Honam Mathematical J. **35** (2013), No. 4, pp. 707–716
http://dx.doi.org/10.5831/HMJ.2013.35.4.707

STUDY ON TOPOLOGICAL SPACES WITH THE SEMI-$T_{\frac{1}{2}}$ SEPARATION AXIOM

Sang-Eon Han

Abstract. The present paper consists of two parts. Since the recent paper [4] proved that an Alexandroff T_0-space is a semi-$T_{\frac{1}{2}}$-space, the first part studies semi-open and semi-closed structures of the Khalimsky nD space. The second one focuses on the study of a relation between the L_S-property of $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ relative to the simple closed k_i-curves $SC_{k_i}^{n_i,l_i}, i \in \{1,2\}$ and its normal k-adjacency. In addition, the present paper points out that the main theorems of Boxer and Karaca's paper [3] such as Theorems 4.4 and 4.7 of [3] cannot be new assertions. Indeed, instead they should be attributed to Theorems 4.3 and 4.5, and Example 4.6 of [10].

1. Introduction

To study mathematical objects from the viewpoint of computer science, applied topology has been rapidly developed and it includes discrete geometry, computational topology, Scott topology, digital topology and so forth. Besides, low dimensional separation axioms such as T_0, $T_{\frac{1}{2}}$, semi-$T_{\frac{1}{2}}$ and so forth have been often used in applied topology supporting computer science such as image analysis, image processing because computers like a finite or at least a locally finite topological structure [1]. The paper [16] established the notions of a semi-open set and a semi-closed set. Motivated from these notions, the recent paper [4] points

Received September 26, 2013. Accepted October 16, 2013.
2010 Mathematics Subject Classification. 54F65, 54A10, 54B05, 54B10.
Key words and phrases. digital topology, domain theory, semi-$T_{\frac{1}{2}}$-separation axiom, semi-open, semi-closed, upper set, normal adjacency, digital product, digital covering space.
This work was supported (in part) by grant from Yu-dang, Ji Sung Yang Memorial Fund and this research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2013R1A1A4A01007577).

out that the notion of semi-$T_{\frac{1}{2}}$-separation axiom (see Definition 2) is very related to both the Alexandroff topological structure and the T_0-separation axiom. The present paper explains that for every mixed point p in the Khalimsky nD space the singleton $\{p\}$ is semi-closed so that the Khalimsky nD space satisfies the semi-$T_{\frac{1}{2}}$-separation axiom [4]. Indeed, the recent paper [4] studies some properties of a semi-$T_{\frac{1}{2}}$ space and its product property. It turns out that the separation axiom semi-$T_{\frac{1}{2}}$ can play an important role in applied topology such as digital topology and domain theory. For instance, the paper [4] proves that Alexandroff T_0-spaces are semi-$T_{\frac{1}{2}}$ spaces and further, the product and the hereditary property of a semi-$T_{\frac{1}{2}}$ space was also studied. Even though the paper [4] referred to the semi-$T_{\frac{1}{2}}$ structure of the Khalimsky nD space, the present paper investigates semi-open and semi-closed structures of the Khalimsky nD space.

The paper [10] studied the product property of two digital covering maps in terms of the L_S-property of a digital product which can play an important role in digital covering and digital homotopy theory. Hence we shows merits of the L_S-property of adjacencies of digital products which corrects the assertion of the paper [3]. Finally, the present paper shows a relation between the L_S-property of $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ and a normal k-adjacency of $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$.

The rest of the present paper proceeds as follows. Section 2 introduces basic notions of digital topological spaces including the Khalimsky nD space. Section 3 investigates semi-open and semi-closed structure of the the Khalimsky nD space related to the semi-$T_{\frac{1}{2}}$-separation axiom of the Khalimsky nD space. Section 4 investigates a relation between the L_S-property of $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ and a normal k-adjacency of $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2})$. Section 5 concludes the paper with a summary. Besides, the section points out that the main theorems of Boxer and Karaca's paper [3] such as Theorems 4.4 and 4.7 of [3] cannot be new assertions because they were already studied in terms of Theorems 4.3 and 4.5, and Example 4.6 of [10].

2. Preliminaries

In applied topology including digital geometry and digital topology one of the interesting areas is the Khalimsky nD space. The space is a locally finite topological space and satisfies the separation axiom T_0

instead of a Hausdorff space if $n \geq 2$, and $T_{\frac{1}{2}}$ if $n = 1$. In relation to the study of objects in $\mathbf{Z}^n$, we have used many tools from combinatorial topology, graph theory, Khalimsky topology and so forth [7, 8, 13]. Motivated from Alexandroff spaces in [1], the Khalimsky nD space was established [13] and the study includes the papers [4, 13, 14]. For a subset $X \subset \mathbf{Z}^n$ we denote by (X, T^n_X) a subspace of $(\mathbf{Z}^n, T^n)$, $n \geq 1$.

Let us now review some basic notions and properties of the Khalimsky nD space. Khalimsky topology can start with the Khalimsky line. More precisely, *Khalimsky line topology* on $\mathbf{Z}$ is induced by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} \,|\, n \in \mathbf{Z}\}$ as a subbase [1] (see also [13]). Namely, the family of the subset $\{\{2n+1\}, [2m-1, 2m+1]_{\mathbf{Z}} \,|\, m, n \in \mathbf{Z}\}$ is a basis of the Khalimsky line topology on $\mathbf{Z}$ denoted by $(\mathbf{Z}, T)$. Furthermore, the product topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}, T)$, denoted by $(\mathbf{Z}^n, T^n)$, is called the *Khalimsky nD space*.

Let us now recall the structure of $(\mathbf{Z}^n, T^n)$. A point $x = (x_1, x_2, \cdots, x_n) \in \mathbf{Z}^n$ is *pure open* if all coordinates are odd; and it is *pure closed* if each of the coordinates is even [14]. The other points in $\mathbf{Z}^n$ are called *mixed* [14]. In each of the spaces of Figure 1, a black jumbo dot means a pure open point and further, the symbols ■ and • mean a pure closed point and a mixed point, respectively. In relation to the further statement of a pure point and a mixed point, we can say that a point x is open if $SN(x) = \{x\}$, where $SN(x)$ means the smallest neighborhood of $x \in \mathbf{Z}^n$. The point $x \in \mathbf{Z}^n$ is called closed if $Cl(x) = \{x\}$, where $Cl(x)$ stands for a closure of the singleton $\{x\}$. Especially, for a mixed point $p = (2m, 2n+1)$ (resp. $p = (2m+1, 2n)$) in $(\mathbf{Z}^2, T^2)$ we call the point p *closed-open* (resp. *open-closed*). Thus, for a point $p := (x, y)$ of $(\mathbf{Z}^2, T^2)$ we can obtain that $SN(p) = \{(x-1, y), p, (x+1, y)\}$ if $p := (x, y)$ is closed-open; $SN(p) = \{(x, y-1), p, (x, y+1)\}$ if $p := (x, y)$ is open-closed (see the proof of Theorem 3.1).

3. Semi-open and semi-closed structures on the Khalimsky nD space

Since the notions of *semi-open* and *semi-closed* has been often used in pure and applied topology and further, they are substantially used in establishing the separation axiom semi-$T_{\frac{1}{2}}$, we need to recall them, as follows.

Definition 1. [16] *Let (X, T) be a topological space. A subset A of X is called semi-open if there is an open set $O \in T$ such that $O \subset$*

$A \subset Cl(O)$, where $Cl(O)$ means the closure of the set O. A subset $F \subset X$ is called a semi-closed set of a topological space (X, T) if $X \setminus F$ is semi-open in (X, T).

The notion of "semi-open" of the subset A in Definition 1 is equivalently represented as follows [16]:

$$A \subset Cl(Int(A)). \tag{3.1}$$

Besides, the notion of "semi-closed" of the subset A in Definition 1 is equivalently represented as follows [16]: there exists a closed set F in T such that

$$Int(F) \subset A \subset F \text{ or } Int(Cl(A)) \subset A. \tag{3.2}$$

It is well known that the Khalimsky line topology [13] and the Marcus Wyse topological structure on $\mathbf{Z}^2$ [17] satisfies the separation axiom $T_{\frac{1}{2}}$. Motivated by this approach, the following notion was developed in [6, 16].

Definition 2. [6] *(see also* [4]*) We say that a topological space (X, T) satisfies the semi-$T_{\frac{1}{2}}$-separation axiom if every singleton of X is either semi-open or semi-closed.*

We now need to declare that the paper [4] proved that an Alexandroff T_0 space is a semi-$T_{\frac{1}{2}}$ space in the theoretical approach. Then the Khalimsky nD space was referred as an example. However, in digital topology since we need the process of the proof per definition, it is meaningful to study what singletons are semi-closed or semi-open, as follows:

Theorem 3.1. *The singletons generated from mixed points of the Khalimsky nD space are semi-closed sets supporting the semi-$T_{\frac{1}{2}}$-separation axiom of $(\mathbf{Z}^n, T^n)$.*

Proof: Let us consider the Khalimsky nD space $(\mathbf{Z}^n, T^n)$, $n \in \mathbf{N}$.
In case $n = 1$, since there is no mixed point in $(\mathbf{Z}, T)$ and further, each singleton from the Khalimsky line topological space $(\mathbf{Z}, T)$ is either open or closed, $(\mathbf{Z}, T)$ obviously satisfies the semi-$T_{\frac{1}{2}}$-separation axiom.
In case $n = 2$, consider each of the pure closed (resp. the pure open) points in $\mathbf{Z}^2$ denoted by p. Then we can observe that the singleton $\{p\}$ is semi-closed (resp. semi-open). To be specific, for a pure open point $p \in \mathbf{Z}^2$ the singleton $\{p\}$ is semi-open because the singleton $\{p\}$ is open. Next, for a pure closed point $p \in \mathbf{Z}^2$ the singleton $\{p\}$ is semi-closed because $\mathbf{Z}^2 \setminus \{p\}$ is open in the Khalimsky plane. Let us now prove that

for every mixed point $p \in \mathbf{Z}^2$ the singleton $\{p\}$ is semi-closed instead of semi-open. Take a mixed point $p := (2m-1, 2n)$ or $p' := (2m, 2n-1)$. For convenience, consider the point $p := (2m-1, 2n)$ which is open-closed. Then we can prove that the singleton $\{p\}$ is semi-closed. More precisely, take the set $\{(2(m-1), 2n), p, (2m, 2n)\}$ including the singleton $\{p\}$ (see Figure 1). Then we observe that $\mathbf{Z}^2 \setminus \{(2(m-1), 2n), p, (2m, 2n)\}$ is an open set in the Khalimsky plane. Put

$$\mathbf{Z}^2 \setminus \{(2(m-1), 2n), p, (2m, 2n)\} := O,$$

which is open in $(\mathbf{Z}^2, T^2)$.
Further, we obtain that

$$O \subset \mathbf{Z}^2 \setminus \{p\} \subset Cl(O), \tag{3.3}$$

because $Cl(O)$ is equal to the total set $\mathbf{Z}^2$ under $(\mathbf{Z}^2, T^2)$. Namely, by (3.3), the singleton $\{p\}$ is proved semi-closed.
Using the same method as above, we can also prove that for the closed-open point $p' := (2m, 2n-1) \in \mathbf{Z}$ the singleton $\{p'\}$ is also semi-closed under $(\mathbf{Z}^2, T^2)$. Concretely, for every point $p \in \mathbf{Z}^2$ it turns out that the singleton $\{p\}$ is either semi-open or semi-closed, which implies that $(\mathbf{Z}^2, T^2)$ satisfies the semi-$T_{\frac{1}{2}}$-separation axiom.
In case $n \geq 3$, using the similar method of the case $n = 2$, we obtain that for every pure open point (resp. pure closed point) $p \in \mathbf{Z}^n$ the singleton $\{p\}$ is semi-open (resp. semi-closed). In addition, for every mixed point $p \in \mathbf{Z}^n$ the singleton $\{p\}$ is proved semi-closed. Thus we obtain that $(\mathbf{Z}^n, T^n)$ satisfies the semi-$T_{\frac{1}{2}}$-separation axiom, $n \geq 3$.
Meanwhile, in view of the property (3.1), for a mixed point $p \in \mathbf{Z}^n$ the singleton $\{p\}$ is not semi-open because $Int(\{p\})$ is an empty set. $\square$

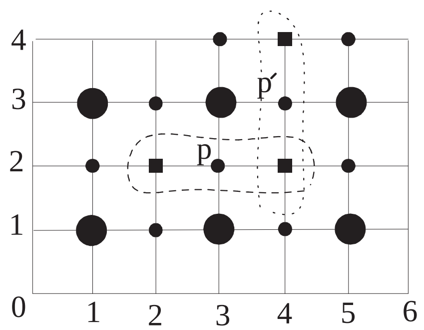

FIGURE 1. Explanation of the semi-closed structure of mixed points in $(\mathbf{Z}^2, T^2)$.

It is well known that a Scott topological space need not have an Alexandroff topological structure [5].

Let us recall [5] that for a poset $(X, \leq)$ an upper set is a subset U of X with the property that, if x is in U and $x \leq y$, then y is in U. As a dual notion of an upper set we say that a lower set of the poset $(X, \leq)$ is a subset L with the property that, if x is in L and $y \leq x$, then y is in L.

For an arbitrary element z of a poset $(X, \leq)$, the smallest lower set containing z is represented by using a down arrow as $\downarrow z = \{x \in X \mid x \leq z\}$. For every $z \in X$ take $\downarrow z$. Then, using the family consisting of the sets $X \setminus \downarrow z$ as a subbase, we can uniquely establish a topology on X, denoted by S_{up} [5].

Proposition 3.2. *The space (X, S_{up}) need not be an Alexandroff topological space.*

Proof: We can prove that the space (X, S_{up}) need not be an Alexandroff topological space in terms of the following example. Let us consider the set $[0, 1]$ which is a proper subset of the set of real numbers. Further consider a typical relation "$\leq$", *i.e.* "less than or equal" in $[0, 1]$. Then, for any point $x \in [0, 1]$ since we can take $\downarrow x := [0, x]$, we can obtain the set $[0, 1] \setminus \downarrow x$ as a member of a subbase of the topological space $([0, 1], S_{up})$. Thus the topological space $([0, 1], S_{up})$ cannot have an Alexandroff topological structure. □

4. Comparison between the L_S-property of $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ and a normal adjacency of the digital product

In relation to the study of multidimensional spaces $X \subset \mathbf{Z}^n$ (or digital spaces), let us now recall the k-adjacency relations of $\mathbf{Z}^n$ as well as some essential terminology such as a normal k-adjacency, a digital covering space and so forth. As a generalization of the k-adjacency relations of $2D$ and $3D$ digital spaces [15, 18], the k-adjacency relations of $\mathbf{Z}^n$ were established in [7] (see also [8]):

For a natural number m where $1 \leq m \leq n$, two distinct points $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n) \in \mathbf{Z}^n$ are called $k(m, n)$- (briefly, k-) adjacent if

- there are at most m indices i such that $|p_i - q_i| = 1$ and
- for all other indices i, $p_i = q_i$.

Concretely, according to the two numbers $m, n \in \mathbf{N}$, the $k(m, n)$ (or

k)-adjacency relations of $\mathbf{Z}^n$ were represented in [7, 8], as follows (for more details, see also [9, 11]).

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{where } C_i^n = \frac{n!}{(n-i)!\ i!}. \tag{4.1}$$

For instance, $(n,m,k) \in \{(2,1,4),(2,2,8);(3,1,6),(3,2,18),(3,3,26);$ $(4,1,8)$, $(4,2,32)$, $(4,3,64)$, $(4,4,80)$; $(5,1,10)$, $(5,2,50)$, $(5,3,130)$, $(5,4,$ $210)$, $(5,5,242)\}$ [7, 8].

In general, a pair (X,k) is assumed to be a (binary) digital space (or digital image) with k-adjacency in a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $(k,\bar{k}) \in \{(k,2n),(2n,\ 3^n-1)\}$ with $k \neq \bar{k}$, k represents an adjacency relation for X, and $\bar{k}$ represents an adjacency relation for $\mathbf{Z}^n \setminus X$ [15]. For $\{a,b\} \subset \mathbf{Z}$ with $a \lneq b$, $[a,b]_{\mathbf{Z}} = \{a \leq n \leq b | n \in \mathbf{Z}\}$ is considered in $(\mathbf{Z},2,2,[a,b]_{\mathbf{Z}})$. However, in this paper we are not concerned with the $\bar{k}$-adjacency between two points in $\mathbf{Z}^n \setminus X$. For a multi-dimensional digital space (X,k) and a point $x \in X \subset \mathbf{Z}^n$, the notion of a digital k-neighborhood of a point x with radius $\varepsilon \in \mathbf{N}$ was established [6, 8] in such a way: $N_k(x_0,\varepsilon) := \{x \in X |\ l_k(x_0,x) \leq \varepsilon\} \cup \{x_0\}$, (2.5) where $l_k(x_0,x)$ is the length of a shortest simple k-path from x_0 to x in X.

The recent paper [10] studies the Cartesian product property of two digital coverings in terms of the properties L_S and L_C. Based on the product property, the paper further focuses on the study of the Cartesian product of the universal covering property. Since the present paper does not concern the property *universal*, in this section we may study the Cartesian product property of two digital coverings. To study this topic, we need to recall the following essential notions such as a digital covering map [8, 12] and a normal adjacency of a digital product [8].

For digital spaces (X,k_1) on $\mathbf{Z}^{n_1}$ and (Y,k_2) on $\mathbf{Z}^{n_2}$, the paper [8] develops a k-adjacency of the Cartesian product (or digital product) $X \times Y = \{(x,y) \,|\, x \in X, y \in Y\} \subset \mathbf{Z}^{n_1+n_2}$, as follows.

Definition 3. [8] *For two digital space (X,k_1) on $\mathbf{Z}^{n_1}$, (Y,k_2) on $\mathbf{Z}^{n_2}$, we say that $(x,y) \in X \times Y \subset \mathbf{Z}^{n_1+n_2}$ is normally k-adjacent to $(x',y') \in X \times Y$ if and only if*

(1) y is equal to y' and x is k_1-adjacent to x', or

(2) x is equal to x' and y is k_2-adjacent to y', or

(3) x is k_1-adjacent to x' and y is k_2-adjacent to y'.

This k-adjacency of Definition 3 has strong merits of studying digital continuities of the corresponding products of both continuous maps and projection maps.

The following simple closed 4- and 8-curves [7, 8] and a simple closed 18- and 26-curves [11] will be often used later in this paper.

$$\left\{\begin{array}{l} SC_8^{2,4} := ((0,0),(1,1),(2,0),(1,-1)), \text{ and} \\ SC_{18}^{3,6} := ((0,0,0),(1,0,1),(1,1,2),(0,2,2),(-1,1,2),(-1,0,1)). \end{array}\right\} \quad (4.2)$$

Besides, more various cases are shown in the paper [12]. In order to study product properties of two digital coverings, the L_S-property of a digital product $(X_1 \times X_2, k)$ was established [10] (see Definition 4 of the present paper).

Definition 4. [12] *For digital spaces (X_i, k_i) in $\mathbf{Z}^{n_i}, i \in \{1,2\}$, we say that the digital product $(X_1 \times X_2, k)$ has the L_S-property (relative to (X_i, k_i)) if each point $(c_i, d_j) \in X_1 \times X_2$ has $N_k((c_i, d_j), 1) \subset X_1 \times X_2$ which is $(k, 8)$-isomorphic with $N_8((0,0),1)$ in $(\mathbf{Z}^2, 8)$.*

Even though the paper [10] referred to a relation between a normal k-adjacency of a digital product and its L_S-property, the present paper proves the assertion more precisely, as follows: Let us now investigate a relation between a normal k-adjacency on the digital product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ and the L_S-property of this product with some k-adjacency, as follows:

Proposition 4.1. *For two $SC_{k_i}^{n_i,l_i}, i \in \{1,2\}$ there is a normal k-adjacency on the digital product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ if and only if the digital product has the L_S-property.*

Proof: The paper [10] proved that the digital product $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ has the L_S-property relative to $SC_{k_i}^{n_i,l_i}$ if and only if $N_k((c_i, d_i), 1) = N_{k_1}(c_i, 1) \times N_{k_2}(d_j, 1)$. Thus we suffice to prove that for $SC_{k_1}^{n_1,l_1} := (c_i)_{i \in [0, l_1 - 1]_{\mathbf{Z}}}$ and $SC_{k_2}^{n_2,l_2} := (d_j)_{j \in [0, l_2 - 1]_{\mathbf{Z}}}$ the k-adjacency satisfying the condition

$$N_k((c_i, d_j), 1) = N_{k_1}(c_i, 1) \times N_{k_2}(d_j, 1) \quad (4.3)$$

is equal to the normal k-adjacency of $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$. Let us now assume that for each point $(c_i, d_j) \in SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ the k-adjacency $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ satisfies the property (4.3). Then the k-adjacency obviously satisfies Definition 3, which implies that a k-adjacency satisfying the property (4.3) is a normal k-adjacency.

Conversely, with the hypothesis that the digital product $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ has a normal k-adjacency, let us prove that the k-adjacency satisfies the property (4.3). Suppose that the k-adjacency of the digital product $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ does not satisfy the property (4.3). To be specific, we may take a point $(c_i, d_j) \in SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ such that

$$N_k((c_i, d_j), 1) \neq N_{k_1}(c_i, 1) \times N_{k_2}(d_j, 1). \tag{4.4}$$

In view of (4.4), we can take either

$$N_k((c_i, d_j), 1) \nsubseteq N_{k_1}(c_i, 1) \times N_{k_2}(d_j, 1) \tag{4.5}$$

or

$$N_{k_1}(c_i, 1) \times N_{k_2}(d_j, 1) \nsubseteq N_k((c_i, d_j), 1). \tag{4.6}$$

In case (4.5), the k-adjacency contradicts the property (3) of Definition 3. In case (4.6), the k-adjacency contradicts the property (1) or (2) of Definition 3. □

Example 4.2. *For the digital product $SC_{18}^{3,6} \times SC_{8}^{2,4}$, we can observe an equivalence between a normal k-adjacency on the digital product $SC_{18}^{3,6} \times SC_{8}^{2,4}$ and the L_S-property of $SC_{18}^{3,6} \times SC_{8}^{2,4}$ relative to $SC_{18}^{3,6}$ and $SC_{8}^{2,4}$. For instance, we can consider the digital product $(SC_{18}^{3,6} \times SC_{8}^{2,4}, k), k \in \{210, 242\}$ guaranteeing Proposition 4.1.*

5. Summary

We have proved that every mixed point in the Khalimsky nD space is semi-closed, which has been substantially used for showing that the Khalimsky nD space satisfies the axiom *semi-*$T_{\frac{1}{2}}$. Further, we have investigated an equivalence between the L_S-property $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ and a normal k-adjacency of $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2})$. This approach can play an important role in applied topology and digital topology including various kinds of product properties of digital spaces.

References

[1] P. Alexandorff, *Diskrete Räume*, Mat. Sb. **2** (1937), 501-518.

[2] F. G. Arenas, *Alexandroff Spaces*, Acta Math. Univ. Comenian., (N.S.) **68** (1999), 501-518.

[3] L. Boxer and I. Karaca, *Fundamental groups for digital products*, Advances and Applications in Mathematical Sciences, **11(4)** (2012), 161-179.

[4] V. A. Chatyrko, S. E. Han, Y. Hattori, *Some remarks concerning semi-*$T_{\frac{1}{2}}$ *spaces*, Filomat, in press.

[5] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *A Compendium of Continuous Lattices*, Springer, Berlin-Heidelberg-New York, 1980.

[6] N. Hamllet, *A correction to the paper "Semi-open sets and semi-continuity in topological spaces" by Norman Levine*, Proceeding of Amer. Math. Soc. **49(2)** (1975), 458-460.

[7] S. E. Han, *Computer topology and its applications*, Honam Math. Jour., **25(1)** (2003), 153-162.

[8] S. E. Han, *Non-product property of the digital fundamental group*, Information Sciences **171(1-3)** (2005), 73-91.

[9] S. E. Han, *The k-homotopic thinning and a torus-like digital image in* $\mathbf{Z}^n$, Journal of Mathematical Imaging and Vision **31(1)** (2008), 1-16.

[10] S. E. Han, *Cartesian product of the universal covering property*, Acta Applicandae Mathematicae **108** (2009), 363-383.

[11] S. E. Han, *KD-(k_0, k_1)-homotopy equivalence and its applications*, Journal of Korean Mathematical Society **47(5)** (2010), 1031-1054.

[12] S. E. Han, *Ultra regular covering space and its automorphism group*, International Journal of Applied Mathematics & Computer Science, **20(4)** (2010), 699-710.

[13] E. D. Khalimsky, *On topologies of generalized segments*, Soviet Math. Dokl., **10** (1969), 1508-1511.

[14] E. Khalimsky, R. Kopperman, P. R. Meyer, *Computer graphics and connected topologies on finite ordered sets*, Topology and Its Applications, **36(1)** (1991), 1-17.

[15] T. Y. Kong, A. Rosenfeld, *Topological Algorithms for the Digital Image Processing*, Elsevier Science, Amsterdam, (1996).

[16] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36-41.

[17] D. Marcus, F. Wyse et al., *Solution to problem 5712*, Am. Math. Monthly **77** (1970), 1119.

[18] A. Rosenfeld, *Arcs and curves in digital pictures*, Jour. of the ACM, **20** (1973), 81-87.

Sang-Eon Han
Faculty of Liberal Education, Institute of Pure and Applied Mathematics, Chonbuk National University,
Jeonju-City Jeonbuk, 561-756, Republic of Korea.
Tel: +82-63-270-4449,
E-mail:sehan@jbnu.ac.kr