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ON THE VOLUMES OF THE ORDERS IN A QUATERNION ALGEBRA

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Abstract. There are several ways to normalize a Haar measure on the orders in a quaternion algebra. In order to understand the arithmetic properties of orders, we study the volumes of orders and compute them for some cases.

1. Introduction

A quaternion algebra over a field k means a semi simple algebra of dimension 4 over k. There are three types of primitive orders in quaternion algebras over a local field. Namely, an order $\mathcal{O}$ of a quaternion algebra H over a local field k is called primitive if it satisfies one of the following conditions: (i) if H is a division algebra, $\mathcal{O}$ contains the full ring of integers of a quadratic extension field of k. This type of order is called a special order in [5]. (ii) if H is isomorphic to $\mathrm{Mat}_{2\times 2}(k)$, then $\mathcal{O}$ contains a subset which is isomorphic either to $\mathfrak{o} \times \mathfrak{o}$, where $\mathfrak{o}$ is the ring of integers in k. This is so called an Eichler order. (iii) if H is isomorphic to $\mathrm{Mat}_{2\times 2}(k)$, then $\mathcal{O}$ contains a subset which is isomorphic to the full ring of integers in a quadratic extension field of k. The arithmetic properties of these primitive orders were studied through the optimal embeddings between orders in [3], [4], [5].

Computing volumes is one of the methods in understanding the arithemetic properties of orders in a quaternion algebra. In this paper, we study a certain Haar measure on the orders and normalize it to compute volumes of orders. We believe that these computation results will help us to study a zeta function defined on these orders.

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2. Primitive orders

2.1. In this section, we summarize the arithmetic theory of a quaternion algebra and its primitive orders. Throughout this paper we assume that k is a local field. Let $\mathfrak{o}$ denote the ring of integers in k, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$ and let L be a quadratic extension field of k. By $\Delta(\alpha)$, we denote the discriminant of α.

$$\Delta(\alpha) = \mathrm{Tr}(\alpha)^2 - 4N(\alpha),$$

where Tr and N are the trace and norm of L over k, respectively.

Let $\mathfrak{o}^2 - 4\mathfrak{o} = \{s^2 - 4n | s, n \in \mathfrak{o}\}$. Then we consider the set of all possible discriminants $(\mathfrak{o}^2 - 4\mathfrak{o})/U^2$.

Definition 1. Let L be a quadratic extension of k. We define

$$t = t(L) = \mathrm{ord}_k(\Delta(L)) - 1.$$

Remark. Note that if L is an unramified extension field of k, then $t = -1$. On the other hand, if L is a ramified extension field of a field k, then $t \geq 0$ (See 1.3 in [3]).

2.2. Special orders. Let L be the quadratic extension field over k and let H be a quaternion algebra ramified at p. That is, $H \otimes k_p$ is a division algebra. Let

$$H = \left\{ \begin{pmatrix} \alpha & \beta \\ p\overline{\beta} & \overline{\alpha} \end{pmatrix} | \alpha, \beta \in L \right\},$$

where $\overline{\alpha}$ means the conjugation of α in L and

$$\mathcal{O}_{2r+1} = \left\{ \begin{pmatrix} \alpha & \beta \\ p^{2r+1}\overline{\beta} & \overline{\alpha} \end{pmatrix} | \alpha, \beta \in \mathcal{O}_L \right\},$$

where $\mathcal{O}_L$ is the ring of integers in L. $\mathcal{O}_{2r+1}$ is called a special order of level p^{2r+1}. For the details, see [2].

2.3. Eichler orders. Let H be a 2×2 matrix algebra defined on a local field k and $\mathfrak{o}$ be the ring of integers in k. Let

$$\mathcal{O}_m = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ p^m\mathfrak{o} & \mathfrak{o} \end{pmatrix}$$

be an order of $M(2, k)$ of level p^m where p is a uniform parameter of k.

2.4. Finally, we now in position to explain the remaining type of primitive orders. Let P_L be the prime ideal of $\mathcal{O}_L$, the ring of integers in L. In [4], we have computed that the possibilities of an order, $\mathcal{O}$ of H containing $\mathcal{O}_L$. We state the results in the following theorem.

Theorem 2.1. *If an order $\mathcal{O}$ of H contains $\mathcal{O}_L$, then the possibilities of $\mathcal{O}$ are one of the followings.*

(i) *If P is a unramified prime in L, $\mathcal{O}_m = \mathcal{O}_L + \xi P_L^m$.*
(ii) *If P is a ramified prime in L, $\mathcal{O}_m = \mathcal{O}_L + (1+\xi)P_L^{m-t-1}$ or $\overline{\mathcal{O}} = \mathcal{O}_L + (1-\xi)P_L^{m-t-1}$.*

for some nonnegative integer ν with some $\xi \in H$.

Proof. See [4]. □

Remark. The order defined in the definition is the remaining type of primitive orders which were mentioned in the introduction. To see this order clearly, we need the following formation. Fix a prime p and let L be a quadratic extension field of k_p. In [4], we have proved that $\left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in L \right\}$ is a quaternion algebra over k_p.

Let $\left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in L \right\} = L + \xi L$, where $\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\xi\alpha = \overline{\alpha}\xi, \xi^2 = 1$ and $\bar{\xi} = -\xi$.

Definition 2. An order $\mathcal{O}$ has level p^m if there exists a quadratic extension field $L(p)$ of k_p and nonnegative integer m (which is even if $L(p)$ is unramified) such that $\mathcal{O} = \mathcal{O}_m$.

3. Volume of Orders

Let X be a subset of a quarternion algebra H over a local field k and $q = |\mathfrak{o}/\mathfrak{p}|$.

Definition 3. We denote by dx the additive Haar measure such that the volume of a maximal order $\mathcal{O}$ is equal to 1. We denote by dx' the multiplicative Haar measure $(1-q^{-1})^{-1}||x||_H^{-1}dx$, where $||x||_H$ is called the modulus of H and equals $N(x)^{-1}$.

Lemma 3.1. *For the multiplicative measure dx, the volume of the unit group $\mathcal{O}^\times$ of an maximal order $\mathcal{O}$ of H is given by*

$$vol(GL(2, \bar{k})) = 1 - q^{-2},$$

where $\mathcal{O}$ is the integer ring of the quarternion H and $\bar{k} = \mathfrak{o}/\mathfrak{p}$.

Proof. The canonical mapping $\mathfrak{o} \to \bar{k}$ induces a surjection from $GL(2, \mathfrak{o})$ to $GL(2, \mathfrak{o}/\mathfrak{p})$. Its kernel Z consists of the matrices congruent to the identity modulo the ideal $\mathfrak{p}$.

Let $\bar{k} = \mathfrak{o}/\mathfrak{p}$ and $|\bar{k}| = q$. Then the number of the elements of $GL(2, \bar{k})$ is

$$(q-1)^2(2q-1+(q-1)(q-2)) + (2q-1)(q-1)^2 = (q^2-1)(q^2-q).$$

The volume of Z for the measure dx is $\mathrm{vol}(\mathfrak{p}) = q^4$. Hence the volume of $GL(2, \mathfrak{o})$ for dx is the product $q^{-4}(q^2-1)(q^2-q)(1-q^{-1})^{-1} = 1-q^{-2}$. □

Definition 4. Let $X \subset \mathbb{R}$ and (e_i) be a $\mathbb{R}$ basis of X. For $x = \sum x_i e_i \in X$, we denote by $T_X(x)$ the common trace of the R- endomorphism of X given by the multiplication by x to the left and to the right. Let dx_X be the additive Haar measure on X such that

$$dx_X = |\det(T_X(e_i e_j))|^{\frac{1}{2}} \prod dx_i.$$

For the multiplicative case, $dx_X = ||x||_X^{-1} dx_X$.

Remark. For the explanations of canonical characters, we use Vigneras's notations as follows. The canonical function Φ of X is

$$\Phi = \begin{cases} \text{the characteristic function of a maximal order if } X \not\supset R \\ e^{-\pi T_X(x^t \bar{x})}, \text{ if } X \supset R \end{cases}.$$

On X, we choose a character ψ_X called a canonical character defined by the following conditions.

- $\psi_{\mathbb{R}}(x) = e^{-2i\pi x}$.
- $\psi_{K'}$ is trivial on the integer ring $R_{K'} = \mathcal{O}_{K'}$ and $\mathcal{O}_{K'}$ is self dual with respect to $\psi_{K'}$ if $\mathcal{O}_{K'}$ is a non archimedean prime field.
- $\psi_K(x) = \psi_{K'} \circ T_X(x)$ if K' is the sub-prime field of K.

The explicit construction of $\psi_{K'}$ will be given in the following lemma.

Remark. The isomorphism $x \to (y \to \psi_X(xy))$ between X and its topological dual can be written as the Fourier transformation on X,

$$f^*(x) = \int_X f(y)\psi_X(xy)dy,$$

where $dy = dy_X$ is the additive measure normalized as above. The dual measure is the measure d^*y such that the following inversion formula is valid.

$$f(x) = \int_X f^*(y)\psi_X(-yx)dy^*.$$

Definition 5. The tamagawa measure on X is the Haar measure which is self dual for the Fourier transformation associated with the canonical character ψ_X.

Lemma 3.2. *The tamagawa measure is the measure $D_X^{-1/2}dx$, where D_X is the discriminant of X, i.e.*

$$D_X = ||det(T_X(e_ie_j)||_K^{-1},$$

where (e_i) is a R' bases of a maximal order of X.

Proof. Let B be the maximal order and Φ denote the characteristic function of B. The Fourier transform of Φ is the characteristic function of the dual B^* of B with respect to trace. Similarly, since the bidual of B equals B itself, we see that $\Phi^{**} = \mathrm{vol}(B^*)\Phi$. The self dual measure of X is thus $\mathrm{vol}(B^*)^{-1/2}dx$. If (e_i) is a basis of B, we denote by (e_i^*) its dual basis defined by $T_X(e_i, e_j) = 0$ if $i \neq j$ and $T(e_i, e_i) = 1$. The dual basis is a B^* basis.

If $e_j^* = \sum a_{ij}e_i$ and let A be a matrix (a_{ij}), then we have $\mathrm{vol}(B^*) = ||\det(A)||\mathrm{vol}(B) = \det(A)^{-1}$ for the measure dx. On the other hand, it is clear that $\det(T_X(e_ie_j)) = \det(A)^{-1}$. We then have $\mathrm{vol}(B) = ||\det(T_X(e_ie_j))||^{-1}$. Similarly, we prove the dual measure of measure dx is $D_X^{-1}dx$. □

Lemma 3.3. *The discriminant of H and K are connected by the relation*

$$D_H = D_K^4 N_K(d(\mathcal{O}))^2,$$

where $d(\mathcal{O})$ is the reduced discriminant of a R' maximal order $\mathcal{O}$ in H.

Proof. See [7]. □

Remark. The discriminant of orders is computed in several references. See [2],[4].

In this section, we will compute the volume of primitive orders in a quaternion algebra.

First, we consider Eichler orders. Let $\mathcal{O}_m = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ p^m\mathfrak{o} & \mathfrak{o} \end{pmatrix}$ be an order of $M(2, k)$ of level p^m. Then

$$\Gamma_0(p^m) = \mathcal{O}_m^1 = SL_2(\mathfrak{o}) \cap \mathcal{O}_m.$$

From the following short exact sequence,

$$1 \to \mathcal{O}_m^1 \to \mathcal{O}_m^\times \to \mathfrak{o}^\times \to 1, \tag{3.1}$$

then we have $\mathrm{vol}(\mathcal{O}_m^\times) = \mathrm{vol}(\mathcal{O}_m^1)\mathrm{vol}(\mathfrak{o}^\times)$.

Theorem 3.4. *If we choose the tamagawa measure $D_X^{-1/2}dx$, the multilplicative measure $dx^{'}$ is defined by $dx^{'} = ||x||^{-1}D_X^{-1/2}dx$ on $X^{\times}$. Then we have*

$$vol(\mathcal{O}_m^1) = D_K^{-3/2}(1-p^{-2})(p+1)^{-1}p^{1-m}.$$

Proof. By the equation (3.1), we have

$$\int_{\mathcal{O}_m^{\times}} dx' = \mathrm{vol}(\mathcal{O}_m^1)\int_{\mathfrak{o}^{\times}} dx'.$$

Then this implies

$$\int_{\mathcal{O}_m^{\times}} D_H^{-1/2}||x||^{-1}dx = \mathrm{vol}(\mathcal{O}_m^1)\cdot D_K^{-1/2}\int_{\mathfrak{o}^{\times}}||x||^{-1}dx'.$$

$$\Longrightarrow \quad (p+1)^{-1}p^{1-m}D_H^{-1/2}\int_{\mathcal{O}_0^{\times}}||x||^{-1}dx = \mathrm{vol}(\mathcal{O}_m^1)D_K^{-1/2}.$$

By Lemma 3.3, $D_H^{-1/2} = D_K^{-2}N_K(n(d(\mathcal{O})))$

$$\Longrightarrow \quad (p+1)^{-1}p^{1-m}D_K^{-3/2}(1-p^{-2}) = \mathrm{vol}(\mathcal{O}_m^1). \qquad \square$$

Second, we consider the volume of groups in the special orders. In this case, let

$$\mathcal{O}_m^1 = \left\{ \begin{pmatrix} \alpha & p^m\beta \\ p^{m+1}\bar{\beta} & \bar{\alpha} \end{pmatrix} | \alpha\bar{\alpha} = 1, \alpha, \beta \in L \right\},$$

where L is a quadratic extension field of K.

Theorem 3.5. *Let $\mathcal{O}_m$ be a special order of level p^m. Then the volume of $\mathcal{O}_m^1$ for the tamagawa measure is*

$$vol(\mathcal{O}_m^1) = D_K^{-3/2}p^{1-m}(p-1)^{-1}(1-p^{-2})$$

for $r \geq 0$.

Proof. As in the Eichler order case, the equation (3.1) implies

$$\int_{\mathcal{O}_m^{\times}} dx' = \int_{\mathcal{O}_m^1} dx' \int_{\mathfrak{o}^{\times}} dx',$$

which is

$$\int_{\mathcal{O}_m^{\times}} D_H^{-1/2}||x||^{-1}dx = \mathrm{vol}(\mathcal{O}_m^1)\cdot\int_{\mathfrak{o}^{\times}} D_K^{-1/2}||x||^{-1}dx'$$

$$\Longrightarrow \quad (p-1)^{-1}p^{1-m}D_H^{-1/2}\int_{\mathcal{O}_0^{\times}}||x||^{-1}dx = \mathrm{vol}(\mathcal{O}_m^1)D_K^{-1/2}.$$

$$\Longrightarrow \quad (p-1)^{-1}p^{1-m}D_K^{-3/2}(1-p^{-2}) = \mathrm{vol}(\mathcal{O}_m^1). \qquad \square$$

Finally, we consider the volume of groups in the remaining primitive orders.

Theorem 3.6. *Let*

$$\mathcal{O}_m = \left\{ \begin{pmatrix} \alpha & p^m\beta \\ p^m\bar{\beta} & \bar{\alpha} \end{pmatrix} |\alpha, \beta \in \mathcal{O}_L \right\}$$

be an order of level p^m and let $\mathcal{O}_m^1 = \{v \in \mathcal{O}_m | N(v) = 1\}$. Then the volume of $\mathcal{O}_m$ for the tamagawa measure is

$$vol(\mathcal{O}_m^1) = D_K^{-3/2}(p^2-1)^{-1}p^{2-2m}(1-p^{-2}),$$

where p is a unramified prime in L for $m \geq 0$.

Proof. By the equation (3.1), since p is unramified, $|\mathcal{O}_L/P_L| = p^2$. Thus

$$\int_{\mathcal{O}_m^\times} dx' = \int_{\mathcal{O}_m^1} dx' \int_{\mathfrak{o}^\times} dx',$$

which is

$$\begin{aligned} & \int_{\mathcal{O}_m^\times} D_H^{-1/2}||x||^{-1}dx = \mathrm{vol}(\mathcal{O}_m^1) \cdot \int_{\mathfrak{o}^\times} D_K^{-1/2}||x||^{-1}dx' \\ \Longrightarrow \quad & (p^2-1)^{-1}p^{2-2m}D_H^{-1/2} \int_{\mathcal{O}_0^\times} ||x||^{-1}dx = \mathrm{vol}(\mathcal{O}_m^1)D_K^{-1/2}. \\ \Longrightarrow \quad & (p^2-1)^{-1}p^{2-2m}D_K^{-3/2}(1-p^{-2}) = \mathrm{vol}(\mathcal{O}_m^1). \end{aligned}$$

□

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