1}b^{-1}\})$ is 1-regular and $\text{Aut}(\mathcal{C}A^1_{(3,3p)}) = G \rtimes Z_2^2$.

Furthermore, if $p \equiv 3 \pmod{4}$, then there is only one connected tetravalent symmetric Cayley graph on the group $G$, that is, $\mathcal{C}A^1_{(3,3p)}$, and if $p \equiv 1 \pmod{4}$ there are exactly two connected tetravalent symmetric Cayley graphs on the group $G$, that is, $\mathcal{C}A^1_{(3,3p)}$ and $\mathcal{C}A^2_{(3,3p)}$, where $\mathcal{C}A^2_{(3,3p)} = \text{Cay}(G, \{b, b^{-1}, ab^w, a^{-1}b^{-w}\})$ and $\text{Aut}(\mathcal{C}A^2_{(3,3p)}) = G \rtimes Z_4$ with $w$ an element of order 4 in $Z_{3p}^*$.

By [27, Theorems 1 and 3], there is only one connected tetravalent symmetric Cayley graph on the cyclic group of order $9p$ for each prime $p \geq 5$.

Example 3.4. Let $p \geq 5$ be a prime and $G = \langle a \rangle \times \langle b \rangle \cong Z_3 \times Z_p$. The unique connected tetravalent symmetric Cayley graph on $G$ is $\mathcal{C}A_{9p} = \text{Cay}(G, \{ab, a^{-1}b^{-1}, a^{-1}b, ab^{-1}\})$, which is 1-regular and its automorphism group $\text{Aut}(\mathcal{C}A_{9p}) = G \rtimes Z_2^2$.

Let $X = \text{Cay}(H, T)$ be a connected tetravalent symmetric Cayley graph on a non-abelian group $H$ of order 27. Then $(T) = H, T^{-1} = T$ and $|T| = 4$. By [7, Corollary 3.2], $X$ is normal, and hence $\text{Aut}(X)_1 = \text{Aut}(H, T)$ by Proposition 2.5. Since $|H| = 27$, we may assume that $T = \{x, x^{-1}, y, y^{-1}\}$. Thus, $\text{Aut}(H, T)$ is a 2-group and faithful on $T$, forcing that $\text{Aut}(H, T) \leq D_8$.

Since $X$ is symmetric, $4 \not| |\text{Aut}(H, T)|$. By the elementary group theory, there are two non-abelian groups of order 27:

$G_1(27) = \langle a, b | a^9 = b^3 = 1, b^{-1}ab = a^4 \rangle$;

$G_2(27) = \langle a, b, c | a^3 = b^3 = c^3 = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$.

If $H = G_1(27)$, then $4 \not| |\text{Aut}(H)|$ because each automorphism $\alpha \in \text{Aut}(H)$ has the following form:

$$\alpha : \begin{cases} 
a \mapsto a^j b^i, & i, 9 \leq j \leq 2; 
b \mapsto a^{2k} b, & 0 \leq k \leq 2. 
\end{cases}$$
This is impossible because $4 \mid \vert \text{Aut}(H,T)\vert$. Thus, $H = G_2(27)$ and $o(x) = o(y) = 3$, where $o(x)$ denotes the order of $x$ in $G_2(27)$. Since $(x, y) = H$ and $[x, y] \in Z(H) = \langle c \rangle$, $a$, $b$ and $c$ have the same relations as do $x$, $y$ and $[x, y]$, which implies that the map $a \mapsto x$, $b \mapsto y$, $c \mapsto [x, y]$ induces an automorphism of $G_2(27)$. It follows that $X \cong \text{Cay}(G_2(27), S)$, where $S = \{a, a^{-1}, b, b^{-1}\}$.

Clearly, the maps $a \mapsto b$, $b \mapsto a$, $c \mapsto c$ and $a \mapsto b$, $b \mapsto a^{-1}$, $c \mapsto c$ induce automorphisms of $G_2(27)$, say $\alpha_1$ and $\alpha_2$, respectively. Then $\alpha_1, \alpha_2 \in \text{Aut}(G_2(27), S)$ and $\langle \alpha_1, \alpha_2 \rangle \cong D_8$, forcing that $X$ is symmetric. On the other hand, since $\text{Aut}(G_2(27), S) \leq D_8$, one has that $\text{Aut}(G_2(27), S) = D_8$ and $\text{Aut}(X) = G_2(27) \rtimes D_8$. Thus, we have the following example.

**Example 3.5.** Let $G = G_2(27) = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$ and $S = \{a, a^{-1}, b, b^{-1}\}$. Define

$$G_{27} = \text{Cay}(G, S).$$

Then $\text{Aut}(G_{27}) = G \rtimes D_8$ and $G_{27}$ is the only connected tetravalent symmetric Cayley graph on non-abelian group of order 27.

Let $X$ be a symmetric graph, and $A$ an arc-transitive subgroup of $\text{Aut}(X)$. Let $\{u, v\}$ be an edge of $X$. Assume that $H = A_u$ is the stabilizer of $u \in V(X)$ and that $g \in A$ interchanges $u$ and $v$. It is easy to see that the core $H_A$ of $H$ in $A$ (the largest normal subgroup of $A$ contained in $H$) is trivial, and that $HgH$ consists of all elements of $A$ which maps $u$ to one of its neighbors in $X$. By [16, 20], the graph $X$ is isomorphic to the coset graph $\text{Cos}(A, H, HgH)$, which is defined as the graph with vertex set $\{Ha \mid a \in A\}$, the set of right cosets of $H$ in $A$, and edge set $\{\{Ha, Hda\} \mid a \in A, d \in HgH\}$. The valency of $\text{Cos}(A, H, HgH)$ is $|HgH|/|H| = |H : H \cap H^g|$, and $\text{Cos}(A, H, HgH)$ is connected if and only if $HgH$ generates $A$. By right multiplication, every element in $A$ induces an automorphism of $\text{Cos}(A, H, HgH)$. Since $H_A = 1$, the induced action of $A$ on $V(\text{Cos}(A, H, HgH))$ is faithful, and hence we may view $A$ as a group of automorphisms of $\text{Cos}(A, H, HgH)$.

From [14], one can see that, up to isomorphism, there is only one primitive tetravalent symmetric graph of order $n$ if $n = 45$ or 153.

**Example 3.6.** Let $G = \text{Aut}(A_6) \cong S_6 \rtimes Z_2$ and let $P$ be a Sylow 2-subgroup of $G$. By [5], $P$ is a maximal subgroup of $G$ and hence $N_G(P) = P$. Let $H$ be an elementary abelian 2-subgroup of $P$ of order 8. Then $N_G(H) \cong S_4 \rtimes Z_2$. Let $d$ be an involution in $N_G(H) \setminus P$. Define

$$G_{45} = \text{Cos}(G, P, PdP).$$

Then $G_{45}$ is a connected tetravalent 1-transitive graph and $\text{Aut}(G_{45}) \cong \text{Aut}(A_6)$.

**Example 3.7.** Let $G = \text{PSL}(2, 17)$ and let $P = \langle a, b \mid a^8 = b^2 = 1, bab = a^{-1} \rangle \cong D_{16}$ be a Sylow 2-subgroup of $G$. By [5], $P$ is a maximal subgroup of $G$ and hence $N_G(P) = P$. Let $H = \langle a^4, b \rangle$. Then $N_G(H) \cong S_4$. Let $d$ be an
involution in $N_G(H) \setminus P$. Define

$$G_{153} = \text{Cos}(G, P, PdP).$$

Then $G_{153}$ is a connected tetravalent 1-transitive graph and Aut($G_{153}$) $\cong$ PSL(2, 17).

Since the automorphism groups of the graphs defined in Examples 3.1-3.7 are pairwise non-isomorphic, we have the following lemma.

**Lemma 3.8.** $G_{9[2K_{1}], G_{18}, CA_{(3,3)p}, CA_{(3,3)p}, G_{27}, G_{45}}$ and $G_{153}$ are connected pairwise non-isomorphic tetravalent symmetric graphs.

4. Classification

This section is devoted to classifying tetravalent symmetric graphs of order 9$p$ for $p$ a prime. First we have the following lemma.

**Lemma 4.1.** Let $p$ be a prime greater than 3 and $G$ a non-abelian group of order 9$p$. Then any connected tetravalent normal Cayley graph on $G$ cannot be symmetric.

**Proof.** Let $X = \text{Cay}(G, S)$ be a connected tetravalent normal Cayley graph. Then $(S) = G$, $S^{-1} = S$ and $|S| = 4$. Since $|G| = 9p$, we may assume $S = \{x, x^{-1}, y, y^{-1}\}$, and since $X$ is normal, Aut($G, S$) $\cong$ Aut($X$) by Proposition 2.5.

Suppose to the contrary that $X$ is symmetric. Then Aut($G, S$) is transitive on $S$, forcing that $o(x) = o(y)$. Note that $p > 3$. By Sylow Theorem, $G$ has a normal Sylow $p$-subgroup, which means that $o(x) \neq p$ because $(S) = G$. Denote by $Z(G)$ the center of $G$. From the elementary group theory, up to isomorphism, there are three non-abelian groups of order 9$p$ for a prime $p > 3$:

- $G_1 = \langle a, b \mid a^p = b^3 = 1, b^{-1}ab = a^r \rangle$, where $r \in Z_p^*$ and $o(r) = 3$;
- $G_2 = \langle a, b \mid a^p = b^3 = 1, b^{-1}ab = a^s \rangle$, where $s \in Z_p^*$ and $o(s) = 9$;
- $G_3 = \langle a, b, c \mid a^p = b^3 = c^3 = [b, c] = [a, b] = 1, c^{-1}ac = a^t \rangle$, where $t \in Z_p^*$ and $o(t) = 3$.

**Case 1:** $G = G_1$

In this case, $Z(G) = \langle b^3 \rangle$ and $Z(G)$ is the unique subgroup of order 3 in $G$. Since $(S) = G$, one has $o(x) \neq 3$ and hence $o(x) = o(y) = 3p$ or 9. Similarly, if $o(x) = 3p$, then $G = \langle S \rangle \subseteq Z(G) \times \langle a \rangle$, a contradiction. Thus, $o(x) = 9$ and $x, y$ have the form $a^i b^{3j+1}$ or $a^i b^{3j-1}$. Each automorphism $\alpha$ in Aut($G$) can be written as follows:

$$\alpha : \begin{cases} 
  a \mapsto a^i, \\
  b \mapsto a^j b^{3k+1}, \quad 1 \leq i \leq p - 1; \\
  b \mapsto a^j b^{3k+1}, \quad 0 \leq j \leq p - 1, \quad 0 \leq k \leq 2.
\end{cases}$$

Clearly, Aut($G$) is transitive on the set $\{\langle g, g^{-1} \rangle \mid g \in G, o(g) = 9\}$. We may assume that $x = b$ and $y = a^i b^{3k+1}$. Since $a \mapsto a^i$, $b \mapsto b$ induces an automorphism of $G$, $S = \{b, b^{-1}, a b^{3k+1}, (a b^{3k+1})^{-1}\}$. Note that every automorphism
of $G$ cannot map $b$ to $a^j b^{3k-1}$. It follows that $\text{Aut}(G, S) \subseteq Z_2$. Thus, $\text{Aut}(G, S)$ cannot be transitive on $S$, a contradiction.

**Case 2:** $G = G_2$

Since $o(x) \neq p$, each element in $S$ has order 3 or 9, and since $\langle a, b \rangle$ is a metacyclic normal subgroup of order $3p$ containing all elements of order 3, one has $o(x) \neq 3$. Thus, $o(x) = o(y) = 9$ and $x, y$ have the form $a^i b^{3j+1}$ or $a^i b^{3j-1}$.

Each automorphism $\alpha$ in $\text{Aut}(G)$ can be written as follows:

$$\alpha : \begin{cases} a \mapsto a^i, & 1 \leq i \leq p - 1; \\ b \mapsto ab, & 0 \leq j \leq p - 1. \end{cases}$$

Note that $a \mapsto a^i$, $b \mapsto b$ and $a \mapsto a^j b^j$ induce automorphisms of $G$. Then $S \equiv \{a^{3k+1}, (a^{3k+1})^{-1}, ab^{3k+1}, (ab^{3k+1})^{-1}\}$. Since every automorphism of $G$ cannot map $b^i$ to $a^j b^{-i}$, one has $\text{Aut}(G, S) \not\subseteq Z_2$. Thus, $\text{Aut}(G, S)$ cannot be transitive on $S$, a contradiction.

**Case 3:** $G = G_3$

Since $o(x) \neq p$, each element in $S$ has order 3 or 9. Since $(a, b)$ contains all elements of order $3p$ in $G$, one has $o(x) = 3$ because $\langle S \rangle = G$. Note that $Z(G) = \langle b \rangle$. Thus, $b, b^2 \not\in S$, and $x, y$ have the form $a^i b^j c$ or $a^i b^j c^{-1}$ with $1 \leq i \leq p$ and $1 \leq j \leq 3$. Each automorphism $\alpha$ in $\text{Aut}(G)$ can be written as follows:

$$\alpha : \begin{cases} a \mapsto a^i, & 1 \leq i \leq p - 1; \\ b \mapsto b^j, & 1 \leq j \leq 2; \\ c \mapsto a^k b^l c, & 0 \leq k \leq p - 1, 0 \leq l \leq 2. \end{cases}$$

Thus, we may assume that $x = c$, and since the map $a \mapsto a^i$, $b \mapsto b^j$, $c \mapsto c$ induces an automorphism of $G$, $S \equiv \{c, c^{-1}, abc, (abc)^{-1}\}$. Since every automorphism of $G$ cannot map $a^i b^j c$ to $(a^i b^j c)^{-1}$, one has $\text{Aut}(G, S) \not\subseteq Z_2$. Thus, $\text{Aut}(G, S)$ cannot be transitive on $S$, a contradiction. \hfill \Box

To state the main theorem, we introduce the so called quotient graph. Let $X$ be a graph and let $G \leq \text{Aut}(X)$ be an arc-transitive subgroup on $X$. Assume that $G$ is imprimitive on $V(X)$ and $B = \{B_1, B_2, \ldots, B_n\}$ is a complete block system of $G$. The block graph or quotient graph $X_G$ of $X$ relative to $B$ is defined as the graph with vertex set the complete block system $B$, and with the two blocks adjacent if and only if there is an edge in $X$ between those two blocks. Clearly, if $X$ is $G$-symmetric, then $X_G$ is $G/K$-symmetric, where $K$ is the kernel of $K$ on $B$. For a normal subgroup $N$ of $G$, the set of the orbits of $N$ forms a complete block system of $G$. In this case we denote by $X_N$ the quotient graph of $X$ relative to the set of the orbits of $N$. The following is the main result of this paper.

**Theorem 4.2.** Let $p$ be a prime. Then any connected tetravalent symmetric graph of order $9p$ is isomorphic to one of the graphs in Table 1. Furthermore, all graphs in Table 1 are pairwise non-isomorphic.
Table 1. Tetravalent $s$-transitive graphs of order $9p$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$s$-transitive</th>
<th>$\text{Aut}(X)$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_9[2K_1]$</td>
<td>1-transitive</td>
<td>$\mathbb{Z}<em>2^2 \times D</em>{18}$</td>
<td>Example 3.1, $p = 2$</td>
</tr>
<tr>
<td>$G_{18}$</td>
<td>1-transitive</td>
<td>$(\mathbb{Z}_3^2 \times \mathbb{Z}_2) \times D_8$</td>
<td>Example 3.2, $p = 2$</td>
</tr>
<tr>
<td>$G_{27}$</td>
<td>1-transitive</td>
<td>$(\mathbb{Z}_3^2 \times \mathbb{Z}_3) \times D_8$</td>
<td>Example 3.5, $p = 3$</td>
</tr>
<tr>
<td>$G_{45}$</td>
<td>1-transitive</td>
<td>$\text{Aut}(A_6)$</td>
<td>Example 3.6, $p = 5$</td>
</tr>
<tr>
<td>$G_{153}$</td>
<td>1-transitive</td>
<td>$\text{PSL(2,17)}$</td>
<td>Example 3.7, $p = 17$</td>
</tr>
<tr>
<td>$CA_{9p}$</td>
<td>1-regular</td>
<td>$\mathbb{Z}_{9p} \rtimes \mathbb{Z}_2^2$</td>
<td>Example 3.4, $p \geq 5$</td>
</tr>
<tr>
<td>$CA_{3(3,3p)}^1$</td>
<td>1-regular</td>
<td>$(\mathbb{Z}<em>3 \times \mathbb{Z}</em>{3p}) \rtimes \mathbb{Z}_2^2$</td>
<td>Example 3.3, $p \geq 3$</td>
</tr>
<tr>
<td>$CA_{3(3,3p)}^{2}$</td>
<td>1-regular</td>
<td>$(\mathbb{Z}<em>3 \times \mathbb{Z}</em>{3p}) \rtimes \mathbb{Z}_4$</td>
<td>Example 3.3, $p \equiv 1(\text{mod } 4)$</td>
</tr>
</tbody>
</table>

Proof. By Lemma 3.8, all graphs in Table 1 are connected pairwise non-isomorphic tetravalent symmetric graphs. Let $X$ be a connected tetravalent symmetric graph of order $9p$. To finish the proof, it suffices to show that $X$ is isomorphic to one of the graphs listed in Table 1.

If $p \leq 7$, then by [17, 24], there are ten connected tetravalent symmetric graphs of order $9p$: two graphs for $p = 2$, two graphs for $p = 3$, four graphs for $p = 5$ and two graphs for $p = 7$. Thus, $X$ is isomorphic to $C_9[2K_2]$, $G_{18}$, $G_{27}$, $CA_{(3,3p)}^1$, $G_{45}$, $CA_{45}$, $CA_{(3,15)}^1$, $CA_{(3,15)}^2$, $CA_{63}$ or $CA_{(3,21)}^1$. Let $p > 7$ and assume that $X$ is a normal Cayley graph. Then by Examples 3.3, 3.4 and Lemma 4.1, $X$ is isomorphic to $CA_{9p}$, $CA_{(3,3p)}^1$ or $CA_{(3,3p)}^2$.

Thus, in what follows one may assume that $p > 7$ and $X$ is not a normal Cayley graph, that is, $A$ has no normal regular subgroup on $V(X)$. Then, to finish the proof it suffices to show that $X \cong G_{153}$.

Set $A = \text{Aut}(X)$ and let $A_v$ be the stabilizer of $v \in V(X)$ in $A$. Since $X$ is symmetric, either $A_v$ is a 2-group or $A_v \cong A_4$, $S_4$, $C_3 \times A_4$, $C_3 \times S_4$ or $S_3 \times S_4$ by Proposition 2.7. It follows that $|A| \leq 2^5 \cdot 3^2 \cdot p$ for some integer $t$. Since $p > 7$, every Sylow 2-subgroup of $A$ is also a Sylow 2-subgroup of a stabilizer of some vertex in $A$, implying that $A$ has no non-trivial normal 2-subgroups.

Suppose that $A$ has an intransitive minimal normal subgroup, say $N$. Since $|V(X)| = 9p$ and $|A| | 2^5 \cdot 3^2 \cdot p$, $N$ is either a non-abelian simple group, or an elementary abelian 3- or $p$-group. Let $B = \{B_1, B_2, \ldots, B_n\}$ be the set of orbits of $N$ and $K$ the kernel of $A$ acting on $B$. Then $N \leq K$. Let $m = |B_1|$. Then $mn = 9p$ with $1 < m, n < 9p$. The quotient graph $X_N$ has vertex set $B$ and $A/K \leq \text{Aut}(X_N)$. Moreover, assume that $B_1$ is adjacent to $B_2$ in $X_N$ with $v \in B_1$ and $u \in B_2$ being adjacent in $X$. Clearly, $X_N$ has valency 2 or 4.

**Case 1:** $X_N$ has valency 2.

In this case, $X_N$ is a cycle and $A/K \cong D_{2m}$. Since $X$ is symmetric, the induced subgraph $\langle B_1 \cup B_2 \rangle$ of $B_1 \cup B_2$ in $X$ is a union of several cycles of the
same length greater than 4, implying that $K_e$ is a 2-group and $K$ acts faithfully on $B_1$. Since $A/K \cong D_{2m}$, one has $|A| = 2^s mn = 2^s9p$ for some integer $s$. This implies that if $A$ has a Hall $\{3,p\}$-subgroup, then it is regular on $V(X)$. Note that $mn = 9p$ with $1 < m, n < 9p$. Thus, $(B_1 \cup B_2) \cong C_{2m}, 3C_6, 3C_{2p}$ or $pC_6$.

Let $(B_1 \cup B_2) \cong C_{2m}$. Since $\text{Aut}(C_{2m}) \cong D_{4m}$, one has $Z_m \trianglelefteq K \trianglelefteq D_{2m}$, and since $A/K \cong D_{2m}$, $A$ has a normal subgroup of order $9p$, which is regular on $V(X)$ because $A_e$ is a 2-group. Thus, $A$ has a normal regular subgroup, a contradiction.

Let $(B_1 \cup B_2) \cong 3C_6$. Then $N$ has blocks of length 3 on $B_1$ and since $K$ acts faithfully on $B_1$, $N$ must be an elementary abelian 3-group and hence $K$ is a $\{2,3\}$-group. By Proposition 2.2, $K$ is solvable, and since $A/K \cong D_{2p}$, $A$ is solvable. Thus, $A$ has a Hall $\{3,p\}$-subgroup, say $G$, which is regular on $V(X)$. Since $N \unlhd G$, $G$ cannot be isomorphic to $G_1, G_2$ or $G_3$ as listed in Lemma 4.1. It follows that $G$ is abelian, and by Proposition 2.6, $X$ is a normal Cayley graph on $G$, a contradiction.

Now let $(B_1 \cup B_2) \cong 3C_{2p}$ or $pC_6$. Then $|B_1| = 3p$ and since $N$ is transitive on $B_1$, $N$ must be a non-abelian simple group, say $T$. By [5, pp. 12–14], $T$ is one of the following groups in Table 2.

<table>
<thead>
<tr>
<th>Group</th>
<th>Order</th>
<th>Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_5$</td>
<td>$2^2 \cdot 3 \cdot 5$</td>
<td>2</td>
</tr>
<tr>
<td>$A_6$</td>
<td>$2^3 \cdot 3^2 \cdot 5$</td>
<td>$2^3$</td>
</tr>
<tr>
<td>$\text{PSL}(2, 7)$</td>
<td>$2^3 \cdot 3 \cdot 7$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{PSL}(2, 8)$</td>
<td>$2^3 \cdot 3^2 \cdot 7$</td>
<td>3</td>
</tr>
<tr>
<td>$\text{PSL}(2, 17)$</td>
<td>$2^4 \cdot 3^2 \cdot 17$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{PSL}(3, 3)$</td>
<td>$2^4 \cdot 3^3 \cdot 13$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{PSU}(3, 3)$</td>
<td>$2^5 \cdot 3^3 \cdot 7$</td>
<td>3</td>
</tr>
<tr>
<td>$\text{PSU}(4, 2)$</td>
<td>$2^6 \cdot 3^3 \cdot 5$</td>
<td>2</td>
</tr>
</tbody>
</table>

If $(B_1 \cup B_2) \cong 3C_{2p}$, then $N$ has a transitive action of degree 3, which is impossible because $N$ is a non-abelian simple group. Thus, $(B_1 \cup B_2) \cong pC_6$. Since $|A| = 2^s mn = 2^s9p$ and $N$ is intransitive, $9p \nmid |N|$. Then by Table 2, one has $N \cong \text{PSL}(2, 7)$. This is impossible because $p > 7$.

**Case 2:** $X_N$ has valency 4.

In this case, $K_e$ fixes the neighborhood of $v$ in $X$ pointwise. Thus, $K = N$ is semiregular on $V(X)$ and $A/N \trianglelefteq \text{Aut}(X_N)$. Since $|V(X)| = 9p$, one has $N \cong \mathbb{Z}_p, \mathbb{Z}_3$ or $\mathbb{Z}_5$.

Let $N \cong \mathbb{Z}_p$. Then the quotient graph $X_N$ has order 9. By Proposition 2.4, $A/N$ contains a regular subgroup, say $B/N$, on $V(X_N)$, that is, $X_N$ is a Cayley graph on $B/N$. It follows that $|B/N| = 9$ and hence $B/N$ is abelian. By
Proposition 2.6, \( B/N \trianglelefteq A/N \) and hence \( B \trianglelefteq A \). Thus, \( B \) is a normal regular subgroup of \( A \) on \( V(X) \), a contradiction.

Let \( N \cong \mathbb{Z}_2^3 \). Then \( X_N \) is a tetravalent \( A/N \)-symmetric graph of order \( p \).

Since \( p > 7 \), \( X_N \) is not a complete graph, and hence \( A/N \) has a normal regular Sylow \( p \)-subgroup by Proposition 2.3. This implies that \( A \) has a normal regular subgroup, a contradiction.

Let \( N \cong \mathbb{Z}_3 \). Then \( X_N \) is a connected tetravalent symmetric graph of order \( 3p \).

Since \( p > 7 \), by Proposition 2.8 one has \( X_N \cong CA_{3p} \). It follows that \( A/N \) has a normal regular subgroup on \( V(X_N) \) because \( \text{Aut}(CA_{3p}) \cong \mathbb{Z}_3 \times \mathbb{Z}_3^2 \), which implies that \( A \) has a normal regular subgroup on \( V(X) \), a contradiction.

Now we may assume that \( A \) has no intransitive minimal normal subgroup. Thus, every non-trivial normal subgroup of \( A \) is transitive on \( V(X) \). Again let \( N \) be a minimal normal subgroup of \( A \). Then \( N \) is transitive on \( V(X) \) and since \( |V(X)| = 9p \), \( N \) is a non-abelian simple group as listed in Table 2.

Recall that \( p > 7 \) and either \( |N_v| = 2^t \) or \( |N_v| = 3 \cdot 2^2, 3 \cdot 2^3, 3^2 \cdot 2^2, 3^2 \cdot 2^3 \) or \( 3^2 \cdot 2^4 \). It follows that \( N \cong \text{PSL}(2, 17) \). Set \( C = C_A(N) \), the centralizer of \( N \) in \( A \). Then \( C \cap N = 1 \) and \( C \) is a \( \{2, 3\} \)-group. If \( C \neq 1 \), then \( C \) is an intransitive normal subgroup of \( A \) because \( |V(X)| = 9p \), which is contrary to our assumption. Thus, \( C = 1 \) and \( A = A/C \cong \text{Aut}(N) \) by Proposition 2.1. Since \( N \cong \text{PSL}(2, 17) \), one has that \( A = \text{PSL}(2, 17) \) or \( \text{PGL}(2, 17) \), and the stabilizer \( A_v \) is a Sylow 2-subgroup of \( A \), which is maximal in \( A \) by [5]. It follows that \( A \) is primitive on \( V(X) \), and by [14, Theorem 1.5] and Example 3.7, \( X \cong G_{15,3} \) and \( A \cong \text{PSL}(2, 17) \).

\( \square \)

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References

TETRAVALENT SYMMETRIC GRAPHS OF ORDER 9


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