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NOTE ON THE CLASSICAL WATSON'S THEOREM FOR THE SERIES $_3F_2$

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Abstract. Summation theorems for hypergeometric series $_2F_1$ and generalized hypergeometric series $_pF_q$ play important roles in themselves and their diverse applications. Some summation theorems for $_2F_1$ and $_pF_q$ have been established in several or many ways. Here we give a proof of Watson's classical summation theorem for the series $_3F_2(1)$ by following the same lines used by Rakha [7] except for the last step in which we applied an integral formula introduced by Choi *et al.* [3].

1. Introduction and Preliminaries

We begin by introducing a response of Michael Atiyah [9] when Michael Atiyah and Isadore Singer were interviewed which took place in Oslo on May 24, 2004, during the Abel Prize celebrations: Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalize in different directions-they are not just repetitions of each other $\cdots$. If you cannot look at a problem from different directions, it is probably not very interesting; the more perspectives, the better.

We recall the well known classical Watson's summation theorem for the generalized hypergeometric series $_3F_2$ (see, *e.g.*, [1, p.16, Eq. (1)]):

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$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c\,; \\ \frac{1}{2}(a+b+1),\ 2c\,; \end{matrix}\ 1\right] = \frac{\Gamma(\frac{1}{2})\,\Gamma(c+\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\,\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})\,\Gamma(c-\frac{1}{2}a+\frac{1}{2})\,\Gamma(c-\frac{1}{2}b+\frac{1}{2})} \tag{1.1}
$$

provided that $\Re(2c-a-b) > -1$. This Watson's summation theorem (1.1) has been established in many different ways (see, *e.g.*, [1, 2, 5, 7, 8, 11, 12]). For concise outlines of various proofs of (1.1), see [7, 8].

Here we present a proof of Watson's summation theorem (1.1) for the series ${}_3F_2(1)$ by following the same lines used by Rakha [7] except for the last step in which we applied an integral formula introduced by Choi *et al.* [3].

For our purpose, we need to recall some known functions and earlier works. The well known Beta function $B(\alpha,\beta)$ is defined by

$$
B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \quad (\Re(\alpha)>0;\ \Re(\beta)>0) \tag{1.2}
$$

or, equivalently,

$$
B(\alpha,\beta) = 2\int_0^{\pi/2} (\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}d\theta \quad (\Re(\alpha)>0;\ \Re(\beta)>0). \tag{1.3}
$$

An integral representation for ${}_3F_2$ is given as follows (see [4]):

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c\,; \\ d,\ e\,; \end{matrix}\ z\right] = \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)}\int_0^1 t^{c-1}(1-t)^{d-c-1}\,{}_2F_1\left[\begin{matrix} a,\ b\,; \\ e\,; \end{matrix}\ zt\right]dt, \tag{1.4}
$$

provided $\Re(c)>0$, $\Re(d-c)>0$ and $\Re(d-a-b)>0$.

A transformation formula for ${}_2F_1$ is as follows (see, *e.g.*, [6, p. 65, Theorem 24]):

$$
{}_2F_1\left[\begin{matrix} a,\ b\,; \\ 2b\,; \end{matrix}\ 2y\right] = (1-y)^{-a}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a,\ \frac{1}{2}a+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\ \left(\frac{y}{1-y}\right)^2\right]dt, \tag{1.5}
$$

provided $|y| < \frac{1}{2}$ and $\left|\frac{y}{1-y}\right| < 1$.

Euler's integral representation for the hypergeometric function ${}_2F_1$ is given as follows (see [10, p. 65]):

$$(1.6)\quad {}_2F_1\left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix}\; z\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}(1-zt)^{-a}dt,$$

provided that $\Re(c) > \Re(b) > 0$ and $|z| < 1$.

An integral representation for ${}_2F_1(1/2)$ is given as follows (see, *e.g.*, [3, p. 510, Eq. (8)]):

$$(1.7)\quad \begin{aligned} {}_2F_1\left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix}\; \frac{1}{2}\right] &= \frac{2^a\,\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \\ &\cdot \int_0^{\pi/2} (\cos\,\theta)^{b-1} \left(\sin\,\frac{\theta}{2}\right)^{2c-2b-1} \left(\cos\,\frac{\theta}{2}\right)^{2a-2c+1} d\theta. \end{aligned}$$

Gauss's second summation theorem is given as follows (see, *e.g.*, [1, p. 10, Eq. (2)]):

$$(1.8)\quad {}_2F_1\left[\begin{matrix} a,\, b\,; \\ \frac{1}{2}(a+b+1)\,; \end{matrix}\; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})}.$$

2. Derivation of Watson's summation theorem (1.1)

Setting $e = 2b$ in (1.4), we have

$$(2.1)\quad \begin{aligned} &{}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ d,\, 2b\,; \end{matrix}\; z\right] \\ &= \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)} \int_0^1 t^{c-1}(1-t)^{d-c-1}\,{}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b\,; \end{matrix}\; zt\right] dt. \end{aligned}$$

Replacing y by $\frac{1}{2}zt$ in (1.5) and applying the resulting identity to the ${}_2F_1$ in (2.1), after a little simplification, we obtain

$$(2.2)\quad \begin{aligned} {}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ d,\, 2b\,; \end{matrix}\; z\right] &= \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)} \\ \cdot \int_0^1 t^{c-1}(1-t)^{d-c-1}\,(1-\frac{1}{2}zt)^{-a}\,&{}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}a+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\; \left(\frac{zt}{2-zt}\right)^2\right] dt. \end{aligned}$$

Expressing the ${}_2F_1$ in (2.2) as a series and changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series on the interval (0, 1), after a little algebra, we have

$$
\tag{2.3}
{}_3F_2\left[\begin{matrix} a,\, b,\, c; \\ d,\, 2b\,; \end{matrix}\; z\right] = \frac{\Gamma(d)}{\Gamma(d-c)\Gamma(c)} \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n\,(\frac{1}{2}a+\frac{1}{2})_n}{(b+\frac{1}{2})_n\, n!} \left(\frac{z}{2}\right)^{2n} \int_0^1 t^{c+2n-1}(1-t)^{d-c-1}\left(1-\frac{1}{2}zt\right)^{-(a+2n)} dt.
$$

Using (1.6) to evaluate the integral in (2.3), after a little simplification, we get

$$
\tag{2.4}
{}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ d,\, 2b\,; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n\,(\frac{1}{2}a+\frac{1}{2})_n\,(c)_{2n}}{(b+\frac{1}{2})_n\,(d)_{2n}\, n!} \left(\frac{z}{2}\right)^{2n} {}_2F_1\left[\begin{matrix} a+2n,\, c+2n\,; \\ d+2n\,; \end{matrix}\; \frac{z}{2}\right].
$$

Interchanging b and c and taking $d = \frac{1}{2}(a+b+1)$ in (2.4), we have

$$
\tag{2.5}
{}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c\,; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n\,(\frac{1}{2}a+\frac{1}{2})_n\,(b)_{2n}}{(c+\frac{1}{2})_n\,(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})_{2n}\, n!} \left(\frac{z}{2}\right)^{2n} {}_2F_1\left[\begin{matrix} a+2n,\, b+2n\,; \\ \frac{1}{2}(a+b+1)+2n\,; \end{matrix}\; \frac{z}{2}\right].
$$

Taking $z = 1$ in (2.5) and using (1.7) in the resulting equation, we obtain

$$
\tag{2.6}
\begin{aligned}
{}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c\,; \end{matrix}\; 1\right] &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n\,(\frac{1}{2}a+\frac{1}{2})_n\,(b)_{2n}}{(c+\frac{1}{2})_n\,(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})_{2n}\, n!} \left(\frac{1}{2}\right)^{2n} \frac{2^{a+2n}\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}+2n)}{\Gamma(b+2n)\Gamma(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})} \\
&\quad \cdot 2^{b-a} \int_0^{\pi/2} (\cos\,\theta)^{b+2n-1}\,(\sin\,\theta)^{a-b}\, d\theta,
\end{aligned}
$$

where we used an elementary trigonometric identity: $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$. Applying (1.3) to evaluate the integral in (2.6) and using Legendre's duplication formula for the Gamma function (see [10, p. 6, Eq. (29)]) in the resulting identity, we get

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c\,; \end{matrix}\; 1\right] \\
&= \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\,\Gamma(\frac{1}{2}b+\frac{1}{2})}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a,\, \frac{1}{2}b\,; \\ c+\frac{1}{2}\,; \end{matrix}\; 1\right],
\end{aligned}
\tag{2.7}
$$

which, upon using the well known Gauss's summation theorem (see, *e.g.*, [10, p. 64, Eq. (7)]), yields (1.1). This completes the proof of Watson's summation theorem (1.1).

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