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$(\in, \in \vee q_k)$-FUZZY IDEALS IN LEFT REGULAR ORDERED $\mathcal{LA}$-SEMIGROUPS

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Abstract. We generalize the idea of $(\in, \in \vee q_k)$-fuzzy ordered semigroup and give the concept of $(\in, \in \vee q_k)$-fuzzy ordered $\mathcal{LA}$-semigroup. We show that $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideals, $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals, $(\in, \in \vee q_k)$-fuzzy interior ideals and $(\in, \in \vee q_k)$-fuzzy $(1, 2)$-ideals need not to be coincide in an ordered $\mathcal{LA}$-semigroup but on the other hand, we prove that all these $(\in, \in \vee q_k)$-fuzzy ideals coincide in a left regular class of an ordered $\mathcal{LA}$-semigroup. Further we investigate some useful conditions for an ordered $\mathcal{LA}$-semigroup to become a left regular ordered $\mathcal{LA}$-semigroup and characterize a left regular ordered $\mathcal{LA}$-semigroup in terms of $(\in, \in \vee q_k)$-fuzzy one-sided ideals. Finally we connect an ideal theory with an $(\in, \in \vee q_k)$-fuzzy ideal theory by using the notions of duo and $(\in, \in \vee q_k)$-fuzzy duo.

1. Introduction and Preliminaries

The concept of fuzzy sets was first proposed by Zadeh [17] in 1965, which has a wide range of applications in various fields such as computer engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. It gives us a tool to model the uncertainty present in a phenomena that does not have sharp boundaries. Many papers on fuzzy sets have been published, showing the importance and their applications to set theory, algebra, real analysis, measure theory and topology etc.

Murali [10] defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. In

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[13], the idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined. The concept of a (α, β)-fuzzy subgroup was first considered by Bhakat and Das in [2] and [3] by using the "belongs to" relation $(\in)$ and "quasi coincident with" relation (q) between a fuzzy poit and a fuzzy subgroup. The idea of a (α, β)-fuzzy subgroup is a viable generalization of Rosenfeld's fuzzy subgroup [14]. The concept of a $(\in, \in \vee q)$-fuzzy sub-near-rings of a near-ring was introduced by Davvaz [4]. Jun et. al. gave the concept of $(\in, \in \vee q)$-fuzzy ordered semigroups [6]. Moreover, $(\in, \in \vee q_k)$-fuzzy ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals of a semigroup are defined in [15]. In [7], Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup.

In mathematics, algebraic structures play an important role with wide range of applications in many fields such as theoretical physics, information sciences and many more. This provides enough inspiration to review various concepts and results from the field of abstract algebra in broader frameworks of fuzzy theory.

The concept of a left almost semigroups ($\mathcal{LA}$-semigroup) was first given by M. A. Kazim and M. Naseeruddin [8] in 1972. An $\mathcal{LA}$-semigroup is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. An $\mathcal{LA}$-semigroup with a right identity becomes a commutative semigroup with an identity [11]. The connection between a commutative inverse semigroup and an $\mathcal{LA}$-semigroup was given in [12] as follows: a commutative inverse semigroup $(S, \circ)$ becomes an $\mathcal{LA}$-semigroup $(S, \cdot)$ where $a \cdot b = b \circ a^{-1}$, for all $a, b \in S$. An $\mathcal{LA}$-semigroup S with a left identity becomes a semigroup under the binary operation "$\circ$" defined as follows: for all $x, y \in S$ and for a fixed element $a \in S$, $x \circ y = (xa)y$ [16]. An $\mathcal{LA}$-semigroup is a generalization of a semigroup [11] and has applications in connection with semigroups as well as with other branches of mathematics.

An $\mathcal{LA}$-semigroup [8] is a groupoid S satisfying the following left invertive law

$$(ab)c = (cb)a, \quad \text{for all } a, b, c \in S. \tag{1}$$

In an $\mathcal{LA}$-semigroup, the medial law [8] holds

$$(ab)(cd) = (ac)(bd), \quad \text{for all } a, b, c, d \in S. \tag{2}$$

If a left identity in an $\mathcal{LA}$-semigroup exists, then it is unique [11]. An $\mathcal{LA}$-semigroup S with a left identity satisfies the following laws

$$(ab)(cd) = (dc)(ba), \quad \text{for all } a, b, c, d \in S. \tag{3}$$

$$a(bc) = b(ac), \quad \text{for all } a, b, c \in S. \tag{4}$$

An ordered $\mathcal{LA}$-semigroup (po-$\mathcal{LA}$-semigroup) [9] is a structure $(S, ., \leq)$ in which the following conditions hold:

(i) $(S, .)$ is an $\mathcal{LA}$-semigroup.

(ii) $(S, \leq)$ is a poset.

(iii) For all a, b and $x \in S$, $a \leq b$ implies that $ax \leq bx$ and $xa \leq xb$.

Example 1. [1] *Consider an $\mathcal{LA}$-semigroup $S = \{a, b, c, d, e\}$ with a left identity d.*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

Then S becomes an ordered $\mathcal{LA}$-semigroup with the order below.

$$\leq := \{(a,a), (a,b), (c,c), (a,c), (d,d), (a,e), (e,e), (b,b)\}.$$

In this paper S denotes an ordered $\mathcal{LA}$-semigroup.

A fuzzy subset or a fuzzy set of a non-empty set S is an arbitrary mapping $f : S \to [0,1]$, where $[0,1]$ is the unit segment of the real line. A fuzzy subset f is a class of objects endowed with membership grades, having the form $f = \{(s,\ f(s)) \mid s \in S\}$.

Set $x \in S$ and $A_x = \{(y,z) \in S \times S \mid x \leq yz\}$.

Assume that S is an ordered $\mathcal{LA}$-semigroup and let $F(S)$ denote the set of all fuzzy subsets of S. Then $(F(S), \circ, \subseteq)$ is an ordered $\mathcal{LA}$-semigroup [9].

For $\emptyset \neq A \subseteq S$, we define

$$(A] = \{t \in S \mid t \leq a, \quad \text{for some } a \in A\}.$$

If $A = \{a\}$, then we usually write $(a]$.

A non-empty subset A of an S is called a left (right) ideal of S if

(i) $SA \subseteq A$ $(AS \subseteq A)$.

(ii) If $a \in A$ and $b \in S$ are such that $b \leq a$, then $b \in A$.

Equivalently, if $(SA] \subseteq A$ $((AS] \subseteq A)$.

A non-empty subset A of an S is called an interior ideal of S if

(i) $(SA)S \subseteq A$.

(ii) If $a \in A$ and $b \in S$ are such that $b \leq a$, then $b \in A$.

Equivalently, if $((SA)S] \subseteq A$.

A subset A of S is called a two-sided ideal of S if it is both a left and a right ideal of S.

For $\emptyset \neq A \subseteq S$ and $k \in [0,1)$, the k-characteristic function $(C_A)_k$ is defined by

$$(C_A)_k = \begin{cases} \frac{1-k}{2} & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Let f and g be any two fuzzy subsets of S. We define the product $f \circ_k g$ by

$$(f \circ_k g)(x) = \left\{ \begin{array}{ll} \bigvee\limits_{(y,z)\in A_x} \{f(y) \wedge g(z) \wedge \frac{1-k}{2}\} & \text{if } A_x \neq \emptyset. \\ 0 & \text{if } A_x = \emptyset. \end{array} \right. , \text{ where } k \in [0,1)$$

For $k \in [0,1)$, the symbols $f \cap_k g$ and $f \cup_k g$ mean the following fuzzy subsets of S :

$$\begin{aligned} (f \cap_k g)(x) &= f(x) \wedge g(x) \wedge \frac{1-k}{2}, \text{ for all } x \in S. \\ (f \cup_k g)(x) &= f(x) \wedge g(x) \wedge \frac{1-k}{2}, \text{ for all } x \in S. \end{aligned}$$

For $k \in [0,1)$, the order relation $\subseteq_k$ between any two fuzzy subsets f and g of S is defined by

$$f \subseteq_k g \text{ if and only if } f(x) \leq g(x) \wedge \frac{1-k}{2}, \text{ for all } x \in S.$$

2. Basic definitions and results

In what follows, $k \in [0,1)$ and $t, r \in (0,1]$ unless otherwise specified.
A fuzzy subset f of S of the form

$$f(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy subset f in a set S, Pu and Liu [13] gave meaning to the symbol $x_t \alpha f$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. quasi-coincident with) a fuzzy set f written $x_t \in f$ (resp. $x_t \in qf$) if $f(x) \geq t$ (resp. $f(x) + t > 1$), and in this case, $x_t \in \vee qf$ (resp. $x_t \in \wedge qf$) means that $x_t \in f$ or $x_t \in qf$ (resp. $x_t \in f$ and $x_t \in qf$). To say that $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold. Generalizing the concept of $x_t qf$, Jun [5] defined $x_t q_k f$ if $f(x) + t + k > 1$ and $x_t \in \vee q_k f$ if $x_t \in f$ or $x_t q_k f$.

In this section, we introduce and study different types of $(\in, \in \vee q_k)$-fuzzy ideals in an ordered $\mathcal{LA}$-semigroup. Note that $(\in, \in \vee q)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy ideal are the particular types of (α, β)-fuzzy ideals. An $(\in, \in \vee q_k)$-fuzzy ideal can be seen as a generalization of an $(\in, \in \vee q)$-fuzzy ideal. Indeed

$$(\in, \in \vee q_k) \Rightarrow (\in, \in \vee q), \text{ for } k = 0.$$

Moreover an $(\in, \in \vee q)$-fuzzy ideal generalizes the notion of a fuzzy ideal.

Definition 1. *A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if*

(i) For all $x,\ y \in S,\ x \leq y,\ y_t \in f \Longrightarrow x_t \in \vee q_k f$.
(ii) For all $x,\ y, z \in S,\ y_t \in f \Longrightarrow (xy)_t \in \vee q_k f \quad (y_t \in f \Longrightarrow (yx)_t \in \vee q_k f)$.

Theorem 1. *If f is a fuzzy subset of S, then f is an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if and only if*
(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, for all $x, y \in S$.
(ii) $f(xy) \geq f(y) \wedge \frac{1-k}{2} \quad \left(f(xy) \geq f(x) \wedge \frac{1-k}{2}\right)$, for all $x, y \in S$.

Proof. It is immediate. □

Corollary 1. *If $k = 0$, then f is an $(\in, \in \vee q)$-fuzzy left (right) ideal of S if and only if*

(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, for all $x, y \in S$.
(ii) $f(xy) \geq f(y) \wedge \frac{1-k}{2} \quad \left(f(xy) \geq f(x) \wedge \frac{1-k}{2}\right)$, for all $x, y \in S$.

Definition 2. *A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup of S if for all $x,\ y \in S,\ x_t \in f$ and $y_r \in f \Longrightarrow (xy)_{t\wedge r} \in \vee q_k f$.*

Theorem 2. *If f is a fuzzy subset of S, then f is an $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup of S if and only if $f(xy) \geq f(x) \wedge f(y) \wedge \frac{1-k}{2}$, for all $x, y \in S$.*

Proof. It is immediate. □

Corollary 2. *If $k = 0$, then f is an $(\in, \in \vee q)$-fuzzy $\mathcal{LA}$-subsemigroup of S if and only if $f(xy) \geq f(x) \wedge f(y) \wedge \frac{1-k}{2}$, for all $x, y \in S$.*

Definition 3. *A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if*

(i) For all $x,\ y \in S,\ x \leq y,\ y_t \in f \Longrightarrow x_t \in \vee q_k f$.
(ii) For all $x,\ y, z \in S,\ x_t \in f$ and $z_r \in f \Longrightarrow ((xy)z)_{t\wedge r} \in \vee q_k f$.

Theorem 3. *If f is a fuzzy subset of S, then f is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if and only if*

(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.

(ii) $f((xy)z) \geq f(x) \wedge f(z) \wedge \frac{1-k}{2}$, *for all* $x, y, z \in S$.

Proof. Assume that f is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. Let $x, y \in S$ be such that $x \leq y$. If $f(y) = 0$, then $f(x) \geq f(y) \wedge \frac{1-k}{2}$. Let $f(y) \neq 0$ and assume on contrary that $f(x) < f(y) \wedge \frac{1-k}{2}$. Choose $t \in (0, 1]$ such that $f(x) < t < f(y) \wedge \frac{1-k}{2}$. If $f(y) < \frac{1-k}{2}$, then $f(x) < t < f(y)$ so $y_t \in f$ but $f(x)+t+k < \frac{1-k}{2}+\frac{1-k}{2}+k = 1$ implies that $x_t \overline{q_k} f$, therefore $x_t \overline{\in \vee q_k} f$, which is a contradiction. Hence $f(x) \geq f(y) \wedge \frac{1-k}{2}$, for all $x, y \in S$.

Suppose on contrary that there exist $x, y, z \in S$ such that $f((xy)z) < f(x) \wedge f(z) \wedge \frac{1-k}{2}$. Choose $t \in (0, 1]$ such that $f((xy)z) < t < f(x) \wedge f(z) \wedge \frac{1-k}{2}$. Then $f(x) > t$ and $f(z) > t$ implies that $x_t \in f$ and $z_t \in f$ but $f((xy)z) < t$ and $f((xy)z) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ implies that $((xy)z)_{t\wedge r} \overline{q_k} f$ so $((xy)z)_{t\wedge r} = ((xy)z)_t \overline{\in \vee q_k} f$, which is a contradiction. Hence $f((xy)z) \geq f(x) \wedge f(z) \wedge \frac{1-k}{2}$, for all $x, y, z \in S$.

Conversely let $x, y \in S$ be such that $x \leq y$ and $y_t \in f$. Then $f(y) \geq t$ and since $x \leq y$ it follows that $f(x) \geq f(y) \geq t$ implies that $f(x) \geq t$, which shows that $x_t \in f$ and therefore $x_t \in \vee q_k f$.

Let $x, y, z \in S$ be such that $x_t \in f$ and $z_r \in f$. Then $f(x) \geq t$ and $f(z) \geq r$. By hypothesis, $f((xy)z) \geq f(x) \wedge f(z) \wedge \frac{1-k}{2} \geq t \wedge r \wedge \frac{1-k}{2}$. If $t \wedge r \leq \frac{1-k}{2}$, then $f((xy)z) \geq t \wedge r$ so $((xy)z)_{t\wedge r} \in f$ and if $t \wedge r > \frac{1-k}{2}$, then $f((xy)z) \geq \frac{1-k}{2}$ and therefore $f((xy)z) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. Thus $((xy)z)_{t\wedge r} q_k f$ implies that $((xy)z)_{t\wedge r} \in \vee q_k f$. This shows that f is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S. □

Corollary 3. *If $k = 0$, then f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S if and only if*

(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.

(ii) $f((xy)z) \geq f(x) \wedge f(z) \wedge \frac{1-k}{2}$, *for all* $x, y, z \in S$.

Definition 4. *If an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S is also an $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup of S, then f is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S*

Theorem 4. *An $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup f of S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if*

(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.

(ii) $f((xy)z) \geq f(x) \wedge f(z) \wedge \frac{1-k}{2}$, *for all* $x, y, z \in S$.

Proof. It can be followed from Theorems 2 and 3. $\square$

Corollary 4. *For $k = 0$, an $(\in, \in \vee q)$-fuzzy $\mathcal{LA}$-subsemigroup f of S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if*

(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.
(ii) $f((xy)z) \geq f(x) \wedge f(z) \wedge \frac{1-k}{2}$, *for all* $x, y, z \in S$.

Definition 5. *A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if*

(i) *For all* x, $y \in S$, $x \leq y$, $y_t \in f \Longrightarrow x_t \in \vee q_k f$.
(ii) *For all* x, $y, z \in S$, $y_t \in f \Longrightarrow ((xy)z)_t \in \vee q_k f$.

Theorem 5. *If f is a fuzzy subset of S, then f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if and only if*
(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.
(ii) $f((xy)z) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y, z \in S$.

Proof. It is similar to the proof of Theorem 3. $\square$

Corollary 5. *If $k = 0$, then f is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if*

(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.
(ii) $f((xy)z) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y, z \in S$.

Definition 6. *A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$-ideal of S if*

(i) *For all* x, $y \in S$, $x \leq y$, $y_t \in f \Longrightarrow x_t \in \vee q_k f$.
(ii) *For all* a, x, $y, z \in S$, $x_t \in f$, $y_r \in f$ *and* $z_s \in f \Longrightarrow ((xa)(yz))_{(t \wedge r) \wedge s}$ $\in \vee q_k f$.

Theorem 6. *An $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup f of S is an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$-ideal S if and only if*
(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.
(ii) $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2}$, *for all* $a, x, y, z \in S$.

Proof. It is similar to the proof of Theorem 3. $\square$

Corollary 6. *For $k = 0$, an $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup f of S is an $(\in, \in \vee q)$-fuzzy $(1, 2)$-ideal S if and only if*

(i) $x \leq y \Longrightarrow f(x) \geq f(y) \wedge \frac{1-k}{2}$, *for all* $x, y \in S$.
(ii) $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2}$, *for all* $a, x, y, z \in S$.

3. $(\in, \in \vee q_k)$-fuzzy ideals

Ideal theory play a very important role in studying and exploring the structural properties of different algebraic structures. Here we study different types of $(\in, \in \vee q_k)$-fuzzy ideals which usually allow us to characterize an $\mathcal{LA}$-semigroup and play the role in an $\mathcal{LA}$-semigroup which is played by $(\in, \in \vee q_k)$-fuzzy normal subgroups in group theory and by $(\in, \in \vee q_k)$-fuzzy ideals in ring theory.

In Theorems 7 and 8 of this section, we characterize $(\in, \in \vee q_k)$-fuzzy (left, right, two-sided, generalized bi-, bi-, interior, $(1, 2)$-) ideals and compare all these $(\in, \in \vee q_k)$-fuzzy ideals with each other to analyze their structural behaviors under a special case by considering them in a left regular class of an ordered $\mathcal{LA}$-semigroup with a left identity.

Lemma 1. *For a fuzzy subset f of S, the following conditions are true.*

(i) f_k is a fuzzy left (right) ideal of S if and only if $x \leq y \Rightarrow f(x) \geq f(y)$, for all $x, y \in S$ and $S \circ_k f \subseteq f_k$ $(f \circ_k S \subseteq f_k)$.

(ii) f_k is a fuzzy $\mathcal{LA}$-subsemigroup of S if and only if $f \circ_k f \subseteq f_k$.

Proof. It is straightforward. □

Definition 7. *An element a of S is called a left (right) regular element of S if there exist any $x, y \in S$ such that $a \leq xa^2$ $(a \leq a^2 y)$ and S is called left (right) regular if every element of S is left (right) regular.*

Remark 1. [9] *The concepts of left and right regularity coincide in an ordered $\mathcal{LA}$-semigroup with a left identity.*

Theorem 7. *The following properties hold in a left regular S with a left identity.*

(i) f is an $(\in, \in \vee q_k)$-fuzzy two-sided ideal f of $S \Leftrightarrow S \circ_k f = f_k = f \circ_k S$.

(ii) f is an $(\in, \in \vee q_k)$-fuzzy bi-(generalized bi-) ideal of $S \Leftrightarrow (f \circ_k S) \circ_k f = f_k = f \circ_k f$.

(iii) f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of $S \Leftrightarrow (S \circ_k f) \circ_k S = f_k$.

(iv) f is an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$-ideal of $S \Leftrightarrow (f \circ_k S) \circ_k (f \circ_k f) = f_k = f \circ_k f$.

Proof. Let S be a left regular ordered $\mathcal{LA}$-semigroup with a left identity.

(i). $\Rightarrow$ Assume that f is an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S. Now for $a \in S$, there exists some $x \in S$ such that $a \leq a^2 x$. Then by (1), we obtain $a \leq (aa)x = (xa)a$. Thus $(xa, a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned}(f \circ_k S)(a) &= (f \circ S)(a) \wedge \frac{1-k}{2} = \bigvee_{(xa,a)\in A_a} \{f(xa) \wedge S(a)\} \wedge \frac{1-k}{2} \\ &\geq f(xa) \wedge S(a) \wedge \frac{1-k}{2} \geq f(a) \wedge 1 \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge \frac{1-k}{2} = f_k(a),\end{aligned}$$

and it is easy to see that $S \circ_k f = f_k$ also holds for all $(\in, \in \vee q_k)$-fuzzy two-sided ideal f of S. $\Leftarrow$ The converse is simple.

(ii). $\Rightarrow$ Let f be an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S. Now for $a \in S$, there exists some $x \in S$ such that $a \leq a^2x$. Then by (1), (4) and (3), we obtain

$$\begin{aligned} a &\leq (aa)x = (xa)a \leq (x((aa)x))a = ((aa)(xx))a = ((xx)(aa))a \\ &= (((aa)x)x)a = (((xa)a)x)a \leq (((x((aa)x))a)x)a \\ &= ((((aa)(xx))a)x)a = ((((xx)(aa))a)x)a(((a(x^2a))a)x)a = pa,\end{aligned}$$

where $p = ((a(x^2a))a)x$ and $p = qx$, where $q = (a(x^2a))a$. Thus $(p, a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned}((f \circ_k S) \circ_k f)(a) &= \bigvee_{(p,a)\in A_a} \left\{(f \circ_k S)(((a(x^2a))a)x) \wedge f(a) \wedge \frac{1-k}{2}\right\} \\ &\geq (f \circ_k S)(((a(x^2a))a)x) \wedge f(a) \wedge \frac{1-k}{2} \\ &= \bigvee_{(q,x)\in A_{qx}} \left\{f((a(x^2a))a) \wedge S(x) \wedge \frac{1-k}{2}\right\} \\ &\quad \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq f((a(x^2a))a) \wedge 1 \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge f(a) \wedge \frac{1-k}{2} \wedge f(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge \frac{1-k}{2} = f_k(a).\end{aligned}$$

Now by (1), (4) and (3), we obtain

$$\begin{aligned} a &\leq (aa)x = (xa)a \leq (x((aa)x))a = ((aa)(xx))a = ((xx)(aa))a \\ &= (a(x^2a))a = pa,\end{aligned}$$

where $p = a(x^2a)$ and $p = aq$, where $q = x^2a$. Thus $(p,a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned}
((f \circ_k S) \circ_k f)(a) &= \bigvee_{(p,a)\in A_a} \left\{ (f \circ_k S)((a(x^2a))) \wedge f(a) \wedge \frac{1-k}{2} \right\} \\
&= \bigvee_{(p,a)\in A_a} \left(\bigvee_{(a,q)\in A_{aq}} \left\{ f(a) \wedge S(q) \wedge \frac{1-k}{2} \right\} \wedge f(a) \right. \\
&\qquad \left. \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{(p,a)\in A_a} \left(\bigvee_{(a,q)\in A_{aq}} f(a) \wedge 1 \wedge f(a) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{(p,a)\in A_a} \left(\bigvee_{(a,q)\in A_{aq}} f(a) \wedge f(a) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{(p,a)\in A_a} \left(f(a) \wedge f(a) \wedge \frac{1-k}{2} \right) \wedge \frac{1-k}{2} \\
&\leq \bigvee_{(p,a)\in A_a} \left\{ f((a(x^2a))a) \right\} \wedge \frac{1-k}{2} \\
&= f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}$$

Thus $(f \circ_k S) \circ_k f = f_k$.

Now by (1), (4) and (3), we obtain

$$\begin{aligned}
a &\leq (aa)x = (xa)a \leq (x((aa)x))a = ((aa)(xx))a = (((xx)a)a)a \\
&\leq (((xx)((aa)x))a)a = (((xx)((xa)a))a)a = (((xx)((ae)(ax)))a)a \\
&= (((xx)(a((ae)x)))a)a = ((a((xx)((ae)x)))a)a = pa,
\end{aligned}$$

where $p = (a((xx)((ae)x)))a$. Thus $(p,a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned}
(f \circ_k f)(a) &= \bigvee_{(p,a)\in A_a} \left\{ f((a((xx)((ae)x)))a) \wedge f(a) \wedge \frac{1-k}{2} \right\} \\
&\geq f((a((xx)((ae)x)))a) \wedge f(a) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge f(a) \wedge f(a) \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}$$

By Lemma 1, $f \circ_k f = f_k$.

$\Leftarrow$ Let f be a fuzzy subset of a left regular ordered $\mathcal{LA}$-semigroup S with a left identity. Then

$$\begin{aligned} f((xy)z) &= ((f \circ_k S) \circ_k f)((xy)z) \\ &= \bigvee_{(xy,z) \in A_{(xy)z}} \left\{ (f \circ_k S)(xy) \wedge f(z) \wedge \frac{1-k}{2} \right\} \\ &\geq \bigvee_{(x,y) \in A_{xy}} \left\{ f(x) \wedge S(y) \wedge \frac{1-k}{2} \right\} \wedge f(z) \wedge \frac{1-k}{2} \\ &\geq f(x) \wedge 1 \wedge f(z) \wedge \frac{1-k}{2} = f(x) \wedge f(z) \wedge \frac{1-k}{2}. \end{aligned}$$

Thus by $f \circ_k f = f_k$ and Lemma 1, it follows that f is an $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup of S. This shows that f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

(iii). It is immediate.

(iv). $\Rightarrow$ Let f be an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S. Now for $a \in S$, there exists some $x \in S$ such that $a \leq a^2 x$. Then by (1) and (4), we obtain

$$\begin{aligned} a &\leq (aa)x = (xa)a \leq (xa)((aa)x) = (aa)((xa)x) \leq (a((aa)x))((xa)x) \\ &= ((aa)(ax))((xa)x) = (((xa)x)(ax))(aa) = (a(((xa)x)x))(aa) \\ &= p(aa), \end{aligned}$$

where $p = a(((xa)x)x) = aq$, where $q = ((xa)x)x$. Thus $(p, (aa)) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned} ((f \circ_k S) \circ_k (f \circ_k f))(a) &= \bigvee_{(p,(aa)) \in A_a} \left\{ (f \circ_k S)(p) \wedge (f \circ_k f)(aa) \wedge \frac{1-k}{2} \right\} \\ &\geq (f \circ_k S)(a(((xa)x)x)) \wedge (f \circ_k f)(aa) \wedge \frac{1-k}{2}. \end{aligned}$$

Now

$$\begin{aligned} (f \circ_k S)(a(((xa)x)x)) &= \left\{ (f \circ S)(a(((xa)x)x)) \wedge \frac{1-k}{2} \right\} \\ &= \bigvee_{(a,q) \in A_{aq}} \left\{ \{ f(a) \wedge S(((xa)x)x) \} \wedge \frac{1-k}{2} \right\} \\ &\geq f(a) \wedge S(((xa)x)x) \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2} \\ &= f_k(a), \end{aligned}$$

and

$$\begin{aligned}(f\circ_k f)(aa) &= (f\circ f)(aa)\wedge\frac{1-k}{2}=\bigvee_{(a,a)\in A_{aa}}\{f(a)\wedge f(a)\}\wedge\frac{1-k}{2}\\ &\geq f(a)\wedge\frac{1-k}{2}=f_k(a).\end{aligned}$$

Thus we get

$$((f\circ_k S)\circ_k(f\circ_k f))(a)\geq f_k(a).$$

Now by (4), (1) and (3), we obtain

$$\begin{aligned}a &\leq (aa)x\leq(((aa)x)((aa)x))x=((aa)(((aa)x)x))x=((aa)((xx)(aa)))x\\ &= ((aa)(x^2(aa)))x=(x(x^2(aa)))(aa)=(x(a(x^2a)))(aa)=(a(x(x^2a)))(aa)\\ &\leq (a(x(x^2((aa)x))))(aa)=(a(x((aa)x^3)))(aa)=p(aa),\end{aligned}$$

where $p=a(x((aa)x^3))=aq$, where $q=x((aa)x^3)$. Thus $(p,(aa))\in A_a$, since $A_a\neq\emptyset$, therefore

$$\begin{aligned}((f\circ_k S)\circ_k(f\circ_k f))(a) &= \bigvee_{(p,(aa))\in A_a}\Big\{(f\circ_k S)(a(x((aa)x^3)))\\ &\qquad\wedge(f\circ_k f)(aa)\wedge\frac{1-k}{2}\Big\}.\end{aligned}$$

Now

$$\begin{aligned}(f\circ_k S)(a(x((aa)x^3))) &= \bigvee_{(a,q)\in A_{aq}}\left\{f(a)\wedge S(x((aa)x^3))\wedge\frac{1-k}{2}\right\}\\ &= \bigvee_{(a,q)\in A_{aq}}\left\{f(a)\wedge\frac{1-k}{2}\right\},\end{aligned}$$

and

$$(f\circ_k f)(aa)=\bigvee_{(a,a)\in A_{aa}}\left\{f(a)\wedge f(a)\wedge\frac{1-k}{2}\right\}=\bigvee_{(a,a)\in A_{aa}}\left\{f(a)\wedge\frac{1-k}{2}\right\}.$$

Therefore

$$\begin{aligned}(f\circ_k S)(a(x((aa)x^3)))\wedge(f\circ_k f)(aa) &= \bigvee_{(a,q)\in A_{aq}}\left\{f(a)\wedge\frac{1-k}{2}\right\}\\ &\quad\wedge\bigvee_{(a,a)\in A_{aa}}\left\{f(a)\wedge\frac{1-k}{2}\right\}\\ &= \bigvee_{(a,q)\in A_{aq}}\left\{f(a)\wedge f(a)\wedge\frac{1-k}{2}\right\}.\end{aligned}$$

Thus we get

$$\begin{aligned}
((f \circ_k S) \circ_k (f \circ_k f))(a) &= \bigvee_{(p,(aa)) \in A_a} \left(\bigvee_{(a,q) \in A_{aq}} \left\{ f(a) \wedge f(a) \wedge \frac{1-k}{2} \right\} \right) \\
&= \bigvee_{(p,(aa)) \in A_a} \left\{ f(a) \wedge f(a) \wedge \frac{1-k}{2} \right\} \\
&\leq \bigvee_{(p,(aa)) \in A_a} \left\{ f((a(x((aa)x^3)))(aa)) \wedge \frac{1-k}{2} \right\} \\
&= f(a) \wedge \frac{1-k}{2} = f_k(a),
\end{aligned}$$

which implies that $(f \circ_k S) \circ_k (f \circ_k f) = f_k$.

Now by (1) and (4), we obtain

$$\begin{aligned}
a &\leq (aa)x = (xa)a \leq (x((aa)x))a = ((aa)(xx))a \leq ((a((aa)x))x^2)a \\
&= (((aa)(ax))x^2)a = ((x^2(ax))(aa))a = ((ax^3)(aa))a = pa,
\end{aligned}$$

where $p = ((ax^3)(aa))$. Thus $(p, (aa)) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned}
(f \circ_k f)(a) &= (f \circ f)(a) \wedge \frac{1-k}{2} \\
&= \bigvee_{(p,(aa)) \in A_a} \left\{ f(((ax^3)(aa))) \wedge f(a) \right\} \wedge \frac{1-k}{2} \\
&\geq f(((ax^3)(aa))) \wedge f(a) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge \frac{1-k}{2} \wedge f(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}$$

By Lemma 1, $f \circ_k f = f_k$.

$\Leftarrow$ Let f be a fuzzy subset of a left regular S. Then by $f \circ_k f = f_k$ and Lemma 1, it follows that f is a $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup of S. Also

$$\begin{aligned}
f((xa)(yz)) &= ((f \circ_k S) \circ_k (f \circ_k f))((xa)(yz)) \\
&= ((f \circ S) \circ (f \circ f))((xa)(yz)) \wedge \frac{1-k}{2} \\
&= ((f \circ S) \circ f)((xa)(yz)) \wedge \frac{1-k}{2}
\end{aligned}$$

$$\begin{aligned}
&= \bigvee_{((xa),(yz))\in A_{(xa)(yz)}} \{(f\circ S)(xa)\wedge f(yz)\}\wedge \frac{1-k}{2}\\
&\geq (f\circ S)(xa)\wedge f(yz)\wedge \frac{1-k}{2}\\
&= \bigvee_{(x,a)\in A_{xa}} \{f(x)\wedge S(a)\}\wedge f(yz)\wedge \frac{1-k}{2}\\
&\geq f(x)\wedge 1\wedge f(y)\wedge f(z)\wedge \frac{1-k}{2}\\
&= f(x)\wedge f(y)\wedge f(z)\wedge \frac{1-k}{2}.
\end{aligned}$$

Thus we get $f((xa)(yz)) \geq f(x)\wedge f(y)\wedge f(z)\wedge \frac{1-k}{2}$ and therefore f is an $(\in,\in\vee q_k)$-fuzzy $(1,2)$-ideal of S. □

Theorem 8. *The $(\in,\in\vee q_k)$-fuzzy left (right, two-sided) ideals, $(\in,\in\vee q_k)$-fuzzy (generalized) bi-ideals, $(\in,\in\vee q_k)$-fuzzy interior ideals and $(\in,\in\vee q_k)$-fuzzy $(1,2)$-ideals coincide with each other in a left regular S with a left identity.*

Proof. Let S be a left regular ordered $\mathcal{LA}$-semigroup with a left identity.

Now for $a,b\in S$, there exists some $x\in S$ such that $a\leq xa^2$. Let f be an $(\in,\in\vee q_k)$-fuzzy left ideal of S. Then by (4) and (1), we obtain

$$\begin{aligned}
f(ab) &\leq f((x(aa))b) = f((a(xa))b) = f((b(xa))a) \Longrightarrow f(ab)\\
&\geq f((b(xa))a) \geq f(a)\wedge \frac{1-k}{2}.
\end{aligned}$$

Similarly we can show that every $(\in,\in\vee q_k)$-fuzzy right ideal of S with a left identity is an $(\in,\in\vee q_k)$-fuzzy left ideal of S.

Clearly an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S is an $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal of S. Now for $a,b\in S$, there exists some $x\in S$ such that $a\leq a^2x$. Let f be an $(\in,\in\vee q_k)$-fuzzy generalized bi-ideal of S. By (3) and (4), it follows that

$$\begin{aligned}
f(ab) &\geq f(((aa)x)b) = f(((aa)(ex))b) = f(((xe)(aa))b)\\
&= f((a((xe)a))b) \geq f(a)\wedge f(b)\wedge \frac{1-k}{2}.
\end{aligned}$$

This shows that f is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S.

It is easy to see that an $(\in,\in\vee q_k)$-fuzzy two-sided ideal of S is an $(\in,\in\vee q_k)$-fuzzy (generalized) bi-ideal of S. Now for $a,b\in S$ there exist some $x, y \in S$ such that $b \leq b^2 y$ and $a \leq a^2 x$. Let f be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Then by (4), (1), (2) and (3), we obtain

$$\begin{aligned} f(ab) &\geq f(a((bb)y)) = f((bb)(ay)) = f(((ay)b)b) \geq f(((ay)((bb)y))b) \\ &= f(((ay)((yb)b))b) = f(((a(yb))(yb))b) = f(((by)((yb)a))b) \\ &= f(((yb)((by)a))b) = f(((a(by))(by))b) \\ &= f((b((a(by))y))b) \geq f(b) \wedge f(b) \wedge \frac{1-k}{2} = f(b) \wedge \frac{1-k}{2}. \end{aligned}$$

This shows that f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S and hence an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S.

It is easy to see that an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S is an $(\in, \in \vee q_k)$-fuzzy $(1,2)$-ideal of S. Now for $a, x \in S$, there exists some $y \in S$ such that $a \leq a^2 y$. Let f be an $(\in, \in \vee q_k)$-fuzzy $(1,2)$-ideal of S. Then by (4), (1) and (3), we obtain

$$\begin{aligned} f(xa) &\geq f(x((aa)y)) = f((aa)(xy)) \geq f((((aa)y)a)(xy)) \\ &= f(((ay)(aa))(xy)) = f(((aa)(ya))(xy)) = f(((xy)(ya))(aa)) \\ &= f(((ay)(yx))a^2) \geq f(((((aa)y)y)(yx))a^2) \\ &= f((((yy)(aa))(yx))a^2) = f((((aa)y^2)(yx))a^2) \\ &= f((((yx)y^2)(aa))a^2) = f((a(((yx)y^2)a))(aa)) \\ &\geq f(a) \wedge f(a) \wedge f(a) \wedge \frac{1-k}{2} = f(a) \wedge \frac{1-k}{2}. \end{aligned}$$

This shows that f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S and hence an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S. Let $a, x, y, z \in S$, then there exists $a^{'} \in S$ such that $a \leq a^2 a^{'}$. Let f be an $(\in, \in \vee q_k)$-fuzzy $(1,2)$-ideal of S. Therefore by (4), (1) and (3), we obtain

$$\begin{aligned} f((xa)(yz)) &\geq f((x((aa)a^{'}))(yz)) = f(((aa)(xa^{'}))(yz)) \\ &= f((((xa^{'})a)a)(yz)) \\ &\geq f((((xa^{'})((aa)a^{'}))a)(yz)) = f((((aa)((xa^{'})a^{'}))a)(yz)) \\ &= f(((yz)a)((aa)((xa^{'})a^{'}))) = f((aa)(((yz)a)((xa^{'})a^{'}))) \\ &= f(((aa)(a^{'}(xa^{'})))(a(yz))) = f(((a(yz))(a^{'}(xa^{'})))(aa)) \\ &\geq f(((((aa)a^{'})(yz))(a^{'}(xa^{'})))(aa)) \\ &= f(((((a^{'}a)a)(yz))(a^{'}(xa^{'})))(aa)) \\ &= f((((xa^{'})a^{'})((yz)((a^{'}a)(ea))))(aa)) \\ &= f((((xa^{'})a^{'})((yz)((ae)(aa^{'}))))(aa)) \end{aligned}$$

$$\begin{aligned} &= f((((xa^{'})a^{'})((yz)(a((ae)a^{'}))))(aa)) \\ &= f((((xa^{'})a^{'})(a((yz)((ae)a^{'}))))(aa)) \\ &= f((a(((xa^{'})a^{'})((yz)((ae)a^{'}))))(aa)) \\ &\geq f(a) \wedge f(a) \wedge f(a) = f(a) \wedge \frac{1-k}{2}. \end{aligned}$$

This shows that f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S. Again let $a, x, y, z \in S$, then there exist some $x^{'}$and $z^{'} \in S$ such that $x \leq x^2 x^{'}$ and $z \leq z^2 z$. Now by using (3), we have

$$f((xa)(yz)) = f((zy)(ax)) \geq f(y) \wedge \frac{1-k}{2},$$

and from (1) and (3), it follows that

$$\begin{aligned} f((xa)(yz)) &\geq f((((xx)x^{'})a)(yz)) = f(((ax^{'})(xx))(yz)) \\ &= f(((xx)(x^{'}a))(yz)) = f((((x^{'}a)x)x)(yz)) \geq f(x) \wedge \frac{1-k}{2}. \end{aligned}$$

Therefore by (4), we obtain

$$\begin{aligned} f((xa)(yz)) &\geq f((xa)(y(((zz)z^{'})))) = f((xa)((zz)(yz^{'}))) \\ &= f((zz)((xa)(yz^{'}))) \geq f(z) \wedge \frac{1-k}{2}. \end{aligned}$$

Thus we get $f((xa)(yz)) \geq f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2}$. If $a, b \in S$, then there exist $a^{'}$ and $b^{'} \in S$ such that $a \leq a^2 a^{'}$ and $b \leq b^2 b^{'}$. Now by (1), (3) and (4), we obtain

$$f(ab) \geq f(((aa)a^{'})b) = f((ba^{'})(aa)) = f((aa)(a^{'}b)) \geq f(a) \wedge \frac{1-k}{2},$$

and

$$f(ab) = f(a((bb)b^{'})) = f((bb)(ab^{'})) \geq f(b) \wedge \frac{1-k}{2}.$$

Thus f is an $(\in, \in \vee q_k)$-fuzzy $(1, 2)$-ideal of S. □

Now let us consider an Example 1 of an ordered $\mathcal{LA}$-semigroup $S = \{a, b, c, d, e\}$ with a left identity d. It is important to note that S is not left regular because for $c \in S$ there does not exists some $x \in S$ such that $c \leq xc^2$.

If we define a fuzzy subset $f : S \longrightarrow [0, 1]$ as follows:

$$f(x) = \begin{cases} 0.7 \text{ for } x = a \\ 0.5 \text{ for } x = b \\ 0.1 \text{ for } x = c \\ 0.3 \text{ for } x = d \\ 0.6 \text{ for } x = e. \end{cases}$$

Then f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S, but it is not an $(\in, \in \vee q_k)$-fuzzy right ideal of S, because

$$f(bd) \ngeq f(b) \wedge \frac{1-k}{2} \text{ for all } k \in [0, 0.8) \tag{5}$$

On the other hand it is easy to see that every $(\in, \in \vee q_k)$-fuzzy right ideal of S with a left identity is an $(\in, \in \vee q_k)$-fuzzy left ideal of S.

If we define a fuzzy subset $f : S \longrightarrow [0, 1]$ as follows:

$$f(x) = \begin{cases} 0.4 \text{ for } x = a \\ 0.4 \text{ for } x = b \\ 0.4 \text{ for } x = c \\ 0.2 \text{ for } x = d \\ 0.5 \text{ for } x = e. \end{cases}$$

Then it is easy to see that f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S but it is not an $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S which can be seen from the following:

$$f(ae) \ngeq f(e) \wedge \frac{1-k}{2} \; (f(ea) \ngeq f(e) \wedge \frac{1-k}{2}) \text{ for all } k \in [0, 0.2).$$

On the other hand it is easy to see that every $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S is an $(\in, \in \vee q_k)$-fuzzy interior (bi- and quasi) ideal of S.

A very major and an abstract conclusion from this section is that all $(\in, \in \vee q_k)$-fuzzy ideals need not to be coincide in S even if S has a left identity but they will only coincide in a left regular class of S with a left identity.

4. ordered $\mathcal{LA}$-semigroups in terms of $(\in, \in \vee q_k)$-fuzzy left (right) ideals

In this section, we establish several conditions for an ordered $\mathcal{LA}$-semigroup to become a left regular ordered $\mathcal{LA}$-semigroup in terms of $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideals and characterize a left regular ordered $\mathcal{LA}$-semigroup by using the properties of $(\in, \in \vee q_k)$-fuzzy left (right) ideals. We also give some counter examples to discuss the converse part of a given problem.

Definition 8. *A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy semiprime if for all $x \in S$, $x_t^2 \in f \Longrightarrow x_t \in \vee q_k f$.*

Lemma 2. *If f is a fuzzy subset of S, then f is an $(\in, \in \vee q_k)$-fuzzy semiprime if and only if $f(x) \geq \{f(x^2) \wedge \frac{1-k}{2}\}$, for all $x \in S$.*

Proof. It is immediate. □

Let us define a fuzzy subset f of S given in Example 1 as follows:

$$f(x) = \begin{cases} 0.2 \text{ for } x = a \\ 0.5 \text{ for } x = b \\ 0.6 \text{ for } x = c \\ 0.1 \text{ for } x = d \\ 0.4 \text{ for } x = e. \end{cases}$$

Then f is an $(\in, \in \vee q_k)$-fuzzy semiprime for all $k \in [0, 0.2)$.

Theorem 9. *A right (left, two-sided) ideal R of S is semiprime if and only if $(C_R)_k$ is $(\in, \in \vee q_k)$-fuzzy semiprime.*

Proof. Let R be a right ideal of S. By Lemma 5, $(C_R)_k$ is a $(\in, \in \vee q_k)$-fuzzy right ideal of S. Now if $a \in S$ then by given assumption $(C_R)_k(a) \geq (C_R)_k(a^2)$. If $a^2 \in R$, then $(C_R)_k(a^2) = \frac{1-k}{2} \Longrightarrow (C_R)_k(a) = \frac{1-k}{2}$, which implies that $a \in R$. Thus every right ideal of S is semiprime. The converse is simple.

Similarly every left or two-sided ideal of S is semiprime if and only if its characteristic functions is $(\in, \in \vee q_k)$-fuzzy semiprime. □

Corollary 7. *If any fuzzy right (left, two-sided) ideal of S is $(\in, \in \vee q_k)$-fuzzy semiprime, then any right (left, two-sided) ideal of S is semiprime.*

Lemma 3. *If S is left regular, then the following assertions hold.*

(i) All $(\in, \in \vee q_k)$-fuzzy right ideals of S are $(\in, \in \vee q_k)$-fuzzy semiprime.

(ii) If S has a left identity, then all $(\in, \in \vee q_k)$-fuzzy left ideals of S are $(\in, \in \vee q_k)$-fuzzy semiprime.

Proof. (i) : It is immediate.

(ii) : If f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S and $a \in S$, then there exists $x \in S$ such that $a \leq a^2 x$. By (3), we have $f(a) \geq f((aa)(ex)) = f((xe)a^2) \geq f(a^2) \wedge \frac{1-k}{2}$, which shows that f is $(\in, \in \vee q_k)$-fuzzy semiprime. □

Theorem 10. *The following statements are equivalent for S with a left identity.*

(i) S is left regular.
(ii) All $(\in, \in \vee q_k)$-fuzzy right (left, two-sided) ideals of S are $(\in, \in \vee q_k)$-fuzzy semiprime.

Proof. $(i) \Longrightarrow (ii)$ It follows from Lemma 3.
$(ii) \Longrightarrow (i)$: Since $(a^2S]$ [9] is a right and also a left ideal of S, therefore by Corollary 7, it follows that $(C_{(a^2S]})_k$ is $(\in, \in \vee q_k)$-semiprime. Now clearly $a^2 \in (a^2S]$, therefore $a \in (a^2S]$, which shows that S is left regular. □

Lemma 4. *Every $(\in, \in \vee q_k)$-fuzzy left ideal of S with a left identity becomes an $(\in, \in \vee q_k)$-fuzzy left ideal of S.*

Proof. Let f be an $(\in, \in \vee q_k)$-fuzzy right ideal of S. By (1), we obtain $f(ab) = f((ea)b) = f((ba)e) \geq f(b) \wedge \frac{1-k}{2}$. Therefore f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. □

The converse of Lemma 4 is not true in general and can be seen from (5).

Theorem 11. *The following statements are equivalent for S with a left identity.*

(i) S is left regular.
(ii) All $(\in, \in \vee q_k)$-fuzzy right ideals of S are $(\in, \in \vee q_k)$-fuzzy semiprime.
(iii) All $(\in, \in \vee q_k)$-fuzzy left ideals of S are $(\in, \in \vee q_k)$-fuzzy semiprime.

Proof. $(i) \Longrightarrow (iii)$ and $(ii) \Longrightarrow (i)$ can be followed from Theorem 10.
$(iii) \Longrightarrow (ii)$: If f is an $(\in, \in \vee q_k)$-fuzzy right ideal of S, then by Lemma 4, f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S and therefore f is an $(\in, \in \vee q_k)$-fuzzy semiprime. □

Lemma 5. *The following properties hold in S.*
(i) A is an $\mathcal{LA}$-subsemigroup of S if and only if $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy $\mathcal{LA}$-subsemigroup of S.
(ii) A is a left (right, two-sided, interior) ideal of S if and only if $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left (right, two-sided, interior) ideal of S.
(iii) For any non-empty subsets A and B of S, $C_A \circ_k C_B = (C_{(AB]})_k$ and $C_A \cap_k C_B = (C_{A \cap B})_k$.

Proof. It is straightforward. □

Lemma 6. *The following conditions are equivalent for S with a left identity.*

(i) S is left regular.

(ii) $f \circ_k f = f_k$, for all $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let f be an $(\in, \in \vee q_k)$-fuzzy left ideal of S, then $f \circ_k f \subseteq f_k$. Let $a \in S$, then by left regularity of S and by (1), it follows that $a \leq (aa)x = (xa)a$. Thus $(xa, a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned} (f \circ_k f)(a) &= \bigvee_{(xa,a)\in A_a} \{f(xa) \wedge f(a) \wedge \frac{1-k}{2}\} \\ &\geq f(a) \wedge f(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \\ &= f(a) \wedge \frac{1-k}{2} = f_k(a) \Longrightarrow f \circ_k f = f_k. \end{aligned}$$

$(ii) \Longrightarrow (i)$: Assume that $f \circ_k f = f_k$ holds for all $(\in, \in \vee q_k)$-fuzzy left ideal of S with a left identity. Since $(Sa]$ [9] is a left ideal of S and by Lemma 5, it follows that $C_{(Sa]})_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Since $a \in (Sa]$, it follows that $(C_{Sa})_k(a) = \frac{1-k}{2}$. By hypothesis and Lemma 5, we obtain $(C_{(Sa]})_k \circ_k (C_{(Sa]})_k = (C_{(Sa]})_k$ and $(C_{(Sa]})_k \circ_k (C_{(Sa]})_k = \left(C_{((Sa](Sa]]}\right)_k$. Thus we have $\left(C_{((Sa](Sa]]}\right)_k (a) = (C_{(Sa]})_k(a) = \frac{1-k}{2}$, which implies that $a \in ((Sa](Sa]]$. Now by (3) and (2), it follows that $a \in ((Sa](Sa]] \subseteq (((Sa)(Sa)]] = ((Sa)(Sa)] = ((aS)(aS)] = (a^2S]$. (see [9]). This shows that S is left regular. □

Theorem 12. *The following conditions are equivalent for S with a left identity.*

(i) S is left regular.

(ii) $f_k = (S \circ_k f) \circ_k (S \circ_k f)$, where f is an arbitrary $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let S be a left regular and let f be an $(\in, \in \vee q_k)$-fuzzy left ideal of S. It is easy to see that $S \circ_k f$ is also an $(\in, \in \vee q_k)$-fuzzy left ideal of S. By Lemma 6, we obtain $(S \circ_k f) \circ_k (S \circ_k f) = S \circ_k f \wedge \frac{1-k}{2} \subseteq f \wedge \frac{1-k}{2} = f_k$. Let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \leq a^2x$ and by (1), we obtain $a \leq (aa)x = (xa)a \leq (xa)((aa)x) = (xa)((xa)a)$. Thus $(xa, (xa)a) \in A_a$, since $A_a \neq \emptyset$, therefore

$$\begin{aligned} ((S \circ_k f) \circ_k (S \circ_k f))(a) &= \bigvee_{(xa,(xa)a)\in A_a} \{(S \circ_k f)(xa) \wedge (S \circ_k f)((xa)a) \\ &\qquad \wedge \frac{1-k}{2}\} \end{aligned}$$

$$
\begin{aligned}
&\geq (S \circ_k f)(xa) \wedge (S \circ_k f)((xa)a) \wedge \frac{1-k}{2} \\
&= \bigvee_{(x,a)\in A_{xa}} \{S(x) \wedge f(a) \wedge \frac{1-k}{2}\} \\
&\quad \wedge \bigvee_{(xa,a)\in A_{(xa)a}} \{S(xa) \wedge f(a) \wedge \frac{1-k}{2}\} \wedge \frac{1-k}{2} \\
&\geq S(x) \wedge f(a) \wedge S(xa) \wedge f(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge \frac{1-k}{2} = f_k(a).
\end{aligned}
$$

Thus we get the required $f_k = (S \circ_k f) \circ_k (S \circ_k f)$.

$(ii) \implies (i)$: Suppose that $f_k = (S \circ_k f) \circ_k (S \circ_k f)$ holds for all $(\in, \in \vee q_k)$-fuzzy left ideal f of S. Then $f_k = (S \circ_k f) \circ_k (S \circ_k f) \subseteq f \circ_k f \subseteq S \circ_k f \leq f_k$. Thus by Lemma 6, it follows that S is left regular. □

5. duo and $(\in, \in \vee q_k)$-fuzzy duo ordered $\mathcal{LA}$-semigroups

Definition 9. *An ordered $\mathcal{LA}$-semigroup S is called a left (right) duo if every left (right) ideal of S is a two-sided ideal of S and is called a duo if it is both left and right duo.*

Lemma 7. *If every $(\in, \in \vee q_k)$-fuzzy left ideal of S with a left identity is a fuzzy interior ideal of S, then S is a left duo.*

Proof. Let $\mathcal{I}$ be a left ideal of S with a left identity. By Lemma 5, $(\mathcal{C}_{\mathcal{I}})_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Thus by hypothesis, $(\mathcal{C}_{\mathcal{I}})_k$ is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S and by Lemma 5, $\mathcal{I}$ is an interior ideal of S. Now

$$(AS] = ((eA)S] \subseteq ((SA)S] \subseteq A,$$

which shows that S is left duo. □

Corollary 8. *Every interior ideal of S with a left identity is a right ideal of S.*

Theorem 13. *The following conditions are equivalent for a left regular S with a left identity.*

(i) S is left duo.

(ii) Every $(\in, \in \vee q_k)$-fuzzy left ideal of S is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S.

Proof. $(i) \Rightarrow (ii)$ Let a left regular S with a left identity be a left duo and assume that f is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. If $a, b, c \in S$, then $b \leq xb^2$ for some $x \in S$. Since $(Sa]$ is a left ideal of S [9], it follows that $(Sa]$ is a two-sided ideal of S. By (4), (3) and (1), we have

$$\begin{aligned} (ab)c &\leq (a(x(bb)))c = (a(b(xb)))c = (b(a(xb)))c \\ &= (c(a(xb)))b \in (S(a(SS)))b \subseteq (S(aS))b \\ &= ((eS)(aS))b = ((Sa)(Se))b \subseteq ((Sa)(SS))b \\ &\subseteq ((Sa)S)b \subseteq ((Sa]S)b \subseteq (Sa]b. \end{aligned}$$

Thus $(ab)c \leq (ta)b$, for some $t \in S$. Now $f((ab)c) \geq f((ta)b) \geq f(b) \wedge \frac{1-k}{2}$ implies that f is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S.

$(ii) \Rightarrow (i)$ cab be followed from Lemma 7. □

Definition 10. *An ordered $\mathcal{LA}$-semigroup S is called an $(\in, \in \vee q_k)$-fuzzy left (right) duo if every $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S is an $(\in, \in \vee q_k)$-fuzzy two-sided ideal of S and is called an $(\in, \in \vee q_k)$-fuzzy duo if it is both $(\in, \in \vee q_k)$-fuzzy left and $(\in, \in \vee q_k)$-fuzzy right duo.*

Remark 2. *If S is left regular with a left identity, then every $(\in, \in \vee q_k)$-fuzzy left duo or $(\in, \in \vee q_k)$-fuzzy right duo is an $(\in, \in \vee q_k)$-fuzzy duo.*

Lemma 8. *Every left ideal of S with a left identity is an interior ideal of S if S is an $(\in, \in \vee q_k)$-fuzzy left duo.*

Proof. It is immediate. □

Theorem 14. *The following conditions are equivalent for a left regular S with a left identity.*

(i) *S is an $(\in, \in \vee q_k)$-fuzzy left duo.*

(ii) *Every left ideal of S is an interior ideal of S.*

Proof. $(i) \Rightarrow (ii)$ cab be followed from Lemma 8.

$(ii) \Rightarrow (i)$ cab be followed from Theorem 8. □

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