Honam Mathematical J. **35** (2013), No. 4, pp. 565–582
http://dx.doi.org/10.5831/HMJ.2013.35.4.565

INTERVAL-VALUED FUZZY SUBGROUPS

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Abstract. We study the conditions under which a given interval-valued fuzzy subgroup of a given group can or can not be realized as a union of two interval-valued fuzzy proper subgroups. Moreover, we provide a simple necessary and sufficient condition for the union of an arbitrary family of interval-valued fuzzy subgroups to be an interval-valued fuzzy subgroup. Also we formulate the concept of interval-valued fuzzy subgroup generated by a given interval-valued fuzzy set by level subgroups. Furthermore we give characterizations of interval-valued fuzzy conjugate subgroups and interval-valued fuzzy characteristic subgroups by their level subgroups. Also we investigate the level subgroups of the homomorphic image of a given interval-valued fuzzy subgroup.

1. Introduction

In 1965, Zadeh[11] introduced the concept of fuzzy sets and in 1975, he[12] suggested interval-valued fuzzy sets as generalization of fuzzy sets. After that time, Biswas[1] applied it to group theory, and Gorzalczany[4] introduced a method of inference in approximate reasoning by using interval-valued fuzzy sets. Moreover, Mondal and Samanta[10] introduced the concept of interval-valued fuzzy topology and investigate some of it's properties. Recently, Choi and Hur[3] introduced the concept of interval-valued smooth topological spaces and investigated some of it's properties. One the other hand, Cheong and Hur[2], and Hur et al.[8] studied interval-valued fuzzy ideals/bi-ideals in a semigroup. In particular, Kang[6], Kang and Hur[7], and Lim et al.[5, 9] applied the notion of interval-valued fuzzy sets to algebra.

Received February 12, 2013. Accepted October 25, 2013.
2010 Mathematics Subject Classification. 03F55, 20N25.
Key words and phrases. interval-valued fuzzy subgroup, level subgroup, interval-valued fuzzy conjugate subgroup, interval-valued fuzzy characteristic subgroup.
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In this paper, we study the conditions under which a given interval-valued fuzzy subgroup of a given group can or can not be realized as a union of two interval-valued fuzzy proper subgroups. Moreover, we provide a simple necessary and sufficient condition for the union of an arbitrary family of interval-valued fuzzy subgroups to be an interval-valued fuzzy subgroup. Also we formulate the concept of interval-valued fuzzy subgroup generated by a given interval-valued fuzzy set by level subgroups. Furthermore we give characterizations of interval-valued fuzzy conjugate subgroups and interval-valued fuzzy characteristic subgroups by their level subgroups. Also we investigate the level subgroups of the homomorphic image of a given interval-valued fuzzy subgroup.

2. Preliminaries

We will list some concepts and results needed in the later sections. Throughout this paper, we will denote the unit interval $[0,1]$ as I.

Let $D(I)$ be the set of all closed subintervals of the unit interval $[0,1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and $\boldsymbol{a} = [a,a]$ for every $a \in (0,1)$. We also note that

(i) $(\forall M, N \in D(I))$ $(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))$ $(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^C, is defined by $M^C = 1 - M = [1 - M^U, 1 - M^L]$(See[10]).

Definition 2.1[**4, 12**]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set*(in short, *IVFS*) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^U \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$[resp $A^U(x)$] is called the *lower*[resp *upper*] *end point of x to A*. For any $[a,b] \in D(I)$, the interval-valued fuzzy A in X defined by $A(x) = [A^L(x), A^U(x)] = [a,b]$ for each $x \in X$ is denoted by $\widetilde{[a,b]}$ and if $a = b$, then the IVFS $\widetilde{[a,b]}$ is denoted by simply $\widetilde{a}$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVFSs in X as $D(I)^X$. It is clear that set $A = [A, A] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2[10]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^C = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(iv)′ $\bigcup_{\alpha\in\Gamma} A_\alpha = [\bigvee_{\alpha\in\Gamma} A^L_\alpha, \bigvee_{\alpha\in\Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(v)′ $\bigcap_{\alpha\in\Gamma} A_\alpha = [\bigwedge_{\alpha\in\Gamma} A^L_\alpha, \bigwedge_{\alpha\in\Gamma} A^U_\alpha]$.

Definition 2.3[10]. Let X and Y be nonempty sets, let $f : X \to Y$ be a mapping. Let $A \in D(I)^X$ and $B \in D(I)^Y$. Then

(i) the *preimage of* B *under* f, denoted by $f^{-1}(B)$, is an IVFS in Y defined as follows: For each $y \in Y$,

$f^{-1}(B)^L(y) = (B^L \circ f)(x) = B^L(f(x))$

and

$f^{-1}(B)^U(y) = (B^U \circ f)(x) = B^U(f(x))$.

(ii) the *image of* A *under* f, denoted by $f(A)$, is the IVFS in Y defined as follows : For each $y \in Y$,

$$f(A)^L(y) = f(A^L)(y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} A^L(x) \text{ if } f^{-1}(y) \neq \emptyset, \\ 0 \text{ if } f^{-1}(y) = \emptyset, \end{cases}$$

$$f(A)^U(y) = f(A^U)(y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} A^U(x) \text{ if } f^{-1}(y) \neq \emptyset, \\ 0 \text{ if } f^{-1}(y) = \emptyset. \end{cases}$$

Result 2.A[10, Theorem 2]. Let $f : X \to Y$ be a mapping and $g : Y \to Z$ be a mapping. Then:

(a) $f^{-1}(B^c) = [f^{-1}(B)]^c$, $\forall B \in D(I)^Y$.

(b) $[f(A)]^c \subset f(A^c)$, $\forall A \in D(I)^Y$.

(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in D(I)^Y$.

(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, where $A_1, A_2 \in D(I)^X$.

(e) $f(f^{-1}(B)) \subset B$, $\forall B \in D(I)^Y$.

(f) $A \subset f(f^{-1}(A))$, $\forall A \in D(I)^Y$.

(g) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, $\forall C \in D(I)^Z$.

(h) $f^{-1}(\bigcup_{\alpha\in\ \Gamma} B_\alpha) = \bigcup_{\alpha\in\ \Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\ \Gamma} \in D(I)^Y$.

(i) $f^{-1}(\bigcap_{\alpha\in\ \Gamma} B_\alpha) = \bigcap_{\alpha\in\ \Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\ \Gamma} \in D(I)^Y$.

Proposition 2.4. Let A, $A_\alpha(\alpha \in \Gamma)$ be IVFSs in X and let B be IVFS in Y. Then :

(a) If f is injective, then $A = f^{-1}(f(A))$.
(b) If f is surjective, then $B = f(f^{-1}(B))$.
(c) $f(\bigcup_{\alpha\in\ \Gamma} A_\alpha) = \bigcup_{\alpha\in\ \Gamma} f(A_\alpha)$.
(d) $f(\bigcap_{\alpha\in\ \Gamma})A_\alpha \subset \bigcap_{\alpha\in\Gamma} f(A_\alpha)$.
If f is injevtive, then $f(\bigcap_{\alpha\in\ \Gamma})A_\alpha = \bigcap_{\alpha\in\Gamma} f(A_\alpha)$.
(e) $f(\tilde{1}) = \tilde{1}$, if f is surjctive and $f(\tilde{0}) = \tilde{0}$.
(f) $f^{-1}(\tilde{1}) = \tilde{1}$ and $f^{-1}(\tilde{0}) = \tilde{0}$.

Proof. The proofs are straightforward. □

Definition 2.5[7]. Let $(G, \cdot)$ be a groupoid and let $A \in D(I)^G$. Then A is called an *interval-valued fuzzy subgroupoid*(in short, *IVGP*) in G if

$$A^L(xy) \geq A^L(x) \wedge A^L(y) \text{ and } A^U(xy) \geq A^U(x) \wedge A^U(y),\ \forall x, y \in G.$$

It is clear that $\widetilde{0}, \widetilde{1} \in$ IVGP(G).

Definition 2.6[1]. Let A be an IVFSs in a group G. Then A is called an *interval-valued fuzzy subgroup* (in short, IVG) in G if it satisfies the conditions : For any $x, y \in G$,

(i) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.
(ii) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$.

We will denote the set of all IVGs of G as IVG(G).

Definition 2.7[1]. $A \in D(I)^X$ is said to *have the sup-property* if for each $T \in P(X)$, $\exists t_0 \in T$ such that $A(t_0) = [\bigvee_{t\in T} A^L(t), \bigvee_{t\in T} A^U(t)]$.

Result 2.B[1, Proposition 3.4; 7, Proposition 4.11(b)]. Let $f : G \to G'$ be a group homomorphism. If $B \in$ IVG(G'), then $f^{-1}(B) \in$ IVG(G).

Result 2.C[1, Proposition 3.5]. Let $f : G \to G'$ be a group homomorphism. If $A \in$ IVG(G), then $f(A) \in$ IVG(G').

Result 2.D[9, Proposition 4.10(a)]. Let $f : G \to G'$ be a group homomorphism, let $A \in$ IVG(G) and let $B \in$ IVG(G'). If A has the sup property, then $f(A) \in$ IVG(G').

Definition 2.8[7]. Let A be an IVFS in a set X and let $\lambda, \mu \in I$ with $\lambda \leq \mu$. Then the set $A^{[\lambda,\mu]} = \{x \in X : A^L(x) \geq \lambda \text{ and } A^U(x) \geq \mu\}$ is called a $[\lambda, \mu]$-*level subset* of A.

Result 2.E[7, Proposition 4.16 and Proposition 4.17]. Let A be an IVFS in a group G. Then $A \in \text{IVG}(G)$ if and only if $A^{[\lambda,\ \mu]}$ is a subgroup of G for each $[\lambda,\ \mu] \in \text{Im}A$.

In this case, we will call the subgroup $A^{[\lambda,\ \mu]}$ as a $[\lambda,\ \mu]$-*level subgroup* of A.

3. Union of interval-valued fuzzy subgroups

It is well known in classical group theory that a group cannot be realized as a union of two proper subgroups. Lim et al.[7] tried to establish the interval-valued fuzzy analog of the above result in the form of the following :

Result 3.A[7, Proposition 4.8]. A group G cannot be the union of two proper IVGs.

A generalization of the above problem can be formulated as follows :

Is it possible for a proper interval-valued fuzzy subgroup to be realized as union of two proper interval-valued fuzzy subgroups such that none is contained in the other?

In this section, we shall demonstrate that the answer of the above question depends on the image set of the interval-valued fuzzy subgroup under consideration.

It is interesting to note that if the image set of the given interval-valued fuzzy subgroup contains at least two nonzero numbers of $D(I)$, then the interval-valued fuzzy subgroup can always be realized as a union of two proper interval-valued fuzzy subgroups, such that none is contained in the other ; a result contrary to the corresponding result of group theory. However, in case the image set consists of

$\{[0,0],[t,s]\}, \text{wher}[t,s] \in D((0,1))$.

then the result of the group theory stated above can be successfully extended to the interval-valued fuzzy setting.

Definition 3.1. Let G be a group. An IVG A of G is said to be *proper* if A is not constant on G, i.e., ImA has at least two elements.

Lemma 3.2. Let G be a group and let $A \in$IVG(G). If for any $x, y \in G$, $A^L(x) < A^L(y)$ and $A^U(x) < A^U(y)$, then $A(xy) = A(x) = A(yx)$.
Proof. Since $A \in$ IVG(G),

$A^L(xy) \geq A^L(x) \wedge A^L(y) = A^L(x)$

and

$A^U(xy) \geq A^U(x) \wedge A^U(y) = A^U(x)$.

On the other hand,

$A^L(x) = A^L(xyy^{-1}) \geq A^L(xy) \wedge A^L(y^{-1}) \geq A^L(xy) \wedge A^L(y)$

and

$A^U(x) = A^U(xyy^{-1}) \geq A^U(xy) \wedge A^U(y^{-1}) \geq A^U(xy) \wedge A^U(y)$.

Since $A^L(x) < A^L(y)$ and $A^U(x) < A^U(y)$,

$$A^L(x) \geq A^L(xy) \text{ and } A^U(x) \geq A^U(xy).$$

Thus $A(xy) = A(x)$. Similarly, we have $A(xy) = A(y)$. This completes the proof. □

In the following example, we show that the union of two IVGs need not be an IVG.

Example 3.3. Let G be the Klein's four group :

$$G = \{e, a, b, ab\},$$

where $a^2 = e = b^2$ and $ab = ba$. Let $[t_i, s_i], 0 \leq i \leq 5$, be the numbers of $D(I)$ such that

$$t_0 > t_1 > \cdots > t_5 \text{ and } s_0 > s_1 > \cdots > s_5.$$

Define two mappings $A, B : G \to D(I)$ as follows :

$A(e) = [t_1, s_1], A(a) = [t_3, s_3]$ and $A(b) = A(ab) = [t_4, s_4]$,

$B(e) = [t_0, s_0], B(a) = [t_5, s_5], B(b) = [t_2, s_2]$ and $B(ab) = [t_5, s_5]$.

Then clearly, $A, B \in$ IVG(G). Moreover, we can easily see that $A \cup B \notin$ IVG(G). □

Now we state without proof the following result. In fact, the proof can be obtained along the lines of Result 2.A.

Proposition 3.4. If C is a constant (improper) IVG of a group G, then C cannot be realized as $A \cup B$, where A and B are proper IVGs of G

with $A \neq C$ and $B \neq C$.

We illustrate through an example that a proper IVG can be realized as a union of two proper IVGs A and B such that $C \neq A, C \neq B$ and $A \neq B$.

Example 3.5. Let G be any non-trivial group. We define a mapping $C : G \to D(I)$ as follows : For each $e \neq x \in G$,

$C(e) = [t_0, s_0]$ and $C(x) = [t_1, s_1]$,

where $[t_0, s_0], [t_1, s_1] \in D((0,1)), t_0 > t_1$ and $s_0 > s_1$.
Then clearly C is a proper IVG of G. We can see that $C = A \cup B$, where A and B are proper IVGs of G defined as follows :

$A(e) = [t'_0, s'_0]$ and $A(x) = [t_1, s_1]$ for each $e \neq x \in G$,

$B(e) = [t_0, s_0]$ and $B(x) = [t'_1, s'_1]$ for each $e \neq x \in G$,

where $t_0 > t'_0 > t_1 > t'_1$ and $[t'_0, s'_0], [t'_1, s'_1] \in D(I)$. But, $A \neq C, B \neq C$ and $A \neq B$. □

Proposition 3.6. Let G be a group and let C be a proper IVG of G such that $\mathrm{Im}C = \{[0,0], [t,s]\}$, where $[t,s] \in D((0,1))$. If $C = A \cup B$, where A and B are IVGs of G, then either $A \subset B$ or $B \subset A$.

Proof. Assume that $A \not\subset B$ and $B \not\subset A$. Then there exist $x_1, x_2 \in G$ such that

$A^L(x_1) > B^L(x_1), A^U(x_1) > B^U(x_1)$,

and

$A^L(x_2) < B^L(x_2), A^U(x_2) < B^U(x_2)$.

Since $C = A \cup B$,

$C^L(x_1) = A^L(x_1) \vee B^L(x_1) = A^L(x_1) > B^L(x_1) \geq 0$,

$C^U(x_1) = A^U(x_1) \vee B^U(x_1) = A^U(x_1) > B^U(x_1) \geq 0$.

and

$C^L(x_2) = A^L(x_2) \vee B^L(x_2) = B^L(x_2) > A^L(x_2) \geq 0$,

$C^U(x_2) = A^U(x_2) \vee B^U(x_2) = B^U(x_2) > A^U(x_2) \geq 0$.

Since $\mathrm{Im}C = \{[0,0], [t,s]\}$,

$C(x_1) = A(x_1) = [t,s] = B(x_2) = C(x_2) = C(x_1x_2)$.

By Lemma 3.2 and the fact that

$A^L(x_2) < t = A^L(x_1), A^U(x_2) < s = A^U(x_1)$

and

$B^L(x_1) < t = A^L(x_1), B^U(x_1) < s = A^U(x_1)$.

We obtain that $A(x_1x_2) = A(x_2)$ and $B(x_1x_2) = B(x_1)$. So

$C^L(x_1x_2) = A^L(x_1x_2) \vee B^L(x_1x_2) = A^L(x_2) \vee B^L(x_1) < t$

and

$$C^U(x_1x_2) = A^U(x_1x_2) \vee B^U(x_1x_2) = A^U(x_2) \vee B^U(x_1) < s.$$

This is a contradiction. Hence either $A \subset B$ or $B \subset A$. □

Proposition 3.7. Let C be a proper IVG of a group G with $3 \leq |\mathrm{Im}C| < \infty$. Then C can always be realized as a union of two proper IVGs A and B of G with $A \neq C, B \neq C$ and $A \neq B$.

Proof. Let $\mathrm{Im}C = \{[t_0, s_0], [t_1, s_1], \cdots, [t_n, s_n]\}$ with $t_0 > t_1 > \cdots > t_n, s_0 > s_1 > \cdots > s_n$ and $2 \leq n < \infty$. Let

$$C^{[t_0,s_0]} \subset C^{[t_1,s_1]} \subset \cdots \subset C^{[t_n,s_n]} = G$$

be the chain of level subgroups of C in G. Let $[\lambda_i,\ \mu_i] \in D((0,1))(i = 1, 2)$ and let

$1 \geq t_0 > \lambda_1 > t_1 > \lambda_2 > t_2 > \cdots > t_n,$

$1 \geq s_0 > \mu_1 > s_1 > \mu_2 > s_2 > \cdots > s_n.$

We define two mappings $A, B : G \to D(I)$ as follows, respectively. For each $x \in G$,

$$A(x) = \begin{cases} [t_1, s_1] & \text{if } x \in C^{[t_0,s_0]} \text{ (i.e., } C(x) = [t_0, s_0]) \\ C(x) & \text{if } x \notin C^{[t_0,s_0]} \text{ (i.e., } C^L(x) < t_0 \text{ and } C^U(x) < s_0) \end{cases}$$

and

$$B(x) = \begin{cases} [t_2, s_2] & \text{if } C(x) = [t_1, s_1], \\ C(x) & \text{if } C(x) \neq [t_1, s_1] \ . \end{cases}$$

Then we can easily see that A and B are proper IVGs of G such that $A \neq C, B \neq C, A \neq B$ and $C = A \cup B$. This completes the proof. □

Now we state without proof the following result.

Corollary 3.8. If C is an IVG of a group G with $\mathrm{Im}C = \{[1, 1], [t, s]\}$, where $[t, s] \in D((0, 1))$, then C can always be realized as a union of two proper interval-valued fuzzy subgroups A and B of G such that $A \neq C, B \neq C$ and $A \neq B$.

Definition 3.9. Let A and B be IVGs of a group G. Then A and B are said to be *equivalent* if they have the same family of level subgroups. Otherwise A and B are said to be *non-equivalent*.

It follows that the union of two proper interval-valued fuzzy subgroups is an IVG.

Example 3.10. Let G be a cyclic group of prime power order p^n, where p is a prime and n is an integer such that $n \geq 1$. Then G has a sequence of subgroups $G_i's$ of order $p^i, i = 0, 1, \cdots, n$. We define two mappings

$A : G \to D(I)$,

and

$B : G \to D(I)$.

as follows, respectively : For each $x \in G$,

$A(e) = [1, 1]$, $A(x) = [\frac{1}{2m}, \frac{1}{2m}]$ if $x \in G_{2m} \setminus G_{2m-2}$,

and

$B(e) = [1, 1]$,

$$B(x) = \begin{cases} [\frac{2}{3}, \frac{2}{3}] & \text{if } x \in G_1 \setminus G_0, \\ [\frac{1}{2m+1}, \frac{1}{2m+1}] & \text{if } x \in G_{2m+1} \setminus G_{2m-1}. \end{cases}$$

Let $x \in G_2$ such that $x \notin G_1$. Then $x \notin G_0$ and $x \in G_3$. Thus $A(x) = [\frac{1}{2}, \frac{1}{2}]$ and $B(x) = [\frac{1}{3}, \frac{1}{3}]$. So $A^L(x) > B^L(x)$ and $A^U(x) > B^U(x)$. Now let $y \in G_1$ such that $y \notin G_0$. Then $y \in G_2$. Thus $A(y) = [\frac{1}{2}, \frac{1}{2}]$ and $B(y) = [\frac{2}{3}, \frac{2}{3}]$. So $A^L(y) < B^L(y)$ and $A^U(y) < B^U(y)$. Hence neither $A \subset B$ nor $B \subset A$. Moreover, it is easily seen that $A, B \in$ IVG(G) and A and B are non-equivalent. Consider the union $A \cup B$. Then $A \cup B$ is given by : For each $x \in G$,

$(A \cup B)(e) = [1, 1]$,

$(A \cup B)(x) = [\frac{2}{3}, \frac{2}{3}]$ if $x \in G_1 \setminus G_0$,

$(A \cup B)(x) = [\frac{1}{2}, \frac{1}{2}]$ if $x \in G_2 \setminus G_1$,

$(A \cup B)(x) = [\frac{1}{3}, \frac{1}{3}]$ if $x \in G_3 \setminus G_2$,

$(A \cup B)(x) = [\frac{1}{n}, \frac{1}{n}]$ if $x \in G_n \setminus G_{n-1}$.

It is clear that $A \cup B \in$ IVG(G). Therefore we have two non-equivalent interval-valued fuzzy subgroups such that their union is an IVG. □

Definition 3.11[7]. Let X and Y be sets, let $f : X \to Y$ be a mapping and let $A \in$ IFS(X). Then A is said to be *interval-valued fuzzy invariant*(in short, *IVF-invariant*) if $f(x) = f(y)$ implies $A(x) = A(y)$ for any $x, y \in X$.

It is obvious that if A is IVF-invariant, then $f^{-1}(f(A)) = A$.

Proposition 3.12. Let $f : G \to G'$ be a group homomorphism and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset$ IVG(G).

(a) If $\bigcup_{\alpha \in \Gamma} A_\alpha \in$ IVG(G), then $\bigcup_{\alpha \in \Gamma} f(A_\alpha) \in$ IVG(G').

(b) If $\bigcup_{\alpha \in \Gamma} f(A_\alpha) \in$ IVG$(G)'$ and each A_α is IVF-invariant, then $\bigcup_{\alpha \in \Gamma} A_\alpha \in$ IVG(G).

Proof. (a) Suppose $\bigcup_{\alpha\in\Gamma} A_\alpha \in \mathrm{IVG}(G)$. Then, by Result 2.C, $f(\bigcup_{\alpha\in\Gamma} A_\alpha) \in \mathrm{IVG}(G')$. By Proposition 2.4$(c)$, $f(\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} f(A_\alpha)$. So $\bigcup_{\alpha\in\Gamma} f(A_\alpha) \in \mathrm{IVG}(G')$.

(b) Suppose $\bigcup_{\alpha\in\Gamma} f(A_\alpha) \in \mathrm{IVG}(G')$. Then, by Result 2.B, $f^{-1}(\bigcup_{\alpha\in\Gamma} f(A_\alpha)) \in \mathrm{IVG}(G)$. By Result 2.A$(h)$, $f^{-1}(\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}(f(A_\alpha))$. Since each A_α is IVF-invariant, $f^{-1}(f(A_\alpha)) = A_\alpha$. So $f^{-1}(\bigcup_{\alpha\in\Gamma} f(A_\alpha)$ $= \bigcup_{\alpha\in\Gamma} A_\alpha$. Hence $\bigcup_{\alpha\in\Gamma} A_\alpha \in \mathrm{IVG}(G)$. □

Proposition 3.13. Let $f : G \to G'$ be a group epimorphism and let $\{B_\alpha\}_{\alpha\in\Gamma} \subset \mathrm{IVG}(G')$. Then $\bigcup_{\alpha\in\Gamma} B_\alpha \in \mathrm{IVG}(G')$ if and only if $\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha) \in \mathrm{IVG}(G)$.

Proof. $(\Rightarrow)$: Suppose $\bigcup_{\alpha\in\Gamma} B_\alpha \in \mathrm{IVG}(G')$. Then, by Result 2.B, $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) \in \mathrm{IVG}(G)$. By Result 2.A$(h)$, $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha)$. Thus $\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha) \in \mathrm{IVG}(G)$.

$(\Leftarrow)$: Suppose $\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha) \in \mathrm{IVG}(G)$. Then, by Result 2.C, $f(\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha)) \in \mathrm{IVG}(G')$. Since f is surjective, by Propositions 2,4(c) and (b),

$$f(\bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha)) = \bigcup_{\alpha\in\Gamma} f(f^{-1}(B_\alpha)) = \bigcup_{\alpha\in\Gamma} B_\alpha.$$

Thus $\bigcup_{\alpha\in\Gamma} B_\alpha \in \mathrm{IVG}(G')$. This completes the proof. □

4. Interval-valued fuzzy subgroup generated by an IVFS

Result 4.A[9, Proposition 2.6]. Let G be a group and let the following be any chain of subgroups

$$G_0 \subset G_1 \subset \cdots \subset G_r = G.$$

Then there exists an interval-valued fuzzy subgroup of G whose level subgroups are precisely the members of this chain.

Proposition 4.1. Let A be an IVFS in a group G with $|\mathrm{Im}A| < \infty$. Define subgroups G_i of G inductively as follows :

$G_0 = \langle\{x \in G : A(x) = [\bigvee_{z\in G} A^L(z), \bigvee_{z\in G} A^U(z)]\}\rangle$,

$G_i = \langle G_{i-1} \cup \{x \in G : A(x) = [\bigvee_{z\in G\setminus G_{i-1}} A^L(z), \bigvee_{z\in G\setminus G_{i-1}} A^U(z)]\}\rangle$

for each $i = 1, \cdots, k$, where $k \le |\mathrm{Im}A|, G_k = G$ and $\langle H\rangle$ denotes the subgroup generated by a subset H of G. Then $A^* \in \mathrm{IVG}(G)$, where $A^* : G \to D(I)$ is a mapping defined as follows : For each $x \in G$,

$$A^*(x) = \begin{cases} [\bigvee_{z\in G} A^L(z), \bigvee_{z\in G} A^U(z)] & \text{if } x \in G_0, \\ [\bigvee_{z\in G\setminus G_{i-1}} A^L(z), \bigvee_{z\in G\setminus G_{i-1}} A^U(z)] & \text{if } x \in G_i \setminus G_{i-1} (1 \le i \le k). \end{cases}$$

In this case, A^* is called the IVG *generated by* A in G.

Proof. By the definition of A^*, it is clear that $A \subset A^*$. Moreover, the $G_i's$ form a chain of subgroups ending at G :

$G_0 \subset G_1 \subset \cdots \subset G_k = G$. $(*)$

By Result 4.A, it follows that A^* is an IVG of G whose level subgroups are precisely the members of the chain $(*)$. Now we shall prove that A^* is the IVG generated by A. Let $B \in \mathrm{IFG}(G)$ such that $A \subset B$. Then, by definition of A^*,

$$(A^*)^L(e) = \bigvee_{z\in G} A^L(z) \le \bigvee_{z\in G} B^L(z) \le B^L(e)$$

and

$$(A^*)^U(e) = \bigvee_{z\in G} A^U(z) \le \bigvee_{z\in G} B^U(z) \le B^U(e).$$

Thus $(A^*)^L(e) \le B^L(e)$ and $(A^*)^U(e) \le B^U(e)$.

Let $K_0 = \{x \in G : A(x) = [\bigvee_{z\in G} A^L(z), \bigvee_{z\in G} A^U(z)]\}$, let $\{B^{[t_i,\, s_i]}\}$ be the chain of level subgroups of B and let $e \neq x \in K_0$. Then

$$\bigvee_{z\in G} A^L(z) = A^L(x) \le B^L(x) \text{ and } \bigvee_{z\in G} A^U(z) = A^U(x) \le B^U(x).$$

Thus

$$\bigvee_{z\in G} A^L(z) \le \bigwedge_{x\in K_0} B^L(x) \text{ and } \bigvee_{z\in G} A^U(z) \le \bigwedge_{x\in K_0} B^U(x).$$

Let $[t_i, s_i] = [\bigwedge_{x\in K_0} B^L(x), \bigwedge_{x\in K_0} B^U(x)]$. Then $B^L(x) \ge t_i$ and $B^U(x) \ge s_i$ for each $x \in K_0$. Thus $K_0 \subset B^{[t_i,\, s_i]}$. Since $G_0 = \langle K_0\rangle, G_0 \subset B^{[t_i,\, s_i]}$. So $B^L(x) \ge t_i$ and $B^U(x) \ge s_i$ for each $x \in G_0$. Let $x \in G_0$. Then

$$(A^*)^L(x) = \bigvee_{z\in G} A^L(z) \le \bigwedge_{x\in K_0} B^L(x) = t_i \le B^L(x)$$

and

$$(A^*)^U(x) = \bigvee_{z\in G} A^U(z) \le \bigwedge_{x\in K_0} B^U(x) = s_i \le B^U(x).$$

Let $x \in G_1 \setminus G_0$. Then $\mu_{A^*}(x) = [\bigvee_{z\in G\setminus G_0} A^L(z), \bigvee_{z\in G\setminus G_0} A^U(z)]$ and $G_1 = \langle K_1\rangle$, where $K_1 = G_0 \cup \{x \in G : A(x) = [\bigvee_{z\in G\setminus G_0} A^L(z), \bigvee_{z\in G\setminus G_0} A^U(z)]\}$. We claim that $G_1 \subset B^{[t_{i_1},\, s_{i_1}]}$, where $[t_{i_1}, s_{i_1}] = [\bigwedge_{x\in K_1\setminus G_0} B^L(x), \bigwedge_{x\in K_1\setminus G_0} B^U(x)]$. Let $x \in K_1 \setminus G_0$. Then

$$A^L(x) = \bigvee_{z\in G\setminus G_0} A^L(z) \text{ and } A^U(x) = \bigvee_{z\in G\setminus G_0} A^U(z).$$

Since $A \subset B$,

$$\bigvee_{z \in G \setminus G_0} A^L(z) \leq \bigwedge_{x \in K_1 \setminus G_0} B^L(x) = t_{i_1} \leq B^L(x)$$

and

$$\bigvee_{z \in G \setminus G_0} A^U(z) \leq \bigwedge_{x \in K_1 \setminus G_0} B^U(x) = s_{i_1} \leq B^U(x).$$

Thus $x \in B^{[t_{i_1},\, s_{i_1}]}$. So $K_1 \setminus G_0 \subset B^{[t_{i_1},\, s_{i_1}]}$. Also $G_0 \subset B^{[t_i,\, s_i]} \subset B^{[t_{i_1},\, s_{i_1}]}$. Hence $G_1 = \langle K_1 \rangle \subset B^{[t_{i_1},\, s_{i_1}]}$. Therefore $B^L(x) \geq t_{i_1}$ and $B^L(x) \geq s_{i_1}$ for each $x \in G_1$. Let $x \in G_1 \setminus G_0$. Then

$$(A^*)^L(x) = \bigvee_{z \in G \setminus G_0} A^L(z) \leq t_{i_1} \leq B^L(x)$$

and

$$(A^*)^U(x) = \bigvee_{z \in G \setminus G_0} A^U(z) \leq s_{i_1} \leq B^U(x).$$

Proceeding as above, we can also see that

$$(A^*)^L(x) \leq B^L(x) \text{ and } (A^*)^U(x) \leq B^U(x)$$

for each $x \in G_i \setminus G_{i-1}, 2 \leq i \leq k$.
Consequentely, $A^* \subset B$. Hence A^* is the IVG generated by A. This completes the proof. □

The following is an example showing that the cardinality of the interval-valued fuzzy set A may not be equal to the cardinality of the image of the interval-valued fuzzy subgroup A^* generated by A.

Example 4.2. Let $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ be the octic group, where $a^4 = e = b^2$ and $ba = a^{-1}b$. Let $[t_i, s_i] \in D(I), 0 \leq i \leq 6$, be such that $t_0 > t_1 > \cdots > t_6$ and $s_0 > s_1 > \cdots > s_6$. We define a mapping $A : G \to D(I)$ as follows :

$A(e) = [t_0, s_0], A(a^2) = [t_1, s_1], A(a) = [t_2, s_2], A(a^3) = [t_3, s_3]$,
$A(b) = [t_4, s_4], A(ab) = A(a^2b) =]t_5, s_5], A(a^3b) = [t_6, s_6]$.

Then clearly $A \in D(I)^G$ but $A \notin \text{IVG}(G)$. Moreover,

$G_0 = \langle e \rangle = \{e\}, G_1 = \langle G_0, a^2 \rangle = \{e, a^2\}$,
$G_2 = \langle G_1, a \rangle = \{e, a, a^2, a^3\}$ and $G_3 = \langle G_2, b \rangle = G$.

By definition of A^*, we have that

$A^*(e) = [t_0, s_0], A^*(a^2) = [t_1, s_1], A^*(a) = A^*(a^3) = [t_2, s_2]$,

and

$A^*(b) = A^*(ab) = A^*(a^2b) = A^*(a^2b) = [t_4, s_4]$.

It is clear that A^* is the IVG generated by A and $|\mathrm{Im}A| > |\mathrm{Im}A^*|$. □

Definition 4.3. Let G be a group, let $A_1, A_2, A \in \mathrm{IVG}(G)$ and let $g \in G$.

(i) A_1 is said to be *conjugate* to A_2 if there exists an $a \in G$ such that $A_1(x) = A_2(a^{-1}xa)$ for each $x \in G$.

(ii) We define a mapping $A_g^* : G \to D(I)$ as follows :

$A_g^*(x) = A(g^{-1}xg)$ for each $x \in G$.

Then A_g^* is called the *interval-valued fuzzy conjugate subgroup* of G determined by A and $g \in G$.

It is clear that $A_g^* \in \mathrm{IVG}(G)$.

Proposition 4.4. Let G be any group and let $A \in \mathrm{IVG}(G)$ such that $\mathrm{Im}A = \{[t_0, s_0], [t_1, s_1], \cdots, [t_n, s_n]\}, t_0 > t_1 > \cdots > t_n$ and $s_0 > s_1 > \cdots > s_n$. If the chain of level subgroup of A is given by :

$A^{[t_0,\, s_0]} \subset A^{[t_1,\, s_1]} \subset \cdots \subset A^{[t_n,\, s_n]} = G,$

then the chain of level subgroups of A_x^* is given by :

$xA^{[t_0,\, s_0]}x^{-1} \subset xA^{[t_1,\, s_1]}x^{-1} \subset \cdots \subset xA^{[t_n,\, s_n]}x^{-1} = G,$

where A_x^* is the interval-valued fuzzy conjugate subgroup of G determined by A and $x \in G$.

Proof. Let $x \in G$. We define a mapping $\varphi_x : G \to G$ as follows :

$\varphi_x(a) = x^{-1}ax$ for each $a \in G$.

Then clearly $\varphi_x \in \mathrm{Aut}G$, where $\mathrm{Aut}G$ denotes the set of all automorphisms on G. Moreover, $A_x^* = A \circ \varphi_x = \varphi_x^{-1}(A)$. Then $\mathrm{Im}A_x^* = \mathrm{Im}A$. On the other hand,

$$\begin{aligned} a \in A_x^{*[t_i,\, s_i]} &\Leftrightarrow (A_x^*)^L(a) \geq t_i \text{ and } (A_x^*)^U(a) \geq s_i \\ &\Leftrightarrow A^L(x^{-1}ax) \geq t_i \text{ and } A^U(x^{-1}ax) \geq s_i \\ &\Leftrightarrow x^{-1}ax \in A^{[t_i,\, s_i]} \\ &\Leftrightarrow \varphi_x(a) \in A^{[t_i,\, s_i]} \\ &\Leftrightarrow a \in \varphi_x^{-1}(A^{[t_i,\, s_i]}) \\ &\Leftrightarrow a \in \varphi_{x^{-1}}(A^{[t_i,\, s_i}) \\ &\Leftrightarrow a \in xA^{[t_i,\, s_i]}x^{-1}. \end{aligned}$$

So $A_x^{*[t_i,\, s_i]} = xA^{[t_i,\, s_i]}x^{-1}$. This completes the proof. □

The following is the converse of Proposition 4.4.

Proposition 4.5. Let G be a group and let $A \in \text{IVG}(G)$ such that $\text{Im}A = \{[t_0, s_0], [t_1, t_1], \cdots, [t_n, s_n]\}, t_0 > t_1 > \cdots > t_n$ and $s_0 > s_1 > \cdots > s_n$. If the chain of level subgroups of A in G is

$$A^{[t_0,\ s_0]} \subset A^{[t_1,\ s_1]} \subset \cdots \subset A^{[t_n,\ s_n]} = G,$$

then there exists a $B \in \text{IVG}(G)$ such that the chain of level subgroups of B is given by :

$$xA^{[t_0,\ s_0]}x^{-1} \subset xA^{[t_1,\ s_1]}x^{-1} \subset \cdots \subset xA^{[t_n,\ s_n]}x^{-1} = G,$$

where $x \in G$ and $B = A^*_x$.

Proof. We define a mapping $B : G \to D(I)$ as follows : For each $g \in G$,

$$B(g) = \begin{cases} [t_0, s_0] & \text{if } g \in xA^{[t_0,\ s_0]}x^{-1}, \\ [t_i, s_i] & \text{if } g \in xA^{[t_i,\ s_i]}x^{-1} \setminus xA^{[t_{i-1},\ s_{i-1}]}x^{-1}, i = 1, \cdots, n. \end{cases}$$

Let $i \in \{1, 2, \cdots, n\}$ and let $g \in G$. Then

$$\begin{aligned} B(g) = [t_i, s_i] &\Leftrightarrow g \in xA^{[t_i,\ s_i]}x^{-1} \setminus xA^{[t_{i-1},\ s_{i-1}]}x^{-1} \\ &\Leftrightarrow x^{-1}gx \in A^{[t_i,\ s_i]} \setminus A^{[t_{i-1},\ s_{i-1}]} \\ &\Leftrightarrow A(x^{-1}gx) = [t_i,\ s_i] \\ &\Leftrightarrow A^*_x(g) = [t_i, s_i]. \end{aligned}$$

On the other hand,

$$\begin{aligned} B(g) = [t_0, s_0] &\Leftrightarrow g \in xA^{[t_0,\ s_0]}x^{-1} \Leftrightarrow x^{-1}gx \in A^{[t_0,\ s_0]} \\ &\Leftrightarrow A(x^{-1}gx) = [t_0, s_0] \\ &\Leftrightarrow A^*_x(g) = [t_0, s_0]. \end{aligned}$$

So $B = A^*_x$.

Now we define a mapping $\varphi_x : G \to G$ as follows:

$\varphi_x(g) = x^{-1}gx$ and for each $g \in G$.

Then clearly $\varphi_x \in \text{Aut}G$. Since $A \in \text{IVG}(G)$, by Result 2.B, $A^*_x = \varphi_x^{-1}(A) \in \text{IVG}(G)$. So $B = A^*_x \in \text{IVG}(G)$. Moreover, it is clear that

$\text{Im}A^*_x = \text{Im}A$ and $A^{*[t_i,\ s_i]}_x = xA^{[t_i,\ s_i]}x^{-1}$.

Hence the chain of level subgroup of B is given by :

$xA^{[t_0,\ s_0]}x^{-1} \subset xA^{[t_1,\ s_1]}x^{-1} \subset \cdots \subset xA^{[t_n,\ s_n]}x^{-1} = G$.

This completes the proof. □

A subgroup H of a group G is called a *characteristic subgroup* of G if $f(H) = H$ for each $f \in \text{Aut}(G)$.

Definition 4.6. Let A be an IVG of a group G. Then A is called an *interval-valued fuzzy characteristic subgroup* of G if $\varphi^{-1}(A) = A$ for each $\varphi \in \text{Aut}G$.

Proposition 4.7. Let G be a finite group and let A be an interval-valued fuzzy characteristic subgroup of G. Then each level subgroup of A is a characteristic subgroup of G.
Proof. Since G is a finite group, $|\text{Im}A| < \infty$. Let $[\lambda, \mu] \in \text{Im}A$ and let $\varphi \in \text{Aut}G$. Since A is an interval-valued fuzzy characteristic subgroup of G, $\varphi^{-1}(A) = A$. Let $x \in A^{[\lambda,\ \mu]}$. Then

$$A^L(\varphi(x)) = \varphi^{-1}(A)^L(x) = A^L(x) \geq \lambda$$

and

$$A^U(\varphi(x)) = \varphi^{-1}(A)^U(x) = A^U(x) \geq \mu.$$

Thus $\varphi(x) \in A^{[\lambda,\ \mu]}$. So $\varphi(A^{[\lambda,\ \mu]}) \subset A^{[\lambda,\ \mu]}$. Hence $A^{[\lambda,\ \mu]}$ is a characteristic subgroup of G. □

The following is the converse of Proposition 4.7.

Proposition 4.8. Let G be a finite group and let $A \in \text{IVG}(G)$. If each level subgroup of A is a characteristic subgroup of G, then A is an interval-valued fuzzy characteristic subgroup of G.
Proof. Since G is a finite, $|\text{Im}A| < \infty$. Let $\text{Im}A = \{[t_0, s_0], [t_1, s_1], \cdots, [t_n, s_n]\}$ such that $t_0 > t_1 > \cdots > t_n$ and $s_0 > s_1 > \cdots > s_n$. Then, by the hypothesis, $A^{[t_i,\ s_i]}$ is a characteristic subgroup of G for each $i = 0, \cdots, n$. Let $\varphi \in \text{Aut}G$. Then clearly, $\text{Im}\varphi^{-1}(A) = \text{Im}A$. On the other hood, for each $i = 0, \cdots, n$,

$$\begin{aligned} x \in (\varphi^{-1}(A))^{[t_i,\ s_i]} &\Leftrightarrow \varphi^{-1}(A)^L(x) = A^L(\varphi(x)) \geq t_i \text{ and} \\ &\quad\ \ \varphi^{-1}(A)^U(x) = A^U(\varphi(x)) \geq s_i \\ &\Leftrightarrow \varphi(x) \in A^{[t_i,\ s_i]} \\ &\Leftrightarrow x \in \varphi^{-1}(A^{[t_i,\ s_i]}) \\ &\Leftrightarrow x \in A^{[t_i,\ s_i]}. \end{aligned}$$

Thus $\varphi^{-1}(A) = A$. So A is an interval-valued fuzzy characteristic subgroup of G. □

Result 4.B[5, Lemma 2.16]. Let A be an IVG of a group G and let $x \in G$. Then $A(x) = [\lambda, \mu]$ if and only if $x \in A^{[\lambda,\ \mu]}$ and $x \notin A^{[t,\ s]}$ for each $[t, s] \in D(I)$ such that $t > \lambda$ and $s > \mu$.

The following is the generalization of Propositions 4.7 and 4.8.

Theorem 4.9. Let G be a group and let $A \in \mathrm{IVG}(G)$. Then A is an interval-valued fuzzy characteristic subgroup of G if and only if each level subgroup of A is a characteristic subgroup of G.

Proof. $(\Rightarrow)$: Suppose A is an interval-valued fuzzy characteristic subgroup of G. Let $[\lambda, \mu] \in \mathrm{Im}A$, let $\varphi \in \mathrm{Aut}G$ and let $x \in A^{[\lambda,\, \mu]}$. Then, by the hypothesis,

$$A^L(\varphi(x)) = A^L(x) \geq \lambda \text{ and } A^U(\varphi(x)) = A^U(x) \geq \mu.$$

Thus $\varphi(x) \in A^{[\lambda,\, \mu]}$. So $\varphi(A^{[\lambda,\, \mu]}) \subset A^{[\lambda,\, \mu]}$. Now let $x \in A^{[\lambda,\, \mu]}$ and let $g \in G$ such that $\varphi(g) = x$. Then

$$A^L(g) = A^L(\varphi(g)) = A^L(x) \geq \lambda$$

and

$$A^U(g) = A^U(\varphi(g)) = A^U(x) \geq \lambda.$$

Thus $g \in A^{[\lambda,\, \mu]}$. So $x \in \varphi(A^{[\lambda,\, \mu]})$. Hence $A^{[\lambda,\, \mu]} \subset \varphi(A^{[\lambda,\, \mu]})$, i.e., $\varphi(A^{[\lambda,\, \mu]}) = A^{[\lambda,\, \mu]}$. Therefore $A^{[\lambda,\, \mu]}$ is a characteristic subgroup of G for each $[\lambda, \mu] \in \mathrm{Im}A$.

$(\Leftarrow)$: Suppose the necessary condition holds. Let $x \in G$, let $\varphi \in \mathrm{Aut}G$ and let $A(x) = [\lambda, \mu]$. Then, by Result 4.B, $x \in A^{[\lambda,\, \mu]}$ but $x \notin A^{[t,\, s]}$ for all $[t, s] \in D(I)$ such that $t > \lambda$ and $s > \mu$. By the hypothesis, $\varphi(A^{[\lambda,\, \mu]}) = A^{[\lambda,\, \mu]}$. Thus $\varphi(x) \in A^{[\lambda,\, \mu]}$. So $A^L(\varphi(x)) \geq \lambda$ and $A^U(\varphi(x)) \geq \mu$. Let $A(\varphi(x)) = [\varphi^{-1}(A)](x) = [t, s]$. If possible, let $t > \lambda$ and $s > \mu$. Then $\varphi(x) \in A^{[t,\, s]} = \varphi(A^{[t,\, s]})$. Since φ is injective, $x \in A^{[t,\, s]}$. This contradicts the fact that $x \notin A^{[t,\, s]}$. So $A(\varphi(x)) = [\lambda, \mu] = A(x)$, i.e., $\varphi^{-1}(A) = A$. Hence A is an interval-valued fuzzy characteristic fuzzy subgroup of G. This completes the proof. □

Theorem 4.10. Let G be a finite group and let $f : G \to G'$ be a group epimorphism. Let $A \in \mathrm{IVG}(G)$ such that $\mathrm{Im}A = \{[t_0, s_0], [t_1, s_1], \cdots, [t_n, s_n]\}, t_0 > t_1 > \cdots > t_n$ and $s_0 > s_1 > \cdots > s_n$. If the chain of level subgroups of A is

$$A^{[t_0,\, s_0]} \subset A^{[t_1,\, s_1]} \subset \cdots \subset A^{[t_n,\, s_n]} = G,$$

then the chain of level subgroups of $f(A)$ is

$f(A^{[t_0,\ s_0]}) \subset f(A^{[t_1,\ s_1]}) \subset \cdots \subset f(A^{[t_n,\ s_n]}) = G'$.

Proof. By Result 2.C, $f(A) \in \text{IVG}(G')$. It is clear that $\text{Im}f(A) \subset \text{Im}A$. Let $[t_i, s_i] \in \text{Im}f(A)$ and let $y \in (f(A))^{[t_i,\ s_i]}$. Then

$f(A)^L(y) = \bigvee_{z \in f^{-1}(y)} A^L(z) \geq t_i$

and

$f(A)^U(y) = \bigvee_{z \in f^{-1}(y)} A^U(z) \geq s_i$.

Since G is a finite group, it is clear that A has sup property. Then there exists a $z_0 \in G$ such that $f(z_0) = y$,

$$\bigvee_{z \in f^{-1}(y)} A^L(z) = A^L(z_0) \geq t_i$$

and

$$\bigvee_{z \in f^{-1}(y)} A^U(z) = A^U(z_0) \geq s_i.$$

Thus $z_0 \in A^{[t_i,\ s_i]}$. So $y = f(z_0) \in f(A^{[t_i,\ s_i]})$. Hence $(f(A))^{[t_i,\ s_i]} \subset f(A^{[t_i,\ s_i]})$.

Now let $f(x) \in f(A^{[t_i,\ s_i]})$. Then there exists $x \in A^{[t_i,\ s_i]}$ such that $f(x) = y$. Thus $A^L(x) \geq t_i$ and $A^U(x) \geq s_i$. So

$$f(A)^L(f(x)) = \bigvee_{z \in f^{-1}(y)} A^L(z) \geq t_i$$

and

$$f(A)^U(f(x)) = \bigvee_{z \in f^{-1}(y)} A^U(z) \geq s_i.$$

Hence $f(x) \in (f(A))^{[t_i,\ s_i]}$, i.e., $f(A^{[t_i,\ s_i]}) \subset (f(A))^{[t_i,\ s_i]}$. Therefore $f(A^{[t_i,\ s_i]}) = (f(A))^{[t_i,\ s_i]}$. This completes the proof. □

Lastly, in view of the study of level subgroups of an interval-valued fuzzy subgroup, we can recast Proposition 4.12 in [7] as follows.

Proposition 4.11. Let G be a finite cyclic group of prime order. Then $A \in \text{IVG}(G)$ if and only if the chain of level subgroups of A consists of only trivial subgroups of G.

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