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NEW EXACT SOLUTIONS OF SOME NONLINEAR EVOLUTION EQUATIONS BY SUB-ODE METHOD

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Abstract. In this paper, an improved $\left(\frac{G'}{G}\right)$-expansion method is proposed for obtaining travelling wave solutions of nonlinear evolution equations. The proposed technique called $\left(\frac{F}{G}\right)$-expansion method is more powerful than the method $\left(\frac{G'}{G}\right)$-expansion method. The efficiency of the method is demonstrated on a variety of nonlinear partial differential equations such as KdV equation, mKdV equation and Boussinesq equations. As a result, more travelling wave solutions are obtained including not only all the known solutions but also the computation burden is greatly decreased compared with the existing method. The travelling wave solutions are expressed by the hyperbolic functions and the trigonometric functions. The result reveals that the proposed method is simple and effective, and can be used for many other nonlinear evolutions equations arising in mathematical physics.

1. Introduction

The nonlinear wave phenomena appears in various scientific and engineering fields such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and so on. In order to understand better the nonlinear phenomena as well as further application in the practical life, it is important to seek their more exact travelling wave solutions. Thus the methods for deriving exact solutions for the governing equations have to be developed. Many

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powerful methods are used to obtain travelling solitary wave solutions to nonlinear partial differential equations (PDEs) such as tanh method [12, 13], the ansatz method [5, 6], the sub-ODE method [3, 4], Jacobi elliptic function method [1], exp-function method [7, 8] and so on.

However, practically there is no unified method that can be used to handle all types of nonlinear partial differential equations. Recently, Wang et al. [2] introduced a new direct method called the $\left(\frac{G'}{G}\right)$-expansion method to look for travelling wave solutions of nonlinear evolution equations. One of the most effective straightforward method to construct exact solutions of PDEs is the $\left(\frac{G'}{G}\right)$-expansion method [9, 10, 11]. The $\left(\frac{G'}{G}\right)$ -expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$. Later Zhang et al. [16] proposed a generalized $\left(\frac{G'}{G}\right)$-expansion method to improve and extend Wang et al.s work [2] for solving variable-coefficient equations and high-dimensional equations.

Motivated by the work in [2], the main purpose of this paper is to introduce a new technique called $\left(\frac{F}{G}\right)$-expansion method to find more exact solutions for nonlinear equations. The main idea behind the $\left(\frac{F}{G}\right)$-expansion method is that the travelling wave solutions of a nonlinear evolution equation can be expressed by a polynomial in $\left(\frac{F}{G}\right)$, where $G = G(\xi)$ and $F = F(\xi)$ satisfy the first order linear ordinary differential system (FLODS) as follows: $F'(\xi) = \lambda G(\xi), G'(\xi) = \mu F(\xi)$, $F' = \frac{dF(\xi)}{d\xi}, G' = \frac{dG(\xi)}{d\xi}, \xi = x - \omega t, \lambda, \mu$ and ω are constants. The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given nonlinear evolution equations. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. The efficiency of the $\left(\frac{F}{G}\right)$-expansion method is demonstrated on a variety of nonlinear partial equations such as KdV equation, mKdV equation and Boussinesq equations. Moreover, the proposed method is capable of greatly minimizing the size of computational work compared to the existing technique. This is due to use of simple first order linear ordinary differential system in the proposed algorithm instead of second order linear ordinary differential equation as in [2].

The rest of this paper is organized as follows. In Section 2, we explain the $\left(\frac{F}{G}\right)$-expansion method for finding travelling wave solutions of nonlinear evolution equations and give some advantage of the method. From Section 3 to Section 5, we illustrate the method in detail with use of KdV equation, mKdV equation and the variant Boussinesq equations. In Section 6, some conclusions are given.

2. Description of the $\left(\frac{F}{G}\right)$-expansion method

The $\left(\frac{F}{G}\right)$-expansion method for finding travelling wave solutions of nonlinear evolution equations is the followings:
Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{xx}, u_{tt}, ...) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the $\left(\frac{F}{G}\right)$-expansion method.

Step 1. Combining the independent variables x and t into one variable $\xi = x - \omega t$, we suppose that

$$u(x,t) = u(\xi), \xi = x - \omega t, \tag{2}$$

the travelling wave variable (2) permits us reducing (1) to an ODE for $u = u(\xi)$

$$Q(u, u', u'', u''', ...) = 0. \tag{3}$$

Step 2. Suppose that the solution of ODE (3) can be expressed by a polynomial in $\left(\frac{F}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{F}{G}\right)^i, \tag{4}$$

where $G = G(\xi)$ and $F = F(\xi)$ satisfy the FLODS in the form

$$F'(\xi) = \lambda G(\xi),\ G'(\xi) = \mu F(\xi). \tag{5}$$

$a_0, a_1, \cdots, a_n$, λ and μ are constants to be determined later, $a_n \neq 0$. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3).

With the aid of Eq.(5), we can find the following solutions $F(\xi)$ and $G(\xi)$, which are listed as follows:

Type 1. If $\lambda > 0$ and $\mu > 0$, then Eq.(5) has the following hyperbolic function solutions:

$$(6)\qquad \begin{cases} F(\xi) = C_1\cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2\frac{\sqrt{\lambda}}{\sqrt{\mu}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi), \\ G(\xi) = C_1\frac{\sqrt{\mu}}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2\cosh(\sqrt{\lambda}\sqrt{\mu}\xi). \end{cases}$$

Type 2. If $\lambda < 0$ and $\mu < 0$, then Eq.(5) has the following hyperbolic function solutions:

$$(7)\qquad \begin{cases} F(\xi) = C_1\cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2\frac{\sqrt{-\lambda}}{\sqrt{-\mu}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi), \\ G(\xi) = -C_1\frac{\sqrt{-\mu}}{\sqrt{-\lambda}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2\cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi). \end{cases}$$

Type 3. If $\lambda > 0$ and $\mu < 0$, then Eq.(5) has the following trigonometric function solutions:

$$(8)\qquad \begin{cases} F(\xi) = C_1\cos(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2\frac{\sqrt{\lambda}}{\sqrt{-\mu}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi), \\ G(\xi) = -C_1\frac{\sqrt{-\mu}}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2\cos(\sqrt{\lambda}\sqrt{-\mu}\xi). \end{cases}$$

Type 4. If $\lambda < 0$ and $\mu > 0$,then Eq.(5) has the following trigonometric function solutions:

$$(9)\qquad \begin{cases} F(\xi) = C_1\cos(\sqrt{-\lambda}\sqrt{\mu}\xi) - C_2\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi), \\ G(\xi) = C_1\frac{\sqrt{\mu}}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi) + C_2\cos(\sqrt{-\lambda}\sqrt{\mu}\xi). \end{cases}$$

Step 3. Substituting (4) and Eq.(5) into Eq.(3) separately yields a set of algebraic equations for $\left(\frac{F}{G}\right)^i$ $(i = 1, 2, \cdots, n)$. Setting the coefficients of $\left(\frac{F}{G}\right)^i$ to zero yields a set of nonlinear algebraic equations in $a_0, a_i(i = 1, 2, \cdots, n)$ and ω. Solving the nonlinear algebraic equations by Maple and Mathematica, we obtain many exact solutions of Eq.(1) according to (2), (4), (6), (7), (8) and (9).

Remark 2.1. *From the above cases, it is concluded that the proposed method can produce more travelling solutions compare with the* $\left(\frac{G'}{G}\right)$*-expansion method. This can be easily seen from the characteristic equation of Eq.(5).*

Remark 2.2. *In this remark, we compare the computation work with the existing method. The* $\left(\frac{F}{G}\right)$*-expansion method gives the derivations*

$$\left(\frac{F}{G}\right)' = \lambda - \mu\left(\frac{F}{G}\right)^2$$

$$\left(\frac{F}{G}\right)'' = -2\lambda\mu\left(\frac{F}{G}\right) + 2\mu^2\left(\frac{F}{G}\right)^3$$

$$\left(\frac{F}{G}\right)''' = -2\lambda^2\mu + 8\lambda\mu^2\left(\frac{F}{G}\right)^2 - 6\mu^3\left(\frac{F}{G}\right)^4$$

$$\vdots$$

The $\left(\frac{G'}{G}\right)$*-expansion method yields the following derivations*

$$\left(\frac{G'}{G}\right)' = -\lambda\left(\frac{G'}{G}\right) - \mu - \left(\frac{G'}{G}\right)^2$$

$$\left(\frac{G'}{G}\right)'' = \lambda\mu + (\lambda^2 + 2\mu)\left(\frac{G'}{G}\right) + 3\lambda\left(\frac{G'}{G}\right)^2 + 2\left(\frac{G'}{G}\right)^3$$

$$\left(\frac{G'}{G}\right)''' = -(\lambda^2\mu + 2\mu^2) - (\lambda^3 + 8\lambda\mu)\left(\frac{G'}{G}\right) - (7\lambda^2 + 8\mu)\left(\frac{G'}{G}\right)^2 - 12\lambda\left(\frac{G'}{G}\right)^3 - 6\left(\frac{G'}{G}\right)^4$$

$$\vdots$$

From the above result we conclude that the proposed method is capable of greatly minimizing the size of computational work compared to the existing technique.

Because of the significant role of the KdV and the mKdV equations in the solitary wave theory, the methods presented above will be applied to these two equations first, then we will proceed to the variant Boussinesq equation.

3. Application to the KdV equation

The KdV equation

$$u_t + uu_x + \delta u_{xxx} = 0, \tag{10}$$

is an important mathematical model with wide application in quantum mechanics and nonlinear optics. Eq.(10) is integrable and it has also been used to describe a number of important physical phenomena such as magnetohydrodynamics waves in a warm plasma, acoustic waves in an harmonic crystal and ion-acoustic waves [17, 19]. Certainly there are many different exact solutions of Eq.(10). This equation has soliton, rational and elliptic solutions [2, 15]. Recently Wang [2] made an effort to obtain some new exact solutions of the Korteweg-de Vries equation. He used $\left(\frac{G'}{G}\right)$-expansion method to solve the Korteweg-de Vries equations.

The KdV equation(10) can be converted to the ODE

$$-\omega u + \frac{1}{2}u^2 + \delta u'' = 0, \tag{11}$$

upon using $u(x,t) = u(\xi), \xi = x - \omega t$ and integrating the resulting ODE once and neglecting the constant of integration.

Suppose that the solution of ODE (11) can be expressed by a polynomial in $\left(\frac{F}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{F}{G}\right)^i, \tag{12}$$

where $F = F(\xi)$ and $G = G(\xi)$ satisfy the FLODS in the form

$$F'(\xi) = \lambda G(\xi),\ G'(\xi) = \mu F(\xi). \tag{13}$$

By using (12) and Eq.(13) it can be easily derived that

$$\begin{cases} u^2 = a_n^2 \left(\frac{F}{G}\right)^{2n} + a_{n-1}^2 \left(\frac{F}{G}\right)^{2(n-1)} + \cdots, \\ u' = -n\mu a_n \left(\frac{F}{G}\right)^{n+1} - (n-1)\mu a_{n-1} \left(\frac{F}{G}\right)^n + \cdots, \\ u'' = n(n+1)\mu^2 a_n \left(\frac{F}{G}\right)^{n+2} + (n-1)n\mu^2 a_{n-1} \left(\frac{F}{G}\right)^{n+1} + \cdots. \end{cases} \tag{14}$$

Considering the homogeneous balance between u'' and u^2 in Eq.(11) gives $n = 2$, so we can write (12) as

$$u(\xi) = a_2 \left(\frac{F}{G}\right)^2 + a_1 \left(\frac{F}{G}\right) + a_0, a_2 \neq 0, \tag{15}$$

and therefore

$$\begin{cases} u^2 = & a_2^2 \left(\frac{F}{G}\right)^4 + 2a_2 a_1 \left(\frac{F}{G}\right)^3 + (a_1^2 + 2a_2 a_0) \left(\frac{F}{G}\right)^2 \\ & +2a_1 a_0 \left(\frac{F}{G}\right) + a_0^2, \\ u'' = & 6\mu^2 a_2 \left(\frac{F}{G}\right)^4 + 2\mu^2 a_1 \left(\frac{F}{G}\right)^3 - 8\lambda\mu a_2 \left(\frac{F}{G}\right)^2 \\ & -2\lambda\mu a_1 \left(\frac{F}{G}\right) + 2\lambda^2 a_2. \end{cases} \tag{16}$$

By substituting (15) and (16) into Eq.(11) and collecting all terms with the same power of $\left(\frac{F}{G}\right)$ together, the left-hand side of Eq.(11) is converted into another polynomial in $\left(\frac{F}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_2, a_1, a_0, \omega, \lambda, \mu$ and δ as follows:

$\left(\frac{F}{G}\right)^0 : -\omega a_0 + \frac{1}{2}a_0^2 + 2\delta a_2 \lambda^2 = 0$

$\left(\frac{F}{G}\right)^1 : -\omega a_1 + a_0 a_1 - 2\delta a_1 \lambda\mu = 0$

$\left(\frac{F}{G}\right)^2 : -\omega a_2 + a_0 a_2 + \frac{1}{2}a_1^2 - 8\delta a_2 \lambda\mu = 0$

$\left(\frac{F}{G}\right)^3 : a_1 a_2 + 2\delta a_1 \mu^2 = 0$

$\left(\frac{F}{G}\right)^4 : \frac{1}{2}a_2^2 + 6\delta a_2 \mu^2 = 0$

Solving the system of algebraic equations using Maple yields the following sets of nontrivial solutions:

Case 1. $\{a_2 = -12\delta\mu^2, a_1 = 0, a_0 = 4\delta\lambda\mu, \omega = -4\delta\lambda\mu\}$,

Case 2. $\{a_2 = -12\delta\mu^2, a_1 = 0, a_0 = 12\delta\lambda\mu, \omega = 4\delta\lambda\mu\}$.

By using the above cases of the coefficients, expression (15) can be written as the following:

$$u(x,t) = a_2 \left(\frac{F}{G}\right)^2 + a_1 \left(\frac{F}{G}\right) + a_0, \tag{17}$$

where $\xi = x - \omega t$.

Substituting the above cases of the coefficients into (17) we can obtain four types of travelling wave solutions of the KdV equation (10) as follows:
First we consider the Case 1. When $\lambda > 0$ and $\mu > 0$, we obtain the hyperbolic function solutions

$$\begin{aligned} u_{11}(x,t) = &- 12\delta\mu^2 \left(\frac{C_1 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \frac{\sqrt{\lambda}}{\sqrt{\mu}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi)}{C_1 \frac{\sqrt{\mu}}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi)} \right)^2 \\ &+ 4\delta\lambda\mu, \end{aligned} \tag{18}$$

where $\xi = x + 4\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be recovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then (18) can be written as

$$u_{11}(x,t) = 12\delta\lambda\mu \operatorname{sech}^2(\sqrt{\lambda\mu}\xi + \xi_0) - 8\delta\lambda\mu,$$

which is the known solitary wave solution of Eq.(10) obtain in [20], where $\xi_0 = \tanh^{-1} \frac{C_1}{C_2\sqrt{\frac{\lambda}{\mu}}}$ and $\xi = x + 4\delta\lambda\mu t$.

When $\lambda < 0$ and $\mu < 0$,we obtain the hyperbolic function solutions

(19)
$$u_{12}(x,t) = -\,12\delta\mu^2 \left(\frac{C_1\cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2\frac{\sqrt{-\lambda}}{\sqrt{-\mu}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{-\lambda}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2\cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi)} \right)^2 + 4\delta\lambda\mu,$$

where $\xi = x + 4\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be recovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then (19) can be written as

$$u_{12}(x,t) = 12\delta\lambda\mu\ \mathrm{sech}^2(\sqrt{\lambda\mu}\xi - \xi_0) - 8\delta\lambda\mu,$$

where $\xi_0 = \tanh^{-1}\frac{C_1}{C_2\sqrt{\frac{\lambda}{\mu}}}$ and $\xi = x + 4\delta\lambda\mu t$.

When $\lambda > 0$ and $\mu < 0$, we obtain the trigonometric function solutions

$$u_{13}(x,t) = -\,12\delta\mu^2 \left(\frac{C_1\cos(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2\frac{\sqrt{\lambda}}{\sqrt{-\mu}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2\cos(\sqrt{\lambda}\sqrt{-\mu}\xi)} \right)^2 + 4\delta\lambda\mu, \tag{20}$$

where $\xi = x + 4\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

When $\lambda < 0$ and $\mu > 0$, we obtain the trigonometric function solutions

$$u_{14}(x,t) = -\,12\delta\mu^2 \left(\frac{C_1\cos(\sqrt{-\lambda}\sqrt{\mu}\xi) - C_2\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi) + C_2\cos(\sqrt{-\lambda}\sqrt{\mu}\xi)} \right)^2 + 4\delta\lambda\mu, \tag{21}$$

where $\xi = x + 4\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

Next we consider the solutions for the Case 2. When $\lambda > 0$ and $\mu > 0$, we obtain the hyperbolic function solutions

$$u_{21}(x,t) = -\,12\delta\mu^2 \left(\frac{C_1\cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2\frac{\sqrt{\lambda}}{\sqrt{\mu}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2\cosh(\sqrt{\lambda}\sqrt{\mu}\xi)} \right)^2 + 12\delta\lambda\mu, \tag{22}$$

where $\xi = x - 4\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be recovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then

(22) can be written as

$$u_{21}(x,t) = 12\delta\lambda\mu\ \mathrm{sech}^2(\sqrt{\lambda\mu}\xi + \xi_0),$$

which is the known solitary wave solution of the KDV equation (10) obtained in [20], where $\xi_0 = \tanh^{-1}\frac{C_1}{C_2\sqrt{\frac{\lambda}{\mu}}}$ and $\xi = x - 4\delta\lambda\mu t$.

When $\lambda < 0$ and $\mu < 0$,we obtain the hyperbolic function solutions

(23)
$$u_{22}(x,t) = -\,12\delta\mu^2\left(\frac{C_1\cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2\frac{\sqrt{-\lambda}}{\sqrt{-\mu}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{-\lambda}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2\cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}\right)^2 + 12\delta\lambda\mu,$$

where $\xi = x - 4\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be recovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then (23) can be written as

$$u_{22}(x,t) = 12\delta\lambda\mu\ \mathrm{sech}^2(\sqrt{\lambda\mu}\xi - \xi_0),$$

where $\xi_0 = \tanh^{-1}\frac{C_1}{C_2\sqrt{\frac{\lambda}{\mu}}}$ and $\xi = x - 4\delta\lambda\mu t$.

When $\lambda > 0$ and $\mu < 0$, we obtain the trigonometric function solutions

$$u_{23}(x,t) = -\,12\delta\mu^2\left(\frac{C_1\cos(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2\frac{\sqrt{\lambda}}{\sqrt{-\mu}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2\cos(\sqrt{\lambda}\sqrt{-\mu}\xi)}\right)^2 + 12\delta\lambda\mu, \tag{24}$$

where $\xi = x - 4\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

When $\lambda < 0$ and $\mu > 0$,we obtain the trigonometric function solutions

$$u_{24}(x,t) = -\,12\delta\mu^2\left(\frac{C_1\cos(\sqrt{-\lambda}\sqrt{\mu}\xi) - C_2\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi) + C_2\cos(\sqrt{-\lambda}\sqrt{\mu}\xi)}\right)^2 + 12\delta\lambda\mu, \tag{25}$$

where $\xi = x - 4\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

4. Application to the mKdV equation

Let us consider the following nonlinear dispersive equation of the form [21]

$$u_t - u^2u_x + \delta u_{xxx} = 0, \delta > 0. \tag{26}$$

This equation is called modified KdV equation, which arises in the process of understanding the role of nonlinear dispersion and in the formation of structures like liquid drops, and it exhibits compactions: solitons with compact support [14]. In order to obtain the travelling wave solutions, we set $u(x,t) = u(\xi), \xi = x - \omega t$, then Eq.(26) reduces to

$$-\omega u - \frac{1}{3}u^3 + \delta u'' = 0, \tag{27}$$

Balancing the highest order of derivative term and nonlinear term in Eq.(27), we can get $n = 1$. According to Eq.(4), the solution of Eq.(27) can be written as

$$u(x,t) = a_1\left(\frac{F}{G}\right) + a_0, a_1 \neq 0, \tag{28}$$

where $F = F(\xi)$ and $G = G(\xi)$ satisfy the FLODS in the form

$$F'(\xi) = \lambda G(\xi),\ G'(\xi) = \mu F(\xi). \tag{29}$$

By using (28) and Eq.(29) it is easily derived that

$$\begin{cases} u^3 = a_1^3\left(\frac{F}{G}\right)^3 + 3a_1^2 a_0\left(\frac{F}{G}\right)^2 + 3a_1 a_0^2\left(\frac{F}{G}\right) + a_0^3, \\ u'' = 2\mu^2 a_1\left(\frac{F}{G}\right)^2 - 2\lambda\mu a_1\left(\frac{F}{G}\right). \end{cases} \tag{30}$$

By substitution (28) and (30) into Eq.(27) and collection all terms with the same power of $\left(\frac{F}{G}\right)$ together, Eq.(27) is converted into another polynomial in $\left(\frac{F}{G}\right)$. Equating the coefficients of this polynomial to zero, we obtain a set of algebraic equations with respect to the unknowns $a_1, a_0, \omega, \lambda, \mu$ and δ as follows:

$$\left(\tfrac{F}{G}\right)^0 : -\omega a_0 - \tfrac{1}{3}a_0^3 = 0$$

$$\left(\tfrac{F}{G}\right)^1 : -\omega a_1 - a_0^2 a_1 - 2\delta\lambda\mu a_1 = 0$$

$$\left(\tfrac{F}{G}\right)^2 : -a_0 a_1^2 = 0$$

$$\left(\tfrac{F}{G}\right)^3 : -\tfrac{1}{3}a_1^3 + 2\delta\mu^2 a_1 = 0$$

Solving the system of algebraic equations using Maple gives a following set of nontrivial solutions:

$$\left\{a_1 = \pm\mu\sqrt{6\delta}, a_0 = 0, \omega = -2\delta\lambda\mu\right\}. \tag{31}$$

Substituting (31) into (28) yields

$$u(x,t) = \pm\mu\sqrt{6\delta}\left(\frac{F}{G}\right),\ \xi = x + 2\delta\lambda\mu. \tag{32}$$

Substituting the general solution of Eq.(29) into the formulae (32), we have four types of travelling wave solutions of the mKdV equation(26) as follows:

When $\lambda > 0$ and $\mu > 0$, we obtain hyperbolic function solutions

$$u_1(x,t) = \pm\mu\sqrt{6\delta}\left(\frac{C_1\cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2\frac{\sqrt{\lambda}}{\sqrt{\mu}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2\cosh(\sqrt{\lambda}\sqrt{\mu}\xi)}\right), \tag{33}$$

where $\xi = x + 2\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

When $\lambda < 0$ and $\mu < 0$, we obtain the following hyperbolic function solutions

$$u_2(x,t) = \pm\mu\sqrt{6\delta}\left(\frac{C_1\cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2\frac{\sqrt{-\lambda}}{\sqrt{-\mu}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{-\lambda}}\sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2\cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}\right), \tag{34}$$

where $\xi = x + 2\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

When $\lambda > 0$ and $\mu < 0$,we have trigonometric function solutions

$$u_3(x,t) = \pm\mu\sqrt{6\delta}\left(\frac{C_1\cos(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2\frac{\sqrt{\lambda}}{\sqrt{-\mu}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2\cos(\sqrt{\lambda}\sqrt{-\mu}\xi)}\right), \tag{35}$$

where $\xi = x + 2\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

When $\lambda < 0$ and $\mu > 0$, we have trigonometric function solutions:

$$u_4(x,t) = \pm\mu\sqrt{6\delta}\left(\frac{C_1\cos(\sqrt{-\lambda}\sqrt{\mu}\xi) - C_2\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi) + C_2\cos(\sqrt{-\lambda}\sqrt{\mu}\xi)}\right), \tag{36}$$

where $\xi = x + 2\delta\lambda\mu t$, C_1 and C_2 are arbitrary constants.

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda\mu > 0$, then (33) and (34) become

$$u(x,t) = \pm\mu\sqrt{6\delta}\sqrt{\frac{\lambda}{\mu}}\tanh\left(\sqrt{\lambda\mu}(x + 2\delta\lambda\mu t)\right), \tag{37}$$

which are the solitary wave solutions of the mKdV equation Eq.(26).

5. Application to the Variant Boussinesq equations

In this section, we take variant Boussinesq equation as an example to illustrate the effectiveness of the proposed method. Let us consider the variant Boussinesq equation

$$\begin{gathered} u_t - (uv)_x + u_{xxx} = 0, \\ v_t + u_x + vv_x = 0. \end{gathered} \tag{38}$$

The Boussinesq equations describes motion of long waves in shallow water under gravity and in one dimensional nonlinear lattice. As a model for water waves, where v is the velocity and u the total depth, and the subscripts denote partial derivatives. For more details about the formulation of Boussinesq equations, see [17, 18, 22]. To look for the travelling wave solutions, we use the transformation $u = u(\xi), v = v(\xi), \xi = x - \omega t$, then Eq.(38)becomes

$$\begin{aligned} &-\omega u - uv + v'' = 0, \\ &-\omega v + u + \frac{1}{2}v^2 = 0. \end{aligned} \tag{39}$$

Considering the homogeneous balance between v'' and uv in the first equation, and u and v^2 in the second equation of Eq.(39) we obtain $m = 2, n = 1$, so we can write (4) as

$$\begin{cases} u(x,t) = a_2\left(\frac{F}{G}\right)^2 + a_1\left(\frac{F}{G}\right) + a_0, a_2 \neq 0, \\ v(x,t) = b_1\left(\frac{F}{G}\right) + b_0, b_1 \neq 0, \end{cases} \tag{40}$$

where $F = F(\xi)$ and $G = G(\xi)$ satisfy the FLODS in the form

$$F'(\xi) = \lambda G(\xi),\ G'(\xi) = \mu F(\xi), \tag{41}$$

By using Eq.(40) and Eq.(41) it can be easily derived that

$$\begin{cases} v^2 = b_1^2\left(\frac{F}{G}\right)^2 + 2b_1b_0\left(\frac{F}{G}\right) + a_0^2, \\ uv = a_2b_1\left(\frac{F}{G}\right)^3 + (a_2b_0 + a_1b_1)\left(\frac{F}{G}\right)^2 + (a_1b_0 + a_0b_1)\left(\frac{F}{G}\right) + a_0b_0 \\ v'' = 2\mu^2 b_1\left(\frac{F}{G}\right)^2 - 2\lambda\mu b_1\left(\frac{F}{G}\right). \end{cases} \tag{42}$$

By substitution (40) and (42) into Eq.(39) and collection all terms with the same power of $\left(\frac{F}{G}\right)$ together, Eq.(39) is converted into another polynomial in $\left(\frac{F}{G}\right)$. Equating the coefficients of this polynomial to zero, yields a set of algebraic equations for $a_2, a_1, a_0, b_1, b_0, \omega, \lambda, \mu$ and δ as follows:

$$\left(\tfrac{F}{G}\right)^0 : -\omega a_0 + a_0b_0 = 0$$

$$\left(\tfrac{F}{G}\right)^1 : -\omega a_1 + a_0b_1 + a_1b_0 - 2\lambda\mu b_1 = 0$$

$$\left(\tfrac{F}{G}\right)^2 : -\omega a_2 + a_1b_1 + a_2b_0 = 0$$

$$\left(\tfrac{F}{G}\right)^3 : a_2b_1 + 2\mu^2 b_1 = 0$$

$$\left(\tfrac{F}{G}\right)^0 : -\omega b_0 + a_0 + \tfrac{1}{2}b_0^2 = 0$$

$\left(\frac{F}{G}\right)^1 : -\omega b_1 + a_1 + b_1 b_0 = 0$

$\left(\frac{F}{G}\right)^2 : a_2 + \frac{1}{2} b_1^2 = 0$

Solving the above algebraic equations using Maple, yields the following sets of nontrivial solutions:

Case 1. $\{a_2 = -2\mu^2, a_1 = 0, a_0 = 2\lambda\mu, b_1 = 2\mu, b_0 = \pm 2\sqrt{\lambda\mu},$
$\omega = \pm 2\sqrt{\lambda\mu}\}.$

Case 2. $\{a_2 = -2\mu^2, a_1 = 0, a_0 = 2\lambda\mu, b_1 = -2\mu, b_0 = \pm 2\sqrt{\lambda\mu},$
$\omega = \pm 2\sqrt{\lambda\mu}\}.$

Substituting the above cases of the coefficients into the formulae (40), we have four types of travelling wave solutions of the variant Boussinesq equation(38) as follows:
First we consider the Case 1. When $\lambda > 0$ and $\mu > 0$,

$$\tag{43} \begin{cases} u_{11}(x,t) = -2\mu^2 \left(\dfrac{C_1 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \frac{\sqrt{\lambda}}{\sqrt{\mu}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi)}{C_1 \frac{\sqrt{\mu}}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi)} \right)^2 + 2\lambda\mu, \\ v_{11}(x,t) = 2\mu \left(\dfrac{C_1 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \frac{\sqrt{\lambda}}{\sqrt{\mu}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi)}{C_1 \frac{\sqrt{\mu}}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}\sqrt{\mu}\xi) + C_2 \cosh(\sqrt{\lambda}\sqrt{\mu}\xi)} \right) \pm 2\sqrt{\lambda\mu}, \end{cases}$$

where $\xi = x \mp 2\sqrt{\lambda\mu}t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be rediscovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then (43) can be written as

$$\begin{cases} u_{11}(x,t) = 2\lambda\mu \ \mathrm{sech}^2(\sqrt{\lambda\mu}\xi + \xi_0), \\ v_{11}(x,t) = 2\sqrt{\lambda\mu} \ \tanh(\sqrt{\lambda\mu}\xi + \xi_0) \pm 2\sqrt{\lambda\mu}, \end{cases}$$

where $\xi_0 = \tanh^{-1}\left(\dfrac{C_1}{C_2\sqrt{\frac{\lambda}{\mu}}} \right)$ and $\xi = x \mp 2\sqrt{\lambda\mu}t.$

When $\lambda < 0$ and $\mu < 0$,

(44)

$$\begin{cases} u_{12}(x,t) = -2\mu^2 \left(\dfrac{C_1 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{-\mu}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{-\lambda}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi)} \right)^2 + 2\lambda\mu, \\ v_{12}(x,t) = 2\mu \left(\dfrac{C_1 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{-\mu}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{-\lambda}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi)} \right) \pm 2\sqrt{\lambda\mu}, \end{cases}$$

where $\xi = x \mp 2\sqrt{\lambda\mu}t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be rediscovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then (44) can be written as

$$\begin{cases} u_{12}(x,t) = 2\lambda\mu \operatorname{sech}^2(\xi_0 - \sqrt{\lambda\mu}\xi), \\ v_{12}(x,t) = -2\sqrt{\lambda\mu} \tanh(\xi_0 - \sqrt{\lambda\mu}\xi) \pm 2\sqrt{\lambda\mu}, \end{cases}$$

where $\xi_0 = \tanh^{-1}\left(\frac{C_1}{C_2\sqrt{\frac{\lambda}{\mu}}}\right)$ and $\xi = x \mp 2\sqrt{\lambda\mu}t$.

When $\lambda > 0$ and $\mu < 0$,

$$\begin{cases} u_{13}(x,t) = -2\mu^2\left(\frac{C_1\cos(\sqrt{\lambda}\sqrt{-\mu}\xi)+C_2\frac{\sqrt{\lambda}}{\sqrt{-\mu}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi)+C_2\cos(\sqrt{\lambda}\sqrt{-\mu}\xi)}\right)^2 + 2\lambda\mu, \\ v_{13}(x,t) = 2\mu\left(\frac{C_1\cos(\sqrt{\lambda}\sqrt{-\mu}\xi)+C_2\frac{\sqrt{\lambda}}{\sqrt{-\mu}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi)}{-C_1\frac{\sqrt{-\mu}}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\sqrt{-\mu}\xi)+C_2\cos(\sqrt{\lambda}\sqrt{-\mu}\xi)}\right) \pm 2\mathrm{i}\sqrt{-\lambda\mu}, \end{cases} \tag{45}$$

where $\xi = x \mp 2\mathrm{i}\sqrt{-\lambda\mu}t$, C_1 and C_2 are arbitrary constants.

When $\lambda < 0$ and $\mu > 0$,

$$\begin{cases} u_{14}(x,t) = -2\mu^2\left(\frac{C_1\cos(\sqrt{-\lambda}\sqrt{\mu}\xi)-C_2\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi)+C_2\cos(\sqrt{-\lambda}\sqrt{\mu}\xi)}\right)^2 + 2\lambda\mu, \\ v_{14}(x,t) = 2\mu\left(\frac{C_1\cos(\sqrt{-\lambda}\sqrt{\mu}\xi)-C_2\frac{\sqrt{-\lambda}}{\sqrt{\mu}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}\sqrt{\mu}\xi)+C_2\cos(\sqrt{-\lambda}\sqrt{\mu}\xi)}\right) \pm 2\mathrm{i}\sqrt{-\lambda\mu}, \end{cases} \tag{46}$$

where $\xi = x + \mp 2\mathrm{i}\sqrt{-\lambda\mu}t$, C_1 and C_2 are arbitrary constants.

Next we consider the solutions for the Case 2. When $\lambda > 0$ and $\mu > 0$,

$$\begin{cases} u_{21}(x,t) = -2\mu^2\left(\frac{C_1\cosh(\sqrt{\lambda}\sqrt{\mu}\xi)+C_2\frac{\sqrt{\lambda}}{\sqrt{\mu}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi)+C_2\cosh(\sqrt{\lambda}\sqrt{\mu}\xi)}\right)^2 + 2\lambda\mu, \\ v_{21}(x,t) = -2\mu\left(\frac{C_1\cosh(\sqrt{\lambda}\sqrt{\mu}\xi)+C_2\frac{\sqrt{\lambda}}{\sqrt{\mu}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi)}{C_1\frac{\sqrt{\mu}}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}\sqrt{\mu}\xi)+C_2\cosh(\sqrt{\lambda}\sqrt{\mu}\xi)}\right) \pm 2\sqrt{\lambda\mu}, \end{cases} \tag{47}$$

where $\xi = x \mp 2\sqrt{\lambda\mu}t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be rediscovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then (47) can be written as

$$\begin{cases} u_{21}(x,t) = 2\lambda\mu \operatorname{sech}^2(\sqrt{\lambda\mu}\xi + \xi_0), \\ v_{21}(x,t) = -2\sqrt{\lambda\mu} \tanh(\sqrt{\lambda\mu}\xi + \xi_0) \pm 2\sqrt{\lambda\mu}, \end{cases}$$

where $\xi_0 = \tanh^{-1}\left(\frac{C_1}{C_2\sqrt{\frac{\lambda}{\mu}}}\right)$ and $\xi = x \mp 2\sqrt{\lambda\mu}t$.

When $\lambda < 0$ and $\mu < 0$,

(48)
$$\begin{cases} u_{22}(x,t) = -2\mu^2 \left(\frac{C_1 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{-\mu}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{-\lambda}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi)} \right)^2 + 2\lambda\mu, \\ v_{22}(x,t) = -2\mu \left(\frac{C_1 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{-\mu}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{-\lambda}} \sinh(\sqrt{-\lambda}\sqrt{-\mu}\xi) + C_2 \cosh(\sqrt{-\lambda}\sqrt{-\mu}\xi)} \right) \pm 2\sqrt{\lambda\mu}, \end{cases}$$

where $\xi = x \mp 2\sqrt{\lambda\mu}t$, C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be rediscovered, for instance, if $C_1 > 0, C_1^2 > C_2^2$, then (48) can be written as

$$\begin{cases} u_{22}(x,t) = 2\lambda\mu \ \mathrm{sech}^2(\sqrt{\lambda\mu}\xi - \xi_0), \\ v_{22}(x,t) = 2\sqrt{\lambda\mu} \ \tanh(\sqrt{\lambda\mu}\xi - \xi_0) \pm 2\sqrt{\lambda\mu}, \end{cases}$$

where $\xi_0 = \tanh^{-1}\left(\frac{C_1}{C_2\sqrt{\frac{\lambda}{\mu}}} \right)$ and $\xi = x \mp 2\sqrt{\lambda\mu}t$.

When $\lambda > 0$ and $\mu < 0$,

$$\begin{cases} u_{23}(x,t) = -2\mu^2 \left(\frac{C_1 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \frac{\sqrt{\lambda}}{\sqrt{-\mu}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi)} \right)^2 + 2\lambda\mu, \\ v_{23}(x,t) = -2\mu \left(\frac{C_1 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \frac{\sqrt{\lambda}}{\sqrt{-\mu}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi)}{-C_1 \frac{\sqrt{-\mu}}{\sqrt{\lambda}} \sin(\sqrt{\lambda}\sqrt{-\mu}\xi) + C_2 \cos(\sqrt{\lambda}\sqrt{-\mu}\xi)} \right) \pm 2\mathrm{i}\sqrt{-\lambda\mu}, \end{cases} \tag{49}$$

where $\xi = x \mp 2\mathrm{i}\sqrt{-\lambda\mu}t$, C_1 and C_2 are arbitrary constants.

When $\lambda < 0$ and $\mu > 0$,

$$\begin{cases} u_{24}(x,t) = -2\mu^2 \left(\frac{C_1 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{\mu}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi)}{C_1 \frac{\sqrt{\mu}}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi) + C_2 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi)} \right)^2 + 2\lambda\mu, \\ v_{24}(x,t) = -2\mu \left(\frac{C_1 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi) - C_2 \frac{\sqrt{-\lambda}}{\sqrt{\mu}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi)}{C_1 \frac{\sqrt{\mu}}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda}\sqrt{\mu}\xi) + C_2 \cos(\sqrt{-\lambda}\sqrt{\mu}\xi)} \right) \pm 2\mathrm{i}\sqrt{-\lambda\mu}, \end{cases} \tag{50}$$

where $\xi = x \mp \mathrm{i}\sqrt{-\lambda\mu}t$, C_1 and C_2 are arbitrary constants.

6. Conclusion

In this paper, $\left(\frac{F}{G}\right)$-expansion method is used to obtain more general exact solutions of the nonlinear evolution equations. The advantages of the $\left(\frac{F}{G}\right)$-expansion method is that it is possible to obtain more travelling wave solutions with distinct physical structures. From our results, some results previously known as traveling wave solutions and soliton-like solutions can be recovered. Moreover, the proposed method is capable of greatly minimizing the size of computational work compared to the existing technique. This is due to use of first order linear ordinary differential equation in the proposed algorithm instead of second order linear ordinary differential equation as in [2]. Finally, it is worth to mention that the implementation of this proposed method is very simple and straightforward, and it can also be applied to other nonlinear evolution equations arising in mathematical physics.

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