Honam Mathematical J. **35** (2013), No. 4, pp. 679–681
http://dx.doi.org/10.5831/HMJ.2013.35.4.679

ON A QUESTION RELATED TO KOLMOGOROV FORWARD EQUATION

Joongul Lee

Abstract. In this paper, we raise a question regarding existence and uniqueness of the steady-state solution of a type of Kolmogorov forward equation. We prove that the question has a positive answer for trivial vector fields, and also provide examples which show that the compactness assumption is necessary.

1. Introduction

Let $\mathcal{M}$ be an orientable Riemannian manifold of dimension n, whose inner product at each point is denoted by $\langle\ ,\ \rangle$. The inner product gives us a canonical isomorphism between the 1-forms on $\mathcal{M}$ and the vector fields on $\mathcal{M}$. For convenience we denote the isomorphism and its inverse by a *tilde.* Then if f is a function on $\mathcal{M}$, its gradient is the vector field

$$\nabla f = \widetilde{df}.$$

And if E is a vector field on $\mathcal{M}$, its divergence is the function

$$\nabla\cdot E = *\,d{*}\widetilde{E}$$

where $*$ is the Hodge star operator. (cf. [4])

Question. Assume $\mathcal{M}$ is compact. For a vector field X on $\mathcal{M}$, is it possible to find unique vector fields E, F with the following properties?

(1) $X = E + F$,
(2) $F = -\nabla\phi$ for some function ϕ on $\mathcal{M}$,
(3) $\langle E, F\rangle + \nabla\cdot E = 0$ at every point of $\mathcal{M}$.

Received August 22, 2013. Accepted September 10, 2013.
2010 Mathematics Subject Classification. Primary 58J65; Secondary 53A45.
Key words and phrases. Kolmogorov forward equation, Fokker-Planck equation.
This work was supported by 2011 Hongik University Research Fund.

2. Background

Kolmogorov forward equation, also known as Fokker-Planck equation, describes the Brownian motion of particles. One may consult [3] for general background.

Given a vector field X on $\mathbb{R}^n$, let us consider the following type of Kolmogorov forward equation

$$\frac{\partial}{\partial t}\rho(x,t) = \nabla\cdot(-X+\nabla)\rho(x,t) \tag{1}$$

where $\rho(x,t)$ is the probability density function at time t. We are interested in the steady-state solution of equation (1), i.e. $\frac{\partial \rho}{\partial t} = 0$.

Let us assume that the steady-state solution is of the form

$$\rho(x) = e^{-\phi(x)}.$$

Let $F = -\nabla\phi$ and $E = X - F$. Then from the steady-state condition, we get

$$\nabla\cdot(-e^{-\phi}X - e^{-\phi}\nabla\phi) = -\nabla\cdot(e^{-\phi}E) = 0,$$

which is equivalent to

$$\langle E, F\rangle + \nabla\cdot E = 0. \tag{2}$$

Conversely, if we can write $X = E + F$ satisfying equation (2) where F is a conservative vector field (i.e. gradient of a function), then we obtain the steady-state solution of equation (1).

Note that we made choices of signs consistent with statistical physics. For more detailed discussion and application to statistical physics, one may consult [1], and [2].

3. Main Result and Examples

Theorem 1. *If $X = 0$, then the question has an affirmative answer for every compact $\mathcal{M}$.*

Proof. $E = F = 0$ is clearly a solution. For uniqueness, the equation becomes $\nabla\cdot E = \langle E, E\rangle$ in this case. By Stokes' theorem $0 = \int_{\mathcal{M}} \nabla\cdot E = \int_{\mathcal{M}}\langle E, E\rangle$ therefore $E = 0$ (and hence $F = 0$) on $\mathcal{M}$. □

The following examples show that the solution of the question is in general not unique without the compactness assumption.

Example 1. Let $\mathcal{M} = \mathbb{R}$ and $X = 0$. Consider $\nabla\cdot E = \langle E, E\rangle$ which in this case becomes

$$\frac{dy}{dx} = y^2. \tag{3}$$

We find that

$$y = -\frac{1}{(x+c)}$$

where c is an arbitrary constant, is a solution of equation (3). Clearly every vector field on $\mathbb{R}$ is conservative, hence we have infinitely many solutions on an open subset of $\mathbb{R}$. Using the product of this solution, we have infinitely many solutions of the question when $\mathcal{M}$ is an open subset of $\mathbb{R}^n$ and $X = 0$.

Example 2. Let $\mathcal{M} = \mathbb{R}^n$ where $n \geq 2$ and let X be a nonzero constant vector field on $\mathcal{M}$. There are infinitely many ways to write a given vector as a sum of two vectors orthogonal to each other. Since a constant vector field is conservative (being the gradient of a linear function) and divergence-free, we conclude that there are infinitely many ways of writing $X = E + F$ satisfying the conditions of the question.

References

[1] Chulan Kwon and Ping Ao, *Nonequilibrium steady state of a stochastic system driven by a nonlinear drift force*, Physical Review E, 84(6):061106, 2011.

[2] Jae Dong Noh, *Stochastic echo phenomena in nonequilibrium systems*, arXiv:1212.2706, **12** (2012).

[3] H. Risken, *The Fokker-Planck equation*, volume 18 of Springer Series in Synergetics, Springer-Verlag, Berlin, second edition, 1989. Methods of solution and applications.

[4] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Scott, Foresman and Co., Glenview, Ill.-London, 1971.

Joongul Lee
Department of Mathematics Education, Hongik University,
Seoul 121-791, Korea.
E-mail: jglee@hongik.ac.kr