Honam Mathematical J. **35** (2013), No. 4, pp. 647–655
http://dx.doi.org/10.5831/HMJ.2013.35.4.647

A NOTE ON DECREASING SCALAR CURVATURE FROM FLAT METRICS

Jongsu Kim

Abstract. We obtain C^∞-continuous paths of explicit Riemannian metrics g_t, $0 \le t < \varepsilon$, whose scalar curvatures $s(g_t)$ decrease, where g_0 is a flat metric, i.e. a metric with vanishing curvature. Most of them can exist on tori of dimension ≥ 3. Some of them yield scalar curvature decrease on a ball in the Euclidean space.

1. Introduction

The total scalar curvature is the integral $\int s(g)d\mu$ of the scalar curvature $s(g)$ of a Riemannian metric g. Let $\mathcal{S}(g) = \int_M s(g)d\mu_g$, defined on the space $\mathcal{M}$ of Riemannian metrics on a closed manifold M. The tangent space $T_g\mathcal{M}$ of $\mathcal{M}$ at g consists of symmetric (2,0)-tensors. The first derivative of $\mathcal{S}$ in the direction of $h \in T_g\mathcal{M}$ is $\mathcal{S}_g'(h) = \frac{d}{dt}\mathcal{S}(g+th)|_{t=0} = \int_M \langle r_g - \frac{s_g}{2}g, h\rangle d\mu_g$. The critical points of $\mathcal{S}$, satisfying $r_g = \frac{s_g}{2}g$, are Einstein metrics and in dimension ≥ 3, they are ricci-flat metrics.

In 1974, Muto in [8] has studied the behavior of the total scalar curvature and shown that for any Einstein metric g the second derivative of total scalar curvature, $\frac{d^2}{dt^2}\mathcal{S}(g+th)|_{t=0}$, can be negative for some symmetric (2,0)-tensor h. This implies that for any metric the total scalar curvature can decrease as one smoothly deforms the metric in some direction $h \in T_g\mathcal{M}$. Moreover, Muto could choose h supported in any ball.

Received August 5, 2013. Accepted October 11, 2013.
2010 Mathematics Subject Classification. 53B20, 53C20, 53C21.
Key words and phrases. negative scalar curvature, total scalar curvature, scalar curvature functional.
This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MOE) (No.NRF-2010-0011704).

Meanwhile, in a previous work [6] we have asked the following question:

Question Let (M^n, g_0), $n \geq 3$, be a manifold and $B \subset M$ a ball. Is there a C^∞-continuous path of Riemannian metrics g_t, $0 \leq t < \varepsilon$ on M with

(i) Scalar curvature of g_t is strictly decreasing in t on B.
(ii) $g_t \equiv g_0$ on $M \backslash B$.

If such a path g_t exists, we call it a *scalar-curvature melting* of g_0 in B. This question is a simpler version of the Lohkamp's conjecture in [7], stating that there exists a path g_t as above but with Ricci curvature replacing the scalar curvature in the condition (i).

If there is a scalar-curvature melting g_t in B, then the total scalar curvature $\int_B s(g_t) d\mu_t$ (therefore $\int_M s(g_t) d\mu_t$ as well) is decreasing in t. So, this existence question of scalar-curvature melting is harder than the total scalar curvature decrease that Muto has solved.

In this article, inspired by Muto's second order approach, we analyzed the second order derivative, $\frac{d^2}{dt^2} s(g_t)|_{t=0}$, of the scalar curvature of some metrics g_t near flat metrics g_0. By deforming flat metrics upto second order in this way, we obtain numerous explicit families of scalar-curvature decrease. Most of them can exist on tori of dimension ≥ 3 but some of them yields genuine melting on a ball. It turns out that the latter families g_t are a slight variation of those found in [5], different only in higher order terms.

But, whereas those metrics in [5] appeared as an isolated construction, our approach here gives a unifying view for generalization.

A technical novel part of this paper is that we focused on the role of the second-order variation tensor, denoted by k in next sections, to obtain various scalar-curvature-decreasing metrics.

2. Derivatives of scalar curvature and scalar curvature melting

We may consider the scalar curvature s as a functional defined on the space $\mathcal{M}$ of Riemannian metrics on a manifold.

Recall that in a Riemannian manifold (M, g) the derivative at g, in the direction of a symmetric (2,0)-tensor h, of the scalar curvature $s(g)$ is given [2] by

$$ds_g(h) = \Delta_g(\mathrm{tr}_g h) + \delta(\delta h) - g(ric_g, h), \tag{1}$$

where ric_g is the Ricci curvature tensor of g, Δ_g is the Laplacian, $\mathrm{tr}_g(h)$ is the trace of h with respect to g, δh is the divergence of h which can be written in local coordinates as $(\delta h)_j = -D^i h_{ij}$ for the Levi-Civita connection D and finally $\delta(\cdot)$ for 1-forms is the formal adjoint of the exterior differential on functions.

For another symmetric (2,0)-tensor k, according to Fisher-Marsden [4, section 7, formulas (3) and (6)], we recall

$$\begin{aligned} &\frac{d^2}{dt^2} s(g + th + \frac{t^2}{2}k)|_{t=0} \\ =& -\frac{1}{2}|Dh|^2 + 2\mathrm{ric}_g \cdot (h \times h) - \frac{1}{2}(d\ \mathrm{tr}_g h)^2 \\ &+ D_k h^{ij} D_i h_j^k + 2h \cdot Dd(\mathrm{tr}_g h) - 2(\delta h)\cdot(d\ \mathrm{tr}_g h) \\ &- \Delta_g |h|^2 - 2\delta\delta(h \times h) + \Delta_g \mathrm{tr}_g k + \delta\delta k - g(\mathrm{ric}_g, k), \end{aligned} \tag{2}$$

where $(h \times h)_{ij} = h_{is} h_j^s$.

Consider the linearization ds_g^B of the scalar curvature functional on the space of Riemannian metrics restricted to a ball B. For generic metrics g, ds_g^B is surjective and there exists a symmetric (2,0)-tensor h with its support in B such that $ds_g(h) < 0$ in B; refer to [1, 3]. Then one can get a scalar-curvature melting by some additional argument.

But for some other metrics ds_g^B is not surjective. For example, ds_g^B can not be surjective for metrics with zero ricci curvature. These ricci flat metrics seem important but difficult to handle. But if one can find a path g_t, with g_0 being a ricci-flat metric, such that

$$\frac{d}{dt} s(g_t)|_{t=0} = 0, \qquad \frac{d^2}{dt^2} s(g_t)|_{t=0} < 0, \tag{3}$$

then we may find scalar-curvature decreasing metrics.

In the next sections we search for such paths on the simplest metrics-the metrics with vanishing curvature.

3. Deforming flat metrics in standard coordinates I

Let g_0 be the Euclidean metric on $\mathbb{R}^n$ with the coordinates $(x_1, \cdots, x_n)$.

We want to find $g_t = g_0 + th + \frac{t^2}{2}k$ such that $\frac{d}{dt}s(g_t)|_{t=0} = 0$ and $\frac{d^2}{dt^2}s(g_t)|_{t=0} < 0$. We do not aim for general metrics with such property. So, considering (1) and (2), we look for h with $\mathrm{tr}_{g_0} h = 0$ and $\delta h = 0$ in this section.

We shall only describe the 3-d case for simplicity, but one should get much more examples in higher dimensions in a similar way.

We choose $h := adx_1^2 + bdx_2^2 - (a+b)dx_3^2$ for two functions a and b. We write a_i for $\frac{da}{dx_i}$ and a_{ij} for $\frac{d^2a}{dx_j dxi}$ etc. One easily computes;

$$
\begin{aligned}
&|h|^2 = a^2 + 2b^2,\\
&h \times h = a^2dx_1^2 + b^2dx_2^2 + (a+b)^2dx_3^2,\\
&\tfrac{1}{2}|Dh|^2 = \tfrac{1}{2}\textstyle\sum_{i,j,k=1}^3 (D_ih_{jk})^2 = \tfrac{1}{2}\sum_{i=1}^3\{a_i^2 + b_i^2 + (a_i+b_i)^2\},\\
&D_kh^{ij}D_ih_j^k = \textstyle\sum_{i=1}^3 D_ih_{ii}D_ih_{ii} = a_1^2 + b_2^2 + (a_3+b_3)^2.
\end{aligned}
$$

We choose $k = h \times h$. Then $\mathrm{tr}_{g_0}k = |h|^2$.

We assume that $\delta h = 0$ holds, which is equivalent to $a_1 = 0, b_2 = 0$ and $a_3 + b_3 = 0$. From (1) we then get the equality $ds_g(h) = 0$ and $\delta\delta(-2h \times h + k) = -\delta\delta\{a^2dx_1^2 + b^2dx_2^2 + (a+b)^2dx_3^2\} = 0.$

Put these into (2), we get $\frac{d^2}{dt^2}s(g_t)|_{t=0} = -\frac{1}{2}\sum_{i=1}^3\{a_i^2+b_i^2+(a_i+b_i)^2\} \leq 0.$

As we look for (3), we only need nonzero smooth functions a and b satisfying $a_1 = 0, b_2 = 0$ and $a_3 + b_3 = 0$. There can be no such a and b supported in a ball. But we may have periodic functions, $a = p(x_2) + q(x_3)$ and $b = r(x_1) - q(x_3)$, where p, q, r are any periodic functions. Then for small t, g_t is a family of scalar-curvature-decreasing metrics on the torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$.

Precisely speaking, $\frac{d^2}{dt^2}s(g_t)|_{t=0}$ may be zero on a thin subset of T^3, upon a choice of a and b. So, g_t may not yet be scalar-curvature decreasing, but one can find a modifying family $\tilde{g}_t$ with the property. This type of argument is explained in [6, Section 4].

4. Deforming flat metrics in standard coordinates II

In the previous section we looked for h with $\mathrm{tr}_{g_0}h = 0, \delta h = 0$ and k with $k = h \times h$. But here let us search for some other case. We now do not require $\delta h = 0$ nor $k = h \times h$.

We shall only do the 4-dimensional case in the standard coordinates (x_1, x_2, x_3, x_4), but higher dimensional generalizations should be immediate.

We choose $h := adx_1^2 - adx_2^2 + bdx_3^2 - bdx_4^2$ for two functions a and b. We get

$$
\begin{aligned}
&|h|^2 = 2(a^2+b^2),\\
&h\times h = a^2dx_1^2 + a^2dx_2^2 + b^2dx_3^2 + b^2dx_4^2,\\
&D_kh^{ij}D_ih_j^k = \textstyle\sum_{i=1}^4 D_ih_{ii}D_ih_{ii} = a_1^2+a_2^2+b_3^2+b_4^2,\\
&\tfrac{1}{2}|Dh|^2 = \tfrac{1}{2}\textstyle\sum_{i,j,k=1}^4 (D_ih_{jk})^2 = \sum_{i=1}^4 (a_i^2+b_i^2),\\
&\delta\delta h = D_iD_jh_{ij} = \textstyle\sum_{i=1}^4 D_iD_ih_{ii} = a_{11}-a_{22}+b_{33}-b_{44}.
\end{aligned}
$$

This time we choose

$$k = 2a^2dx_1^2 + 2b^2dx_3^2.$$

Then $\mathrm{tr}_{g_0}k = |h|^2$ and $\delta\delta(-2h\times h+k) = -2\delta\delta(a^2dx_2^2+b^2dx_4^2) = -4\{aa_{22}+(a_2)^2+bb_{44}+(b_4)^2\}$.
Summarizing the above, we get;

$$
\begin{aligned}
\frac{d}{dt}s(g_0+th+\frac{t^2}{2}k)|_{t=0} &= \delta\delta h = a_{11}-a_{22}+b_{33}-b_{44}.\\
\frac{d^2}{dt^2}s(g_0+th+\frac{t^2}{2}k)|_{t=0} &= -(a_3^2+a_4^2+b_1^2+b_2^2)\\
&\quad -4\{aa_{22}+(a_2)^2+bb_{44}+(b_4)^2\}.
\end{aligned}
$$

We require that $a_{22}=0$ and $b_{44}=0$. Then (3) is possible if $a_{11}+b_{33}=0$. There can be no such nonzero smooth functions a and b supported in a ball. But we can get periodic functions $a=a(x_1,x_3)$ and $b=b(x_1,x_3)$ by integrating $a_{11}=\alpha(x_1)\beta(x_3)$ and $b_{33}=-\alpha(x_1)\beta(x_3)$, with proper choice of two functions α and β. For example, we get $a=\sin(x_1)\sin(x_3)=-b$. This would produce a family of scalar-curvature-decreasing metrics on the 4-d torus, just as in the previous section.

5. Deforming Euclidean metrics in polar coordinates

In the previous sections we have obtained scalar-curvature-decreasing metrics on tori, but not a scalar-curvature melting in a ball of Euclidean space.

So, we shall try an alternative way, which is to use another coordinates: let $(r,\theta),(\rho,\sigma)$ be the polar coordinates for each summand of $\mathbb{R}^4:=\mathbb{R}^2\times\mathbb{R}^2$ respectively. Then the Euclidean metric g_0 becomes

$$g_0 = dr^2 + r^2d\theta^2 + d\rho^2 + \rho^2d\sigma^2, \tag{4}$$

We write $\partial_r := \frac{\partial}{\partial r}$, $\partial_\theta := \frac{\partial}{\partial \theta}$, etc.. It is easy to compute for the Levi-Civita connection D;

$$
\begin{aligned}
&D_{\partial_r}\partial_r = 0, && D_{\partial_r}\partial_\theta = D_{\partial_\theta}\partial_r = \frac{\partial_\theta}{r}, && D_{\partial_\theta}\partial_\theta = -r\partial_r,\\
&D_{\partial_\rho}\partial_\rho = 0, && D_{\partial_\rho}\partial_\phi = D_{\partial_\phi}\partial_\rho = \frac{\partial_\phi}{\rho}, && D_{\partial_\phi}\partial_\phi = -\rho\partial_\rho,\\
&Ddr = rd\theta \otimes d\theta, && Dd\theta = -\frac{1}{r}(dr \otimes d\theta + d\theta \otimes dr).
\end{aligned} \tag{5}
$$

We choose $h = adr^2 - ar^2 d\theta^2 + bd\rho^2 - b\rho^2 d\phi^2$. We shall choose a and b to be functions of r and ρ only. Then $\mathrm{tr}_{g_0} h = 0$. From (5), we compute Dh and δh;

$$
\begin{aligned}
Dh =& D(adr^2 - ar^2 d\theta^2 + bd\rho^2 - b\rho^2 d\phi^2)\\
=& a_r dr \otimes dr \otimes dr + a_\rho d\rho \otimes dr \otimes dr + 2ard\theta \otimes d\theta \otimes dr\\
&- a_r r^2 dr \otimes d\theta \otimes d\theta - a_\rho r^2 d\rho \otimes d\theta \otimes d\theta + 2ard\theta \otimes dr \otimes d\theta\\
&+ b_\rho d\rho \otimes d\rho \otimes d\rho + b_r dr \otimes d\rho \otimes d\rho + 2b\rho d\phi \otimes d\phi \otimes d\rho\\
&- b_\rho \rho^2 d\rho \otimes d\phi \otimes d\phi - b_r \rho^2 dr \otimes d\phi \otimes d\phi + 2b\rho d\phi \otimes d\rho \otimes d\phi.\\
\delta h =& - a_r dr - \frac{2a}{r} dr - b_\rho d\rho - \frac{2b}{\rho} d\rho.
\end{aligned}
$$

Note that if we require $\delta h = 0$, we do not get a good solution a or b. So we shall require $\delta\delta h = 0$. We also set $k = 2(a^2 dr^2 + b^2 d\rho^2)$. Then,

$$
\begin{aligned}
&\mathrm{tr}_{g_0}(k) = |h|^2 = 2(a^2 + b^2),\\
&h \times h = a^2 dr^2 + a^2 r^2 d\theta^2 + b^2 d\rho^2 + b^2 \rho^2 d\phi^2,\\
&D_k h^{ij} D_i h^k_j = a_r^2 - 4\tfrac{aa_r}{r} + 4\tfrac{a^2}{r^2} + b_\rho^2 - 4\tfrac{bb_\rho}{\rho} + 4\tfrac{b^2}{\rho^2},\\
&\tfrac{1}{2}|Dh|^2 = a_\rho^2 + a_r^2 + 4\tfrac{a^2}{r^2} + b_r^2 + b_\rho^2 + 4\tfrac{b^2}{\rho^2},\\
&\delta\delta h = a_{rr} + 3\tfrac{a_r}{r} + b_{\rho\rho} + 3\tfrac{b_\rho}{\rho}\\
&\delta\delta(-2h \times h + k) = -2\delta\delta(a^2 r^2 d\theta^2 + b^2 \rho^2 d\phi^2) = 4(\tfrac{aa_r}{r} + \tfrac{bb_\rho}{\rho}).
\end{aligned}
$$

Summarizing the above for $g_t = g_0 + th + \frac{t^2}{2}k$,

$$
\begin{aligned}
\frac{d}{dt}s(g_t)|_{t=0} &= \delta\delta h\\
&= a_{rr} + 3\tfrac{a_r}{r} + b_{\rho\rho} + 3\tfrac{b_\rho}{\rho}
\end{aligned}
$$

$$\begin{aligned}\frac{d^2}{dt^2}s(g_t)|_{t=0} =& -\frac{1}{2}|Dh|^2 + D_k h^{ij} D_i h_j^k - \Delta_{g_0}|h|^2 - 2\delta\delta(h\times h)\\ &+ \Delta \mathrm{tr}_{g_0} k + \delta\delta k\\ =& -\frac{1}{2}|Dh|^2 + D_k h^{ij} D_i h_j^k + \delta\delta(-2h\times h + k)\\ =& -a_\rho^2 - a_r^2 - 4\frac{a^2}{r^2} - b_r^2 - b_\rho^2 - 4\frac{b^2}{\rho^2} + a_r^2 - 4\frac{aa_r}{r} + 4\frac{a^2}{r^2} + b_\rho^2\\ &- 4\frac{bb_\rho}{\rho} + 4\frac{b^2}{\rho^2} + 4\frac{aa_r}{r} + 4\frac{bb_\rho}{\rho}\\ =& -a_\rho^2 - b_r^2\end{aligned}$$

Let $\mathcal{D}$ be the 2-dimensional unit disc in $\mathbb{R}^2$. We want smooth functions a and b on $\mathbb{R}^4$ with support in $\mathcal{D}\times\mathcal{D}\subset\mathbb{R}^4$ such that

$$a_{rr} + 3\frac{a_r}{r} + b_{\rho\rho} + 3\frac{b_\rho}{\rho} = 0, \tag{6}$$

$$-a_\rho^2 - b_r^2 < 0.$$

Owing to the support condition, a strict negativity of $-a_\rho^2 - b_r^2$ in $\mathcal{D}\times\mathcal{D}$ is impossible. We may only get that $-a_\rho^2 - b_r^2 < 0$ except a thin subset of $\mathcal{D}\times\mathcal{D}$. As we want $a_{rr} + 3\frac{a_r}{r} + b_{\rho\rho} + 3\frac{b_\rho}{\rho} = 0$ in (6), we rediscovered the equation (3.2) of [5]. So, we may recall the solution there.

Setting $a_{rr} + \frac{3}{r}a_r = \alpha(r)\beta(\rho)$ and $b_{\rho\rho} + 3\frac{b_\rho}{\rho} = -\alpha(r)\beta(\rho)$ where α, β are smooth functions on $\mathbb{R}$, we do integration to get

$$a(r,\rho) = \beta(\rho)\int_0^r \left(\frac{1}{y^3}\int_0^y x^3\alpha(x)\,dx\right)dy,$$

$$b(r,\rho) = -\alpha(r)\int_0^\rho \left(\frac{1}{y^3}\int_0^y x^3\beta(x)\,dx\right)dy.$$

The function α is specified as follows; first consider a smooth function $p(y)$ on $\mathbb{R}$ such that

$$\begin{cases} a)\ p(y) = 0 \quad \text{for} \quad y\le 0, y\ge 1,\\ b)\ |\frac{p'(y)}{y^3}| \ll 1, \ \text{for } y>0\\ c)\ \int_0^1 \frac{p(y)}{y^3}\,dy = 0,\\ d)\ 0 < \int_0^r \frac{p(y)}{y^3}\,dy < 1 \quad \text{for any} \quad r \quad \text{with} \quad 0<r<1\end{cases}$$

and then define $\alpha(y) = \frac{p'(y)}{y^3}$. β can be quite similarly defined.

Then the functions $a(r,\rho)$ and $b(r,\rho)$ satisfy the equation (6) and

$$a,\ b \equiv 0 \quad \text{for } r \le 0,\ r \ge 1 \ \text{ or } \ \rho \le 0,\ \rho \ge 1.$$

For such functions a and b, we have

$$\begin{aligned} &\frac{d}{dt} s(g_t)|_{t=0} = 0 \\ &\frac{d^2}{dt^2} s(g_t)|_{t=0} = -\beta'(\rho)^2 \left\{ \int_0^r \left(\frac{1}{y^3} \int_0^y x^3 \alpha(x)\, dx \right) dy \right\}^2 \\ &\qquad\qquad - \alpha'(r)^2 \left\{ \int_0^\rho \left(\frac{1}{y^3} \int_0^y x^3 \beta(x)\, dx \right) dy \right\}^2 . \end{aligned}$$

Note that now h and k has support in $\mathcal{D} \times \mathcal{D}$. We proved;

Theorem 1. *There exists a C^∞-continuous one-parameter family of Riemannian metrics g_t on $\mathbb{R}^4$ for $0 \le t < \varepsilon$ for some number ε with the following property: g_0 is the Euclidean metric on $\mathbb{R}^4$, g_t is isometric to g_0 in the complement of the polydisc $\mathcal{D} \times \mathcal{D} \subset \mathbb{R}^4$, the derivative $\frac{ds(g_t)}{dt}|_{t=0}$ of scalar curvatures of g_t is identically zero and $\frac{d^2 s(g_t)}{dt^2}|_{t=0}$ is negative in the polydisc except a thin subset.*

The above g_t is not yet a scalar curvature melting because $\frac{d^2 s(g_t)}{dt^2}|_{t=0}$ is zero somewhere in the polydisc. As explained in Section 2, one can apply the "modification" process of the section 4 of [6]. In fact, we diffuse the negativity of $s(g_t)$ onto a ball by conformally deforming the metric to $\tilde{g}_t = e^{2\phi_t} g_t$, where ϕ_t is a family of functions with $\phi_0 = 0$. Then we get a scalar curvature melting $\tilde{g}_t$ of the Euclidean metric in a ball.

Remark 1. Here we have obtained explicit scalar-curvature decreasing metrics by finding the deforming tensors h and k. Of course, it should be understood that much more examples can be gotten by resorting to other general methods. By working on this explicit method, we only intend to give some insight into the geometry. I hope to address its generalization in near future.

References

[1] R. Beig, P. T. Chruściel and R. Schoen, *KIDs are non-generic*, Ann. Henri Poincare. **6(1)** (2005), 155-194.

[2] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik. 3 Folge, Band 10, Springer-Verlag, 1987.

[3] J. Corvino, *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Comm. Math. Phys. **214** (2000), 137-189.
[4] A. E. Fisher and J. E. Marsden, *Deformation of scalar curvture*, Duke Math. Journal. **42(3)** (1975), 519-547.
[5] Y. Kang and J. Kim, *Almost Kähler metrics with non-positive scalar curvature which are Euclidean away from a compact set*, Jour. Korean. Math. Soc. **41(5)** (2004), 809-820.
[6] J. Kim, *Melting of Euclidean metric to negative scalar curvature*, Bull. Korean Math. Soc. **50(4)** (2013), 1087-1098.
[7] J. Lohkamp, *Curvature h-principles*, Ann. of Math. (2) **142** (1995), 457-498.
[8] Y. Muto, *On Einstein metrics*, Jour. Differential Geom. **9** (1974), 521-530.

Jongsu Kim
Department of Mathematics, Sogang University,
Seoul 121-742, KOREA.
E-mail: jskim@sogang.ac.kr