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GENERALIZED DERIVATIONS AND DERIVATIONS OF RINGS AND BANACH ALGEBRAS

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Abstract. We investigate anti-centralizing and skew-centralizing mappings involving generalized derivations and derivations on prime and semiprime rings. We also obtain some range inclusion results for generalized linear derivations and linear derivations on Banach algebras by applying the algebraic techniques. Some results in this note are to improve the ones in [22].

1. Introduction

Throughout, R will represent an associative ring. The commutator $xy - yx$ will be denoted by $[x, y]$ and the anti-commutator $xy + yx$ by $\langle x, y\rangle$. Let us $Z(R)$ be the center of R and $N(R) = \{x \in R : \langle x, y\rangle = 0 \text{ for all } y \in R\}$. For $x_i, y \in R$, $i = 1, 2, \cdots, m$, we define the $(m+1)$-tuple $\langle y, x_1, \cdots, x_m\rangle$ as follows: $\langle y, x_1\rangle := yx_1 + x_1y$ and $\langle y, x_1, \cdots, x_{m-1}, x_m\rangle := \langle\langle y, x_1, \cdots, x_{m-1}\rangle, x_m\rangle$. In particular, in the case $x_1 = x_2 = \cdots = x_m = x$, $\langle y, x\rangle_m$ will stand for the $(m+1)$-tuple $\langle y, x, \cdots, x\rangle$ and let $\langle y, x\rangle_0 = y$. The relation $\Delta_i(y, x, e)$ stands for the tuple $\langle y, x_1, \cdots, x_m\rangle$ such that $x_i = x$ and $x_j = e$ for all $j \neq i$ and all $x_i, y \in R$, where $i, j = 1, 2, \cdots, m$. We often make use of the following basic properties: for any $x, y, z \in R$, $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = [x, y]z + y[x, z]$ and $[\langle y, x\rangle, x] = \langle [y, x], x\rangle$.

A mapping $f : R \to R$ is said to be *commuting* on R if $[f(x), x] = 0$ for all $x \in R$. Similarly f is called *skew-commuting* (resp. *skew-centralizing*) on R if $\langle f(x), x\rangle = 0$ (resp. $\langle f(x), x\rangle \in Z(R)$) for all $x \in R$. By analogy with the definition of n-commutativity introduced in [7], for convenience' sake, for $n \geq 2$, a mapping $f : R \to R$ will be called *n-anti-centralizing* on R if $\langle f(x), x^n\rangle \in N(R)$ for all $x \in R$. An 1-anti-centralizing is

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simply an anti-centralizing. We define a mapping $f : R \to R$ to be *n-skew-commuting* (resp. *n-skew-centralizing*) on R if $\langle f(x), x^n\rangle = 0$ (resp. $\langle f(x), x^n\rangle \in Z(R)$) for all $x \in R$. An 1-skew-commuting mapping (resp. 1-skew-centralizing) is called simply a skew-commuting mapping (resp. skew-centralizing).

The study of (skew-)centralizing and (skew-)commuting mappings was initiated by a well-known theorem of Posner [13] which states that the existence of a nonzero centralizing derivation on a prime ring implies that the ring is commutative. This theorem has been extended by several authors in different ways ([21], [22], etc).

An additive mapping $\mu : R \to R$ is called a *left* (resp. *right*) *centralizer* (or *multiplier*) if $\mu(xy) = \mu(x)y$ (resp. $\mu(xy) = x\mu(y)$) holds for all $x, y \in R$. An additive mapping $\delta : R \to R$ is called a *derivation* if the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all $x, y \in R$.

In [3], M. Brešar defined the following concept. Let $\delta : R \to R$ be a derivation. An additive mapping $f : R \to R$ is called a *generalized derivation* associated with δ if $f(xy) = f(x)y + x\delta(y)$ holds for all $x, y \in R$. This notion is a generalization of both derivations and centralizers. Other properties of generalized derivations were given by B. Hvala [9] and M. A. Quadri et al. [14].

For example, let $a, b \in R$ be such that one of them is not zero. Define a mapping $f : R \to R$ by $f(x) = ax + xb$ for all $x \in R$. Then for all $x, y \in R$, we have $f(x + y) = f(x) + f(y)$ and

$$\begin{aligned} f(xy) &= axy + xyb \\ &= (ax + xb)y + x(-by - y(-b)) \\ &= f(x)y + x\delta(y), \end{aligned}$$

where δ is an inner derivation on R induced by the element b. That is, f is a generalized derivation on R.

In this note, we investigate anti-centralizing and skew-centralizing mappings involving generalized derivations and derivations on prime rings, semiprime rings and Banach algebras, and some results are to improve the ones in [22].

2. Generalized Derivations and Derivations of Prime and Semiprime Rings

To obtain our main results in this section, we need the following basic facts: let R be a semiprime ring and U the *left Utumi quotient ring* of R. Then U can be characterized as a ring satisfying the next properties:

(1) R is a subring of U.
(2) For each $u \in U$, there exists a dense left ideal I_u of R such that $I_u u \subseteq R$.
(3) If $u \in U$ and $Iu = 0$ for some dense left ideal I of R, then $u = 0$.
(4) If $\varphi : I \to R$ is a left R-module mapping from a dense left ideal I of R into R, then there exists an element $u \in U$ such that $\varphi(x) = xu$ for all $x \in I$.

Up to isomorphisms, U is uniquely determined by the above four properties. If R is a (semi-)prime ring, then U is also a (semi-)prime ring. U always has the identity element e. The center of U is called the extended centroid of R. The set of all idempotents in the extended centroid of R is denoted by E. The element of E is called a central idempotent.

A mapping $\Lambda : R\times R \to R$ is said to be *symmetric* if $\Lambda(x, y) = \Lambda(y, x)$ for all $x, y \in R$. A mapping $\lambda : R \to R$ defined by $\lambda(x) = \Lambda(x, x)$ for all $x, y \in R$, where $\Lambda : R \times R \to R$ is a symmetric mapping, is called the *trace* of Λ. It is obvious that, in case when $\Lambda : R\times R \to R$ is a symmetric mapping which is also *bi-additive* (i.e., additive in both arguments), the trace λ of Λ satisfies the relation

$$\lambda(x + y) = \lambda(x) + \lambda(y) + 2\Lambda(x, y) \tag{2.1}$$

for all $x, y \in R$.

We precede the following lemmas to prove the main results.

Lemma 2.1 ([5])**.** *Let n be a fixed positive integer and R a $n!$-torsion free ring. Suppose that $y_1, y_2, \cdots, y_n \in R$ satisfy $ty_1 + t^2y_2 + \cdots + t^ny_n = 0$ for $t = 1, 2, \cdots, n$. Then $y_i = 0$ for all i.*

Lemma 2.2. *Let $m \geq 0$ and $n \geq 1$. Let R be a $(m+n+1)!$-torsion-free ring with identity. Let $\Lambda : R \times R \to R$ be a symmetric bi-additive mapping and λ the trace of Λ. If the mapping $x \mapsto \langle\lambda(x), x\rangle_m$ is n-skew-commuting on R, then we have $\lambda = 0$ on R.*

Proof. Since R has the identity element e, it follows from (2.1) that $\lambda(0) = 0$ and so $\lambda(e) = 0$. Suppose that

$$\langle\langle\lambda(x), x\rangle_m, x^n\rangle = 0 \tag{2.2}$$

for all $x \in R$.

Let t be any positive integer. Replacing x by $x + te$ in (2.2) and considering $\lambda(x + te) = \lambda(x) + t^2\lambda(e) + 2t\Lambda(x, e) = \lambda(x) + 2t\Lambda(x, e)$ for all $x \in R$, we obtain

$$tP_1(x, e) + t^2P_2(x, e) + \cdots + t^{m+n+1}P_{m+n+1}(x, e) = 0$$

for all $x \in R$, where $P_k(x, e)$ is the sum of terms involving x and e such that $P_k(x, te) = t^k P_k(x, e)$, $k = 1, 2, \cdots, m+n+1$. By Lemma 2.1, we see that for each $k = 1, 2, \cdots, m+n+1$,

$$P_k(x, e) = 0$$

for all $x \in R$.

By utilizing $\lambda(e) = 0$ and $e^n = e$, we have, in particular,

$$0 = P_{m+n+1}(x, e) = 2\Delta_{m+1}(\Lambda(x, e), e, e) \tag{2.3}$$

and

$$\begin{aligned} 0 &= P_{m+n}(x, e) \\ &= 2\Delta_{m+1}(\lambda(x), e, e) + \sum_{i=1}^{m} 2\Delta_i(\Lambda(x, e), x, e) \\ &\quad + 2n\Delta_{m+1}(\Lambda(x, e), x, e) \end{aligned} \tag{2.4}$$

for all $x \in R$. By inspecting (2.3), we arrive at $2^{m+2}\Lambda(x, e) = 0 = \Lambda(x, e)$ for all $x \in R$. This forces (2.4) to

$$\Delta_{m+1}(\lambda(x), e, e) = \langle \lambda(x), \overbrace{e, \cdots, e}^{m\text{times}} \rangle = 0 \tag{2.5}$$

for all $x \in R$. Calculating (2.5), we get

$$2^{m+1}\lambda(x) = 0 = \lambda(x)$$

for all $x \in R$ which is the conclusion of the lemma. □

Now we are ready to prove our main results. First, we prove the following result which is a generalization of [22, Theorem 3.1].

Theorem 2.3. *Let $m \geq 0$ and $n \geq 1$. Let R be a $(m+n+1)!$-torsion-free prime ring. If there exist generalized derivations $d, g : R \to R$ such that the mapping $x \mapsto \langle d^2(x) + g(x), x\rangle_m$ is n-anti-centralizing on R, then d and g are either left centralizers or right centralizers.*

Proof. Let us write h instead of $d^2 + g$. We define a mapping $\Lambda : R \times R \to R$ by

$$\Lambda(x, y) = \langle h(x), y\rangle + \langle h(y), x\rangle$$

for all $x, y \in R$. Then it is clear that Λ is a symmetric bi-additive mapping and the mapping $\lambda : R \to R$ defined by $\lambda(x) = \Lambda(x, x) = 2\langle h(x), x\rangle$ for all $x, y \in R$, is the trace of Λ.

Since it follows from the hypothesis that $\langle\langle h(x), x\rangle_m, x^n\rangle \in N(R)$ is valid for all $x \in R$, we have

$$\langle\langle h(x), x\rangle_m x^n + x^n\langle h(x), x\rangle_m, x\rangle = 0$$

for all $x \in R$ which reduces to $\langle\langle h(x), x\rangle_m, x\rangle x^n + x^n\langle\langle h(x), x\rangle_m, x\rangle = 0$ for all $x \in R$. This implies that $\langle h(x), x\rangle_{m+1}x^n + x^n\langle h(x), x\rangle_{m+1} = 0$ for all $x \in R$, that is, $\langle \lambda(x), x\rangle_m x^n + x^n\langle \lambda(x), x\rangle_m = 0$ for all $x \in R$. Note that R and U satisfy the same differential identities [10, Theorem 2] and so satisfy the same generalized differential identities. Hence we see that

$$\langle \lambda(x), x\rangle_m x^n + x^n\langle \lambda(x), x\rangle_m = 0$$

for all $x \in U$. From Lemma 2.2, it follows that $\lambda = 0$ on R, that is, $h = d^2 + g$ is skew-commuting on U. Applying [2, Theorem 2] yields

$$d^2(x) + g(x) = 0 \tag{2.6}$$

for all $x \in U$ since U is prime and so is semiprime. The equation (2.6) means that d^2 is a generalized derivation on U and hence

$$d^2(xy) = d^2(x)y + x\delta(y) \tag{2.7}$$

for all $x, y \in U$, where δ is the associated derivation of d^2.

On the other hand, we see that

$$d^2(xy) = d(d(x)y + x\tau(y)) = d^2(x)y + 2d(x)\tau(y) + x\tau^2(y) \tag{2.8}$$

for all $x, y \in U$, where τ is the associated derivation of d. From (2.7) and (2.8), we obtain

$$x\delta(y) = 2d(x)\tau(y) + x\tau^2(y) \tag{2.9}$$

for all $x, y \in U$. Putting $x = e$ in (2.9), we get

$$\delta(y) = 2d(e)\tau(y) + \tau^2(y) \tag{2.10}$$

for all $y \in U$. The substitution yx for y in (2.10) gives

$$\begin{aligned} \delta(y)x + y\delta(x) = {} & 2d(e)\tau(y)x + 2d(e)y\tau(x) \\ & + \tau^2(y)x + 2\tau(y)\tau(x) + y\tau^2(x) \end{aligned} \tag{2.11}$$

for all $x, y \in U$. Right multiplication of (2.10) by x leads to

$$\delta(y)x = 2d(e)\tau(y)x + \tau^2(y)x \tag{2.12}$$

for all $x, y \in U$. Subtracting (2.12) from (2.11), we have

$$y\delta(x) = 2d(e)y\tau(x) + 2\tau(y)\tau(x) + y\tau^2(x) \tag{2.13}$$

for all $x, y \in U$. Combining (2.13) with (2.10), we deduce that

$$\big\{[d(e), y] + \tau(y)\big\}\tau(x) = 0$$

for all $x, y \in U$. From [13, Lemma 1], it follows that

$$\tau(x) = 0 \quad \text{or} \quad \tau(y) = [y, d(e)]$$

for all $x, y \in U$ since R is prime. In case $\tau = 0$ on U, relations (2.8) and (2.6) tell us that d, d^2 and g are all left centralizers. In case $\tau(y) = [y, d(e)]$ for all $y \in U$, we obtain

$$d(y) = d(ey) = d(e)y + \tau(y) = yd(e)$$

for all $y \in U$. Now it is immediate that d, d^2 and g are all right centralizers. This completes the proof. □

Since the same method as in the proof of Theorem 2.3 with $d = 0$ leads to the relation (2.6), i.e., $g = 0$ on R, we obtain the following result.

Theorem 2.4. *Let $m \geq 0$ and $n \geq 1$. Let R be a $(m+n+1)!$-torsion-free semiprime ring. If there exists a generalized derivation $g : R \to R$ such that the mapping $x \mapsto \langle g(x), x\rangle_m$ is n-anti-centralizing on R, then we have $g = 0$ on R.*

Theorem 2.5. *Let $m \geq 0$ and $n \geq 1$. Let R be a $(m + n + 1)!$-torsion-free semiprime ring. If there exist derivations $d, g : R \to R$ such that the mapping $x \mapsto \langle d^2(x) + g(x), x\rangle_m$ is n-anti-centralizing on R, then we have $d = g = 0$ on R.*

Proof. In view of the same process as in the proof of Theorem 2.3, the relation (2.6) yields that d^2 is a derivation on R. From [13], it follows that $d = 0$ on R and so the relation (2.6) gives $g = 0$ on R, i.e., $d = g = 0$ on R. □

Remark 1. Theorems 2.4 and 2.5 are to improve Corollaries 3.2 and 3.3 of [22], respectively.

We continue the next result.

Theorem 2.6. *Let $m \geq 0$ and $n \geq 1$. Let R be a $(m+n+1)!$-torsion-free noncommutative prime ring. If there exist derivations $d, g : R \to R$ such that the mapping $x \mapsto \langle d^2(x) + g(x), x\rangle_m$ is n-skew-centralizing on R, then we have $d = g = 0$ on R.*

Proof. Set $h = d^2 + g$ as in the proof of Theorem 2.2. Defining a mapping $\Lambda : R \times R \to R$ by

$$\Lambda(x, y) = [h(x), y] + [h(y), x]$$

for all $x, y \in R$, it is obvious that Λ is a symmetric bi-additive mapping and the mapping $\lambda : R \to R$ defined by $\lambda(x) = \Lambda(x, x) = 2[h(x), x]$ for all $x, y \in R$, is the trace of Λ.

From the hypothesis $\langle\langle h(x), x\rangle_m, x^n\rangle \in Z(R)$ for all $x \in R$, we get, by recalling $[\langle y, x\rangle, x] = \langle [y, x], x\rangle$,

$$[\langle h(x), x\rangle_m x^n + x^n \langle h(x), x\rangle_m, x] = 0$$

for all $x \in R$ which implies that $[\langle h(x), x\rangle_m, x]x^n + x^n[\langle h(x), x\rangle_m, x] = 0$ for all $x \in R$. This reduces to $\langle [h(x), x], x\rangle_m x^n + x^n\langle [h(x), x], x\rangle_m = 0$ for all $x \in R$, that is, $\langle \lambda(x), x\rangle_m x^n + x^n\langle \lambda(x), x\rangle_m = 0$ for all $x \in R$. Again, using the fact that R and U satisfy the same differential identities and so satisfy the same generalized differential identities, we obtain, by Lemma 2.2, $\lambda(x) = 0$ for all $x \in U$ which means that $h = d^2 + g$ is commuting on R, namely,

$$[h(x), x] = 0 \tag{2.14}$$

for all $x \in R$. We claim that $d = g = 0$ on R.

To show the claim, the arguments used in the proof of [4, Theorem 1] carry over almost verbatim, but we will proceed the proof for the sake of completeness. The linearization of the relation (2.14) gives

$$[h(x), y] + [h(y), x] = 0 \tag{2.15}$$

for all $x, y \in R$. Putting xy for y in (2.15), we obtain

$$h(x)[y, x] + x[h(y), x] + 2[d(x)d(y), x] + x[h(x), y] = 0 \tag{2.16}$$

for all $x, y \in R$. From (2.15), the relation (2.16) reduces to

$$h(x)[y, x] + 2[d(x)d(y), x] = 0 \tag{2.17}$$

for all $x, y \in R$. In (2.17), we replace y by yx to obtain

$$h(x)[y, x]x + 2[d(x)d(y), x]x + 2[d(x)yd(x), x] = 0 \tag{2.18}$$

for all $x, y \in R$. By (2.17), the relation (2.18) becomes

$$d(x)yd(x)x - xd(x)yd(x) = 0 \tag{2.19}$$

for all $x, y \in R$. Substituting $yd(x)z$ for y in (2.19) yields

$$d(x)yd(x)zd(x)x - xd(x)yd(x)zd(x) = 0 \tag{2.20}$$

for all $x, y, z \in R$. According to (2.19), we can write, in relation (2.20), $xd(x)zd(x)$ for $d(x)zd(x)x$ and $d(x)yd(x)x$ instead of $xd(x)yd(x)$, which gives

$$d(x)y[d(x), x]zd(x) = 0$$

for all $x, y, z \in R$. From the primness of R, it follows that, for all $x \in R$, we have either $[d(x), x] = 0$ or $d(x) = 0$. In any case $[d(x), x] = 0$ for all $x \in R$. Posner's theorem guarantees that $d = 0$ on R. Now the initial hypothesis yields that $[g(x), x] = 0$ for all $x \in R$ so $g = 0$ on R. This completes the proof of the theorem. □

We apply the orthogonal completeness method to extend Theorem 2.6 to the case of semiprime rings.

Theorem 2.7. *Let $m \geq 0$ and $n \geq 1$. Let R be a $(m+n+1)!$-torsion-free noncommutative semiprime ring. If there exist derivations $d, g : R \to R$ such that the mapping $x \mapsto \langle d^2(x) + g(x), x\rangle_m$ is n-skew-centralizing on R, then both d and g map R into $Z(R)$.*

Proof. Let B be the complete Boolean algebra of E. We choose a maximal ideal M of B. According to [1], MU is a prime ideal of U, which is invariant under any derivation of U. It was well-known that any derivation on R can be uniquely extended to a derivation on U. Let $\bar{d}$ and $\bar{g}$ be derivations on $\bar{U} = U/MU$ induced by d and g, respectively. Since R and U satisfy the same differential identities, the hypothesis implies that

$$\langle\langle d^2(x) + g(x), x\rangle_m, x^n\rangle \in Z(U)$$

for all $x \in U$. Hence this yields that

$$\langle\langle \bar{d}^2(\bar{x}) + \bar{g}(\bar{x}), \bar{x}\rangle_m, \bar{x}^n\rangle \in Z(\bar{U})$$

for all $\bar{x} \in \bar{U}$. By Theorem 2.6, we see that either $\bar{d} = \bar{g} = 0$ on $\bar{U}$ or $[\bar{U}, \bar{U}] = 0$. In any case we have

$$d(U)[U, U] \in MU$$

and

$$g(U)[U, U] \in MU$$

for all M. We observe that $\bigcap\{MU : M \text{ is any maximal ideal of } B\} = \{0\}$. Thus we obtain $d(U)[U, U] = 0$ and $g(U)[U, U] = 0$. In particular, we get $d(R)[R, R] = 0$ and $g(R)[R, R] = 0$ which imply that

$$0 = d(R)[R^2, R] = d(R)R[R, R] + d(R)[R, R]R = d(R)R[R, R]$$

and

$$0 = g(R)[R^2, R] = g(R)R[R, R] + g(R)[R, R]R = g(R)R[R, R].$$

Hence we obtain $[R, d(R)]R[R, d(R)] = 0$ and $[R, g(R)]R[R, g(R)] = 0$. From the semiprimness of R, it follows that $[R, d(R)] = 0$ and $[R, g(R)] = 0$ which give the conclusion of the theorem. □

Remark 2. Theorems 2.6 and 2.7 are to improve Theorems 3.4 and 3.5 of [22], respectively.

3. Applications to Banach Algebras

In 1955 I.M. Singer and J. Wermer [18] proved that a continuous linear derivation on a commutative Banach algebra maps the algebra into its Jacobson radical. I.M. Singer and J. Wermer conjectured in [18] that the continuity assumption in their result is superfluous. It took more than thirty years until this conjecture was finally proved by Thomas [19].

In this section we investigate the ranges of generalized linear derivations and linear derivations on complex Banach algebras to discuss some problems which are related to the well-known noncommutative Singer-Wermer conjecture from the point of view of ring theory.

Let A be a Banach algebra. The *Jacobson radical* (resp. the *nil radical*) of A will be denoted by $rad(A)$ (resp. $nil(A)$). Note that $rad(A)$ (resp. $nil(A)$) is the intersection of all primitive ideals (resp. all prime ideals) of A. A is said to be semisimple (resp. semiprime) if $rad(A) = \{0\}$ (resp. $nil(A) = \{0\}$). For a linear mapping $f : A \to A$, the set

$$S(f)=\{y \in A\colon \text{there is a sequence } (x_n) \text{ in } A \text{ with } x_n \to 0 \text{ and } f(x_n) \to y\}$$

is said to be the *separating space* of f. By the Closed Graph Theorem, f is continuous if and only if $S(f) = \{0\}$ ([17, Lemma 1.2]).

Let I be a closed ideal of A. π_I will denote the canonical quotient map from A onto A/I. Here, Banach algebras will be over the complex field.

Our first result in this section is about continuous generalized derivations on Banach algebras.

Theorem 3.1. *Let $m \geq 0$ and $n \geq 1$. Let A be a unital Banach algebra. If there exists a continuous generalized linear derivation $g : A \to A$ such that the mapping $x \mapsto \langle g(x), x\rangle_m$ is n-anti-centralizing on A, then g maps A into $rad(A)$.*

Proof. Let Q be any primitive ideal of A. From [11, Theorem 3], it follows that $g = L_a + \delta$, where L_a ($a \in A$) is a left multiplication and δ is a derivation on A. Under the hypothesis that g is continuous, it is well known that the left multiplication is also continuous, hence we have that the derivation δ is continuous. In [16], Sinclair proved that any continuous linear derivation on Banach algebras leaves each primitive ideal invariant. Therefore, we see that $g(Q) = aQ + \delta(Q) \subseteq Q$, that is, also the continuous generalized derivation g leaves each primitive ideal invariant. Then the generalized derivation g induces the generalized derivation $\bar{g}$ on the semiprime Banach algebra A/Q, defined by $\bar{g}(\bar{x}) = g(x) + Q$ for all $\bar{x} = x + Q \in A/Q$ and $x \in A$. Since the assumption that the mapping $x \mapsto \langle g(x), x\rangle_m$ is n-anti-centralizing on A implies that $\bar{x} \mapsto \langle \bar{g}(\bar{x}), \bar{x}\rangle_m$ is n-anti-centralizing on A/Q, Theorem 2.4 gives $\bar{g} = 0$ on A/Q in either case A/Q is commutative or noncommutative. Therefore we have $g(A) \subseteq rad(A)$ since Q is arbitrary. □

Lemma 3.2 ([17]). *Let A be a Banach algebra and $d : A \to A$ be a linear derivation. If P is a minimal prime ideal of A such that $S(d) \not\subseteq P$, then P is closed.*

Lemma 3.3 ([20]). *Let d be a linear derivation on a Banach algebra A and Q a primitive ideal of A. If there exists a constant $C > 0$ such that $\|\pi_Q \circ d^n\| \leq C^n$ for all $n \in N$, then we have $d(Q) \subseteq Q$.*

Now we may prove the following:

Theorem 3.4. *Let $m \geq 0$ and $n \geq 1$. Let A be a Banach algebra. If there exist linear derivations $d, g : A \to A$ such that the mapping $x \mapsto \langle d^2(x) + g(x), x\rangle_m$ is n-skew-centralizing on A, then both d and g map A into $rad(A)$.*

Proof. Let Q be any primitive ideal of A. Using Zorn's lemma, we find a minimal prime ideal P contained in Q, and hence $d(P) \subseteq P$ and $g(P) \subseteq P$ by [12, Lemma 1]. Suppose first that P is closed. Then we can define derivation $\bar{d}, \bar{g} : A/P \to A/P$ by $\bar{d}(\bar{x}) = d(x) + P$ and $\bar{g}(\bar{x}) = g(x) + P$, respectively, for all $x \in A$. If A/P is commutative, then both $\bar{d}(A/P)$ and $\bar{g}(A/P)$ are contained in $rad(A/P)$ by [18]. Hence $\bar{d}(A/P) \subseteq Q/P$ and $\bar{g}(A/P) \subseteq Q/P$. Consequently we see that $d(A) \subseteq Q$ and $g(A) \subseteq Q$. We consider the case when A/P is noncommutative. The assumption that the mapping $x \mapsto \langle d^2(x) + g(x), x\rangle_m$ is n-skew-centralizing on A leads to the fact that the mapping $x \mapsto \langle \bar{d}^2(\bar{x}) + \bar{g}(\bar{x}), \bar{x}\rangle_m$ is n-skew-centralizing on A/P. Since A/P is prime, it follows from Theorem 2.6 that both $\bar{d} = 0$ and $\bar{g} = 0$ on A/P. Consequently, we see that both $d(A) \subseteq Q$ and $g(A) \subseteq Q$.

If P is not closed, then we see that $S(d) \subseteq P$ by Lemma 3.2. Denoting $\pi_{\bar{P}} : A \to A/\bar{P}$ the canonical epimorphism, we have, by [17, Lemma 1.3] $S(\pi_{\bar{P}} \circ d) = \overline{\pi_{\bar{P}}(S(d))} = \{0\}$ whence $\pi_{\bar{P}} \circ d$ is continuous. So $(\pi_{\bar{P}} \circ d)(\bar{P}) = \{0\}$, that is, $d(\bar{P}) \subseteq \bar{P}$. Hence we can also define a continuous derivation $\widetilde{d} : A/\bar{P} \to A/\bar{P}$ by $\widetilde{d}(\widetilde{x}) = d(x) + \bar{P}$ for all $x \in A$. This shows that we may also define a mapping

$$\Phi \circ \widetilde{d}^n \circ \pi_{\bar{P}} : A \to A/\bar{P} \to A/\bar{P} \to A/Q$$

by $(\phi_{\bar{P}} \circ \widetilde{d}^n \circ \pi_{\bar{P}})(x) = (\pi_Q \circ d^n)(x)$ for all $x \in A$, where ϕ is the canonical inclusion mapping from $A/\bar{P}$ onto A/Q (which exists since $\bar{P} \subseteq Q$). We therefore conclude that $\|\pi_Q \circ d^n\| \leq \|\widetilde{d}\|^n$ for all $n \in \mathbb{N}$, since the other mappings are norm depressing. From Lemma 1.4, the continuity of $\widetilde{d}$ is clear and so yields that $\|\pi_Q d^n\| \leq \|\widetilde{d}\|^n$ for all $n \in \mathbb{N}$. Now, according to Lemma 3.2, we obtain that $d(Q) \subseteq Q$. Following the same argument with g, we see that $g(Q) \subseteq Q$. Then the derivations d and g on A induce the derivations $\hat{d}$ and $\hat{g}$ on the Banach algebra A/Q, defined by $\hat{d}(\hat{x}) = d(x) + Q$ and $\hat{d}(\hat{x}) = d(x) + Q$ for all $x \in A$. The rest follows as when P is closed since the primitive algebra A/Q is prime. Thus we also obtain that $d(A) \subseteq Q$ and $g(A) \subseteq Q$. Since Q is arbitrary, we arrive at the conclusion that $d(A) \subseteq rad(A)$ and $g(A) \subseteq rad(A)$. □

Corollary 3.5. *Let $m \geq 0$ and $n \geq 1$. Let A be a semisimple Banach algebra. If there exist linear derivations $d, g : A \to A$ such that the mapping $x \mapsto \langle d^2(x) + g(x), x\rangle_m$ is n-skew-centralizing on A, then we have $d = g = 0$ on A.*

Our final results are related to the so-called *automatic continuity* for derivations on Banach algebras.

Theorem 3.6. *Let $m \geq 0$ and $n \geq 1$. Let A be a Banach algebra such that every prime ideal is closed. If there exist linear derivations $d, g : A \to A$ such that the mapping $x \mapsto \langle d^2(x) + g(x), x\rangle_m$ is n-skew-centralizing on A, then both $S(d)$ and $S(g)$ are contained in $nil(A)$.*

Proof. Note that $d(nil(A)) \subseteq nil(A)$ and $g(nil(A)) \subseteq nil(A)$ by [6, Lemma 4.1]. Since every prime ideal of A is closed, the nilradical $nil(A)$ is closed. Then the derivations d and g on A induce the derivations $\bar{d}$ and $\bar{g}$ on the Banach algebra $A/nil(A)$, defined by $\bar{d}(\bar{x}) = d(x) + nil(A)$ and $\bar{g}(\bar{x}) = g(x) + nil(A)$ for all $x \in A$. If $A/nil(A)$ is commutative, then $\bar{d}$ and $\bar{g}$ are continuous by [8, Theorem 3.2]. Thus [17, Lemma 1.4] yields that $S(d) \subseteq nil(A)$ and $S(g) \subseteq nil(A)$. In case $A/nil(A)$ is noncommutative, the hypothesis of the theorem means that the mapping $x \mapsto \langle \bar{d}^2(\bar{x}) + \bar{g}(\bar{x}), \bar{x}\rangle_m$ is n-skew-centralizing on $A/nil(A)$. Since $A/nil(A)$ is prime, it follows from Theorem 2.6 that both $\bar{d} = 0$ and $\bar{g} = 0$ on $A/nil(A)$. Hence we see that both $d(A) \subseteq nil(A)$ and $g(A) \subseteq nil(A)$. Therefore we have $S(d) \subseteq nil(A)$ and $S(g) \subseteq nil(A)$ since $S(d)$ and $S(g)$ are contained in the closure $\overline{d(A)}$ of $d(A)$. The proof of the theorem is complete. □

Corollary 3.7. *Let $m \geq 0$ and $n \geq 1$. Let A be a semiprime Banach algebra such that every prime ideal is closed. If there exist linear* *derivations $d, g : A \to A$ such that the mapping $x \mapsto \langle d^2(x) + g(x), x\rangle_m$ is n-skew-centralizing on A, then both d and g are continuous on A.*

Remark 3 ([15]). Let $A = \{a = \sum_{n=0}^{\infty} a_n x^n : \|a\| = \sum_{n=0}^{\infty} |a_n| w^n < \infty\}$ in one indeterminant x with complex coefficients where $\{w_n : n = 0, 1, 2, \cdots\}$ is a sequence in $(0, \infty)$ such that $w_0 = 1, w_{n+m} \leq w_n w_m$ and $\lim_{n\to\infty}(w_n)^{\frac{1}{n}} = 0$. Then A is a Banach algebra of power series. Furthermore, A has a unique maximal ideal $M = \{\sum_{n=0}^{\infty} a_n x^n : a_0 = 0\}$. If $\{w_n\}$ is chosen properly, then the only prime ideals of A are $\{0\}$ and A. Hence every prime ideal in A is closed.

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