CERTAIN INTEGRAL REPRESENTATIONS FOR THE RIEMANN ZETA FUNCTION $\zeta(s)$ AT POSITIVE INTEGER ARGUMENT

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Abstract. We aim at presenting certain integral representations for the Riemann Zeta function $\zeta(s)$ at positive integer arguments by using some known integral representations of $\log \Gamma(1 + z)$ and $\psi(1 + z)$.

1. Introduction and Preliminaries

The Riemann Zeta function $\zeta(s)$ is defined by (see, *e.g.*, [4, Section 2.3])

$$(1.1) \quad \zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; \ s \neq 1), \end{cases}$$

which is an obvious special case of the Hurwitz (or generalized) Zeta function $\zeta(s, a)$ defined by

$$(1.2) \qquad \zeta(s, a) := \sum_{k=0}^{\infty} (k + a)^{-s} \quad \left(\Re(s) > 1; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^- \right),$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers and nonpositive integers, respectively. Both the Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s, a)$ can be continued meromorphically to the

whole complex s-plane, except for a simple pole only at $s = 1$ with their respective residue 1, in many different ways.

The Gamma function $\Gamma(z)$ developed by Leonhard Euler (1707-1783) is usually defined by

$$(1.3) \qquad \Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} \quad (\Re(z) > 0).$$

Among several equivalent forms of the Gamma function $\Gamma(z)$, we choose to recall its canonical product form due to Karl Weierstrass (1815-1897):

$$(1.4) \qquad \Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^\infty \left\{ \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} \right\} \quad \left(z \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

where γ denotes the Euler-Mascheroni constant defined by

$$(1.5) \qquad \gamma := \lim_{n \to \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) \cong 0.57721\,56649\,01532\,86060\,6512 \cdots.$$

We also recall the Polygamma functions $\psi^{(n)}(s)$ ($n \in \mathbb{N} := \{1, 2, 3, \ldots\}$) defined by

$$(1.6) \qquad \begin{aligned} \psi^{(n)}(s) &:= \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(s) = \frac{d^n}{ds^n} \psi(s) \\ & \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \end{aligned}$$

where $\psi(s)$ denotes the Psi (or Digamma) function defined by

$$(1.7) \qquad \psi(s) := \frac{d}{ds} \log \Gamma(s) \quad \text{and} \quad \psi^{(0)}(s) = \psi(s) \quad \left(s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

A well-known (and potentially useful) relationship between the Polygamma functions $\psi^{(n)}(s)$ and the generalized Zeta function $\zeta(s, a)$ is given by

$$(1.8) \qquad \begin{aligned} \psi^{(n)}(s) &= (-1)^{n+1}\, n! \sum_{k=0}^\infty \frac{1}{(k+s)^{n+1}} = (-1)^{n+1}\, n!\, \zeta(n+1, s) \\ & \left(n \in \mathbb{N};\ s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \end{aligned}$$

In particular, we have

$$(1.9) \qquad \psi^{(n)}(1) = (-1)^{n+1}\, n!\, \zeta(n+1) \quad (n \in \mathbb{N}).$$

Since $\log \Gamma(1+z)$ is analytic at $z=0$, we have the Maclaurin series expansion of $\log \Gamma(1+z)$ in the following form:

$$(1.10) \qquad \log \Gamma(1+z) = \sum_{n=0}^{\infty} a_n\, z^n \quad (|z| < 1),$$

where

$$(1.11) \qquad a_n = \frac{1}{n!}\, \frac{d^n}{dz^n}\left(\log \Gamma(1+z)\right)\bigg|_{z=0} \quad (n \in \mathbb{N}_0).$$

From (1.9), we find explicit expressions of a_n:

$$(1.12) \quad a_0 = 0, \quad a_1 = -\gamma, \quad \text{and} \quad a_n = (-1)^n\, \frac{\zeta(n)}{n!} \quad (n \in \mathbb{N}\setminus\{1\}).$$

Differentiating both sides of (1.10), we obtain the Maclaurin series expansion of $\psi(1+z)$:

$$(1.13) \qquad \psi(1+z) = \sum_{n=0}^{\infty} b_n\, z^n \quad (|z| < 1),$$

where, in view of (1.9), we have

$$b_0 = -\gamma \quad \text{and}$$

$$(1.14) \qquad b_n = \frac{1}{n!}\, \psi^n(1+z)\bigg|_{z=0} = (-1)^{n+1}\, \zeta(n+1) \quad (n \in \mathbb{N}).$$

Here, in this paper, we aim at presenting certain integral representations for the Riemann Zeta function $\zeta(s)$ at positive integer arguments by using *both* some known integral representations of $\log \Gamma(1+z)$ and $\psi(1+z)$ *and* the relation (1.14).

2. Known representations and Faà di Bruno formula

For our purpose, here, we recall some known integral presentations of $\log \Gamma(1+z)$ and $\psi(1+z)$, and Faà di Bruno formula for the nth derivative of a composite function (see, *e.g.*, [3, p. 44, Problem 2.1.38]) in the following lemmas.

Lemma 1.   *Each of the following integral formulas holds true*:

$$\log \Gamma(1+z) = \left(z + \frac{1}{2}\right) \log(z+1) - z - 1 + \frac{1}{2}\log(2\pi)$$

$$(2.1)$$

$$+ 2 \int_0^{\infty} \frac{\arctan\left(\frac{t}{z+1}\right)}{e^{2\pi t} - 1}\, dt \quad (\Re(z) > -1);$$

(2.2) $\psi(z+1) = \displaystyle\int_0^\infty \left[t^{-1}e^{-t} - \left(1 - e^{-t}\right)^{-1} e^{-t}\, e^{-tz}\right] dt \quad (\Re(z) > -1);$

(2.3) $\qquad \psi(z+1) = \displaystyle\int_0^\infty \left[e^{-t} - (1+t)^{-z-1}\right] t^{-1}\, dt \quad (\Re(z) > -1);$

(2.4) $\qquad \psi(z+1) = \log(z+1) + \displaystyle\int_0^\infty \left[t^{-1} - \left(1 - e^{-t}\right)^{-1}\right] e^{-t}\, e^{-tz}\, dt$
$$\quad (\Re(z) > -1);$$

(2.5) $\qquad \psi(z+1) = \log(z+1) - \dfrac{1}{2(z+1)}$
$$- \int_0^\infty \left[\left(1 - e^{-t}\right)^{-1} - t^{-1} - \frac{1}{2}\right] e^{-t}\, e^{-tz} \quad (\Re(z) > -1);$$

(2.6) $\qquad \psi(z+1) = \log(z+1)$
$$+ \int_0^\infty \left[\left(1 - e^{-t}\right)^{-1} + t^{-1} - 1\right] e^{-t}\, e^{-tz}\, dt \quad (\Re(z) > -1);$$

(2.7) $\qquad \psi(z+1) = \log(z+1) - \dfrac{1}{2(z+1)}$
$$- \int_0^\infty \left[\left(e^t - 1\right)^{-1} - t^{-1} + \frac{1}{2}\right] e^{-t}\, e^{-tz}\, dt \quad (\Re(z) > -1).$$

Lemma 2. *Let I, J be open intervals and let $f : J \to \mathbb{R}$, $g : I \to J$ be infinitely differentiable on J and I, respectively. Then Faà di Bruno formula for the nth derivative of $h = f \circ g$ is given as follows:*

(2.8)
$$h^{(n)}(t) = \sum \frac{n!}{k_1! k_2! \cdots k_n!} f^{(k)}(g(t)) \left(\frac{g^{(1)}(t)}{1!}\right)^{k_1} \left(\frac{g^{(2)}(t)}{2!}\right)^{k_2}$$
$$\cdots \left(\frac{g^{(n)}(t)}{n!}\right)^{k_n},$$

where $k = k_1 + k_2 + \cdots + k_n$ and the summation is taken over all k_1, k_2, ..., $k_n \in \mathbb{N}_0$ such that $k_1 + 2\, k_2 + \cdots + n\, k_n = n$.

3. Integral representations for $\zeta(n+1)$ $(n \in \mathbb{N})$

Here we present some integral representations for $\zeta(n+1)$ $(n \in \mathbb{N})$, for convenience, as in the following two theorems.

Theorem 1. *The following integral formula holds true*:

$$(3.1) \qquad \zeta(n+1) = \frac{1}{2} + \frac{1}{n} + 2\,(-1)^n \int_0^\infty \frac{t\,\alpha_n(t)}{e^{2\pi t} - 1}\, dt \quad (n \in \mathbb{N}),$$

where, for convenience,

$$(3.2) \qquad \alpha_n(t) := \sum \frac{(-1)^k\, k!}{k_1!\, k_2!} \frac{2^{k_1}}{\left(1 + t^2\right)^{k+1}},$$

$k = k_1 + k_2$ *and summation being taken over all* k_1 *and* k_2 $(k_1,\, k_2 \in \mathbb{N}_0)$ *such that* $k_1 + 2\,k_2 = n$. *The first several explicit formulas of* (3.1) *are given here*:

$$(3.3) \qquad \zeta(2) = \frac{3}{2} + 4 \int_0^\infty \frac{t}{\left(t^2 + 1\right)^2 \left(e^{2\pi t} - 1\right)}\, dt;$$

$$(3.4) \qquad \zeta(3) = 1 - 2 \int_0^\infty \frac{t\left(t^2 - 3\right)}{\left(t^2 + 1\right)^3 \left(e^{2\pi t} - 1\right)}\, dt;$$

$$(3.5) \qquad \zeta(4) = \frac{5}{6} - 8 \int_0^\infty \frac{t\left(t^2 - 1\right)}{\left(t^2 + 1\right)^4 \left(e^{2\pi t} - 1\right)}\, dt;$$

$$(3.6) \qquad \zeta(5) = \frac{3}{4} + 2 \int_0^\infty \frac{t\left(t^4 - 10\,t^2 + 5\right)}{\left(t^2 + 1\right)^5 \left(e^{2\pi t} - 1\right)}\, dt.$$

Proof. Differentiating both sides of (2.1), we obtain a known integral representation for $\psi(z+1)$:

$$\psi(z+1) = \log(z+1) - \frac{1}{2(z+1)}$$

$$(3.7)$$

$$- 2 \int_0^\infty \frac{t}{\left\{(z+1)^2 + t^2\right\} \left(e^{2\pi t} - 1\right)}\, dt \quad (\Re(z) > -1).$$

Now differentiating both sides of (3.7) n times by using easily derivable formulas:

$$(3.8) \qquad \frac{d^n}{dz^n} \log(z+1) = \frac{(-1)^{n-1} (n-1)!}{(z+1)^n} \quad (n \in \mathbb{N})$$

and

$$(3.9) \qquad \frac{d^n}{dz^n} \frac{1}{z+1} = \frac{(-1)^n n!}{(z+1)^{n+1}} \quad (n \in \mathbb{N}_0),$$

and the Faà di Bruno formula (2.8) by setting

$$f(z) = \frac{1}{z} \quad \text{and} \quad g(z,t) = (z+1)^2 + t^2$$

under the integral sign, and putting $z = 0$ in the resulting identity, in view of (1.13) and (1.14), we are led to (3.1). $\square$

Theorem 2. *Each of the following integral formulas holds true*:

$$(3.10) \qquad \zeta(n+1) = \frac{1}{n!} \int_0^\infty \frac{t^n}{e^t - 1}\, dt \quad (n \in \mathbb{N});$$

$$(3.11) \qquad \zeta(n+1) = \frac{1}{n!} \int_0^\infty \frac{\log^n(1+t)}{t\,(1+t)}\, dt \quad (n \in \mathbb{N});$$

$$(3.12) \quad \zeta(n+1) = \frac{1}{n} - \frac{1}{n!} \int_0^\infty \left[t^{-1} - \left(1 - e^{-t}\right)^{-1} \right] e^{-t}\, t^n\, dt \quad (n \in \mathbb{N});$$

$$(3.13) \quad \zeta(n+1) = \frac{1}{n} + \frac{1}{2} + \frac{1}{n!} \int_0^\infty \left[\left(1 - e^{-t}\right)^{-1} - t^{-1} - \frac{1}{2} \right] e^{-t}\, t^n\, dt \quad (n \in \mathbb{N});$$

$$(3.14) \quad \zeta(n+1) = \frac{1}{n} - \frac{1}{n!} \int_0^\infty \left[\left(1 - e^{-t}\right)^{-1} + t^{-1} - 1 \right] e^{-t}\, t^n\, dt \quad (n \in \mathbb{N});$$

$$(3.15) \quad \zeta(n+1) = \frac{1}{n} + \frac{1}{2} + \frac{1}{n!} \int_0^\infty \left[\left(e^t - 1\right)^{-1} - t^{-1} + \frac{1}{2} \right] e^{-t}\, t^n\, dt \quad (n \in \mathbb{N}).$$

Proof. Differentiating both sides of each of the formulas (2.2) to (2.7) by using, if necessary, (3.8) and (3.9), in view of (1.13) and (1.14), we can obtain our desired integral formulas (3.10) to (3.15). $\square$

Remark.   Setting $z = 0$ in (3.7) and (2.2) to (2.7), and using the relation

$$(3.16) \qquad\qquad \psi(1) = -\gamma,$$

we obtain known integral representations for the Euler-Mascheroni constant γ (see, *e.g.*, [1], [2], and [4, Section 1.2]). (2.1) is called Binet's second expression for $\log \Gamma(z)$. The integral formulas (2.2) and (2.3) are due to Gauss and Dirichlet, respectively.

References

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