Honam Mathematical J. **35** (2013), No. 4, pp. 747–755
http://dx.doi.org/10.5831/HMJ.2013.35.4.747

# LOCAL SPLITTING PROPERTIES OF ENDOMORPHISM RINGS OF PROJECTIVE MODULES

Sang Cheol Lee

**Abstract.** This paper deals with the unit groups of the endomorphism rings of projective modules over polynomial rings and further over formal power series rings. A normal subgroup of the unit group is defined and discussed. The local splitting properties of elements of endomorphism rings of projective modules over polynomial rings are given.

## 1. Introduction

Throughout this paper every *ring* will be a commutative ring with identity, unless otherwise indicated, and every *module* will be a finitely generated unitary module.

Many algebraists, such as Quillen [5], Suslin, Mandal [4], Bhatwadekar, Sridharan [1], Ischebeck, and Ravi Rao [3], have worked on finitely generated projective modules over commutative Noetherian rings.

Let $R$ be a (not necessarily commutative) ring and let $x$ be an indeterminate. Consider the polynomial ring $R[x]$. In section 2, $(1 + xR[x])^*$ is defined. So, if $P$ is a projective module over a ring $A$, then $(id + xEnd_{A[x]}(P[x]))^*$ is constructed. This is a normal subgroup of the unit group of endomorphism rings of a projective module over a polynomial ring. If $(A, \mathfrak{m})$ is a local ring with $dim(A) \geq \mu(\mathfrak{m})$, then we show that $(id + xEnd_{A[x]}(P[x]))^*$ is a normal subgroup of $SL_{A[x]}(P[x])$. Also, we will show that under these conditions the similar conclusion can be drawn for the ring of formal power series.

In section 3, let $P$ be a projective $A$-module. Let $s_1, s_2 \in A$ be such that $As_1 + As_2 = A$. Then we use the splitting lemma of Quillen to show that every element of $(id_{P_{s_1 s_2}} + xEnd_{A_{s_1 s_2}[x]}(P_{s_1 s_2}[x]))^*$ has two decompositions. Finally, we generalize the result to Theorem 3.4.

Received October 14, 2013. Accepted November 22, 2013.
2010 Mathematics Subject Classification. Primary 13C10, 13B25, 13B30.
Key words and phrases. projective module, endomorphism ring, polynomial ring.

## 2. A Normal Subgroup of the Unit Group

Let $R$ be a (not necessarily commutative) ring with identity. An element of $R$ is called a *unit* if it has a multiplicative inverse. Let $U(R)$ be the set of all units of $R$. Then $(U(R), \cdot)$ forms a group, which is called the *unit group* of the ring $R$.

We now consider the polynomial ring $R[x]$ over a ring $R$ with an indeterminate $x$. Then $U(R) \subseteq U(R[x])$. However, the converse does not hold, in general. For example, $(1 + 2x)^2 = 1$ in $\mathbb{Z}_4[x]$, so $1 + 2x \in U(\mathbb{Z}_4[x]) \backslash U(\mathbb{Z}_4)$. Hence $U(\mathbb{Z}_4) \subsetneq U(\mathbb{Z}_4[x])$. (Of course, if $R$ is an integral domain, then it is well-known that $U(R[x]) = U(R)$.)

For a ring $R$, define a map $\varphi : U(R[x]) \to U(R)$ by $\varphi(f(x)) = f(0)$, where $f(x) \in U(R[x])$. Then $\varphi$ is a group epimorphism with

$$Ker(\varphi) = \{f(x) \in U(R[x]) \mid f(0) = 1\}.$$

Hence $\{f(x) \in U(R[x]) \mid f(0) = 1\}$ is a normal subgroup of the unit group $U(R[x])$ of the polynomial ring $R[x]$ and

$$U(R[x])/\{f(x) \in U(R[x]) \mid f(0) = 1\} \cong U(R).$$

Write $1+xR[x] = \{1+xg(x) \mid g(x) \in R[x]\}$. Then the following inclusion does not hold in general:

$$1 + xR[x] \subseteq U(R[x]).$$

For example, let $\mathbb{Z}$ be the ring of integers. Consider a polynomial $1+2x \in \mathbb{Z}[x]$. Then $1+2x$ has no inverse in the polynomial ring $\mathbb{Z}[x]$. (Of course, it has an inverse $1 + (-2)x + 4x^2 + \cdots + (-2)^n x^n + \cdots\cdots$ in the formal power series ring $\mathbb{Z}[[x]]$.) For our convenience, write

$$(1 + xR[x])^* = (1 + xR[x]) \cap U(R[x]).$$

Notice that $(1 + xR[x]) \cap U(R[x]) = \{f(x) \in U(R[x]) \mid f(0) = 1\}$. Then $(1 + xR[x])^* = \{f(x) \in U(R[x]) \mid f(0) = 1\}$. Hence $(1 + xR[x])^*$ is a normal subgroup of the unit group $U(R[x])$ and

$$U(R[x])/(1 + xR[x])^* \cong U(R).$$

Let $A$ be a ring. For an $A$-module $M$, we write $M[x]$ for $M \otimes_A A[x]$. Let $P$ be a projective $A$-module. Since the identity of the endomorphism ring $End_A(P)$ is the identity map $id_P : P \to P$, we can get a normal subgroup $(id_P + xEnd_A(P)[x])^*$ of the unit group $U(End_A(P)[x])$ of the polynomial ring $End_A(P)[x]$ over the endomorphism ring $End_A(P)$. By [3, Lemma 4.3.5] there exists a ring isomorphism

$$\varphi : End_A(P)[x] \to End_{A[x]}(P[x]).$$

We can restrict the isomorphism $\varphi$ to the unit group $U(End_A(P)[x])$ to get a group isomorphism

$$\varphi|_{U(End_A(P)[x])} : U(End_A(P)[x]) \to U(End_{A[x]}(P[x])).$$

The normal subgroup $(id_P + xEnd_A(P)[x])^*$ of the unit group $U(End_A(P)[x])$ corresponds to a normal subgroup $(id_{P[x]} + xEnd_{A[x]}(P[x]))^*$ of the unit group $U(End_{A[x]}(P[x]))$ under the isomorphism $\varphi|_{U(End_A(P)[x])}$. The latter group is the automorphism group $Aut_{A[x]}(P[x])$. Hence

$$Aut_{A[x]}(P[x])/(id_{P[x]} + xEnd_{A[x]}(P[x]))^* \cong U(End_A(P)) = Aut_A(P)$$

and

$$\begin{aligned}(id_{P[x]} \quad + \quad & xEnd_{A[x]}(P[x]))^* \\ = \quad & \{id_{P[x]} + x\beta(x) \mid \beta(x) \in End_{A[x]}(P[x])\} \cap U(End_{A[x]}(P[x])) \\ = \quad & \{\alpha(x) \in End_{A[x]}(P[x]) \mid \alpha(0) = id_P\} \cap Aut_{A[x]}(P[x]) \\ = \quad & \{\alpha(x) \in Aut_{A[x]}(P[x]) \mid \alpha(0) = id_P\}.\end{aligned}$$

**Lemma 2.1.** *Let $A$ be a ring. Let $P$ be a projective $A$-module with finite rank. Then for every element $\alpha(x) \in (id + xEnd_{A[x]}(P[x]))^*$ there exists a nilpotent polynomial $f(x)$ in $A[x]$ such that $det(\alpha(x)) = 1 + xf(x)$ in $A[x]$.*

*Proof.* Let $P$ be a projective $A$-module of rank $n < \infty$. Then $P[x]$ is a projective $A[x]$-module of rank $n < \infty$. Then

$$Aut_{A[x]}(P[x]) = \{\alpha(x) \in End_{A[x]}(P[x]) \mid det(\alpha(x)) \in U(A[x])\}$$

and

$$(id + xEnd_{A[x]}(P[x]))^* = \{\alpha(x) \in Aut_{A[x]}(P[x]) \mid \alpha(0) = id\}.$$

Now let $\alpha(x) \in (id + xEnd_{A[x]}(P[x]))^*$. Since $det(\alpha(x)) \in U(A[x])$, there exists $h(x) \in A[x]$ such that $(det(\alpha(x)))h(x) = 1$. Since $\alpha(0) = id$, we have $det(\alpha(0)) = 1$. Hence $det(\alpha(x)) = 1 + xf(x)$ for some $f(x) \in A[x]$. Also, $h(0) = det(\alpha(0))h(0) = 1$, so $h(x) = 1 + xg(x)$ for some $g(x) \in A[x]$. $1 = (det(\alpha(x)))h(x) = (1 + xf(x))(1 + xg(x))$. So, we get

$$\begin{cases} f(x) + g(x) &= 0, \\ f(x)g(x) &= 0 \end{cases}$$

From these equations, we can get $f(x)^2 = 0$. □

We adopt the notation $\mu(M)$ in [4, Notations 4.1.1] in the following.

**Theorem 2.2.** *Let $P$ be a projective $A$-module with finite rank. If $(A, \mathfrak{m})$ is a local ring with $dim(A) \geq \mu(\mathfrak{m})$, then $(id + xEnd_{A[x]}(P[x]))^*$ is a normal subgroup of $SL_{A[x]}(P[x])$.*

*Proof.* By our assumption and [6, Lemma 4.12],

$$dim(A) \geq \mu(\mathfrak{m}) = dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \geq dim(A).$$

So, $dim(A) = dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. Hence $A$ is a regular local ring. By [3, Proposition 8.2.2], $A$ is an integral domain. So, $A[x]$ is an integral domain. Therefore the result follows from Lemma 2.1. □

For an $A$-module $M$, we write $M[[x]]$ for $M \otimes_A A[[x]]$.

**Theorem 2.3.** *Let $A$ be a ring. Let $P$ be a projective $A$-module with finite rank. Then the following are true.*

(1) $(id + xEnd_{A[[x]]}(P[[x]]))^* = \{\alpha(x) \in Aut_{A[[x]]}(P[[x]]) \mid \alpha(0) = id\}$.

(2) $(id+xEnd_{A[[x]]}(P[[x]]))^*$ *is a normal subgroup of* $Aut_{A[[x]]}(P[[x]])$.

(3) *If $(A, \mathfrak{m})$ is a local ring with $dim(A) \geq \mu(\mathfrak{m})$, then*

$$(id + xEnd_{A[[x]]}(P[[x]]))^*$$

*is normal in* $SL_{A[[x]]}(P[[x]])$.

*Proof.* It is easy to see that (1) and (2) are true. (3) If $A$ is an integra domain, then so is $A[[x]]$. As in the proof of Theorem 2.2, we can show that (3) is true. □

## 3. Local Splitting Properties of Endomorphism Rings of Projective Modules over Polynomial Rings

Daniel Quillen proved the following lemma in [5, Lemma 1]. This is called the *Quillen splitting lemma* (see [4, Lemma 3.1.1, Lemma 4.3.1].)

**Lemma 3.1.** *Let $R$ be an algebra over a ring $A$, let $f \in A$, and let $\theta \in (1 + xR_f[x])^*$. Then there exists an integer $k \geq 0$ such that for any $g_1, g_2 \in A$ with $g_1 - g_2 \in f^k A$, there exists $\psi \in (1 + xR[x])^*$ such that*

$$\psi_f(x) = \theta(g_1 x)\theta(g_2 x)^{-1}.$$

The following result is a local version of the Quillen splitting lemma above. We can see its proof in [5, Theorem 1] and [3, Lemma 4.3.8], but we use Lemma 3.1 to prove it. We state the proof for our records. The matrix form of Theorem 3.2 (1) is given in [2, Lemma 2.5.1].

In the localization $A_s$ of a ring $A$ at $T = \{1, s, s^2, \cdots, s^n, \cdots\}$, we sometimes write an element 1 of $T$ by $s^0$ for our clarification.

**Theorem 3.2.** *Let $P$ be a projective module over a ring $A$ and let $s_1, s_2 \in A$ be such that $As_1 + As_2 = A$. Let $\sigma(x) \in Aut_{A_{s_1 s_2}[x]}(P_{s_1 s_2}[x])$ such that $\sigma(0) = id_{P_{s_1 s_2}}$. Then $\sigma(x)$ splits in two ways:*

(1) *There exist* $\alpha_1(x) \in Aut_{A_{s_1}[x]}(P_{s_1}[x])$ *and* $\alpha_2(x) \in Aut_{A_{s_2}[x]}(P_{s_2}[x])$ *with* $\alpha_1(0) = id_{P_{s_1}}$ *and* $\alpha_2(0) = id_{P_{s_2}}$*, respectively, such that*

$$\sigma(x) = \alpha_1(x)_{s_2}\alpha_2(x)_{s_1}.$$

(2) *There exist* $\beta_1(x) \in Aut_{A_{s_1}[x]}(P_{s_1}[x])$ *and* $\beta_2(x) \in Aut_{A_{s_2}[x]}(P_{s_2}[x])$ *with* $\beta_1(0) = id_{P_{s_1}}$ *and* $\beta_2(0) = id_{P_{s_2}}$*, respectively, such that*

$$\sigma(x) = \beta_2(x)_{s_1}\beta_1(x)_{s_2}.$$

*Moreover, if* $\sigma(x) \neq \sigma(ax)^2$ *for any non-zero element* $a \in A$*, then the two decompositions are distinct.*

*Proof.* Let $\sigma(x) \in Aut_{A_{s_1s_2}[x]}(P_{s_1s_2}[x])$ such that $\sigma(0) = id_{P_{s_1s_2}}$. Then since $U(End_{A_{s_1s_2}[x]}(P_{s_1s_2}[x])) = Aut_{A_{s_1s_2}[x]}(P_{s_1s_2}[x])$, we have

$$\sigma(x) \in (id + xEnd_{A_{s_1s_2}[x]}(P_{s_1s_2}[x]))^*.$$

Notice that

$$(id + xEnd_{A_{s_1s_2}[x]}(P_{s_1s_2}[x]))^* = (id + xEnd_{(A_{s_1})_{s_2}[x]}((P_{s_1})_{s_2}[x])))^*.$$

Then by the Quillen splitting lemma, there exists an integer $k_1 \geq 0$ such that for any $f_1, f_2 \in A$ with $f_1 - f_2 \in As_2^{k_1}$, we have $f_1/s_1^0, f_2/s_1^0 \in A_{s_1}$ with $f_1/s_1^0 - f_2/s_1^0 \in A_{s_1}(s_2/s_1^0)^{k_1}$, so that there exists $\psi_1(x) \in (id + xEnd_{A_{s_1}[x]}(P_{s_1}[x]))^*$ such that $\psi_1(x)_{s_2/s_1^0} = \sigma((f_1/s_1^0)x)\sigma((f_2/s_1^0)x)^{-1}$. For our simplicity, we write this equation by $\psi_1(x)_{s_2} = \sigma(f_1x)\sigma(f_2x)^{-1}$. Since

$$(id + xEnd_{A_{s_1s_2}[x]}(P_{s_1s_2}[x]))^* = (id + xEnd_{(A_{s_2})_{s_1}[x]}((P_{s_2})_{s_1}[x])))^*,$$

it follows from the Quillen splitting lemma again that there exists an integer $k_2 \geq 0$ such that for any $g_1, g_2 \in A$ with $g_1 - g_2 \in As_1^{k_2}$, there exists $\psi_2(x) \in (id + xEnd_{A_{s_2}[x]}(P_{s_2}[x]))^*$ such that $\psi_2(x)_{s_1} = \sigma(g_1x)\sigma(g_2x)^{-1}$.

Now take $k = max\{k_1, k_2\}$. $As_1 + As_2 = A$, so $As_1^k + As_2^k = A$. There exist $\lambda, \mu \in A$ such that $\lambda s_1^k + \mu s_2^k = 1$.

(1) Consider the equation

$$\sigma(x) = (\sigma(x)\sigma(\lambda s_1^k x)^{-1})(\sigma(\lambda s_1^k x)\sigma(0)^{-1}).$$

$1 - \lambda s_1^k = \mu s_2^k \in As_2^k \subseteq As_2^{k_1}$, so there exists

$$\alpha_1(x) \in (id + xEnd_{A_{s_1}[x]}(P_{s_1}[x]))^*$$

such that $\alpha_1(x)_{s_2} = \sigma(x)\sigma(\lambda s_1^k x)^{-1}$. $\lambda s_1^k \in As_1^k \subseteq As_1^{k_2}$, so there exists

$$\alpha_2(x) \in (id + xEnd_{A_{s_2}[x]}(P_{s_2}[x]))^*$$

such that $\alpha_2(x)_{s_1} = \sigma(\lambda s_1^k x)\sigma(0)^{-1}$. Hence $\sigma(x) = \alpha_1(x)_{s_2}\alpha_2(x)_{s_1}$.

(2) Consider the equation

$$\sigma(x) = (\sigma(\lambda s_1^k x)\sigma(0)^{-1})(\sigma(\lambda s_1^k x)^{-1}\sigma(x)).$$

Let $\tau(x) = \sigma(x)^{-1}$. Then $\sigma(\lambda s_1^k x)^{-1}\sigma(x) = \tau(\lambda s_1^k x)\tau(x)^{-1}$. So,

$$\sigma(x) = (\sigma(\lambda s_1^k x)\sigma(0)^{-1})(\tau(\lambda s_1^k x)\tau(x)^{-1}).$$

$\lambda s_1^k - 1 = -\mu s_2^k \in As_2^k \subseteq As_2^{k_1}$, so there exists $\beta_1(x) \in (id + xEnd_{A_{s_1}[x]}(P_{s_1}[x]))^*$ such that $\beta_1(x)_{s_2} = \tau(\lambda s_1^k x)\tau(x)^{-1}$. Take $\beta_2(x) = \alpha_2(x)$ with the notation as in (1). Then $\sigma(x) = \beta_2(x)_{s_1}\beta_1(x)_{s_2}$.

Moreover, assume that the two decompositions are identical. Then

$$\alpha_1(x)_{s_2} = \beta_2(x)_{s_1} \text{ and } \alpha_2(x)_{s_1} = \beta_1(x)_{s_2}.$$

So, by (1) and (2),

$$\sigma(x)\sigma(\lambda s_1^k x)^{-1} = \alpha_1(x)_{s_2} = \beta_2(x)_{s_1} = \sigma(\lambda s_1^k x)\sigma(0)^{-1}.$$

Since $\sigma(0) = id$, it follows that $\sigma(x) = \sigma(\lambda s_1^k x)^2$. □

**Corollary 3.3.** *Let $P$ be a projective $A$-module. Let $s_1, s_2 \in A$ be such that $As_1 + As_2 = A$. Then*

$$\begin{aligned}(id_{P_{s_1 s_2}} &+ xEnd_{A_{s_1 s_2}[x]}(P_{s_1 s_2}[x]))^* \\ &= (id_{P_{s_1}} + xEnd_{A_{s_1}[x]}(P_{s_1}[x]))^*_{s_2}(id_{P_{s_2}} + xEnd_{A_{s_2}[x]}(P_{s_2}[x]))^*_{s_1}\end{aligned}$$

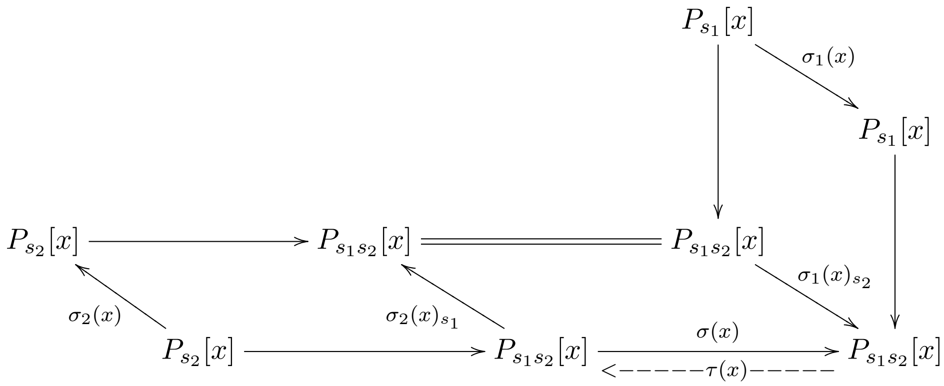
*and*

$$\begin{aligned}(id_{P_{s_1 s_2}} &+ xEnd_{A_{s_1 s_2}[x]}(P_{s_1 s_2}[x]))^* \\ &= (id_{P_{s_2}} + xEnd_{A_{s_2}[x]}(P_{s_2}[x]))^*_{s_1}(id_{P_{s_1}} + xEnd_{A_{s_1}[x]}(P_{s_1}[x]))^*_{s_2}.\end{aligned}$$

**Theorem 3.4.** *Let $P$ be a projective $A$-module and let $s_1, s_2 \in A$ be such that $As_1 + As_2 = A$. Let $\sigma(x) \in End_{A_{s_1 s_2}[x]}(P_{s_1 s_2}[x])$. Then the following statements are true:*

(1) *If $\sigma(x)$ has a left inverse and $\sigma(0) = id_{P_{s_1 s_2}}$, then there exists $\sigma_1(x) \in End_{A_{s_1}[x]}(P_{s_1}[x])$ which has a right inverse and $\sigma_1(0) = id_{P_{s_1}}$ and there exists $\sigma_2(x) \in End_{A_{s_2}[x]}(P_{s_2}[x])$ which has a left inverse and $\sigma_2(0) = id_{P_{s_2}}$ such that $\sigma(x) = \sigma_1(x)_{s_2}\ \sigma_2(x)_{s_1}$.*
(2) *If $\sigma(x)$ has a right inverse and $\sigma(0) = id_{P_{s_1 s_2}}$, then there exists $\sigma_1(x) \in End_{A_{s_1}[x]}(P_{s_1}[x])$ which has a left inverse and $\sigma_1(0) = id_{P_{s_1}}$ and there exists $\sigma_2(x) \in End_{A_{s_2}[x]}(P_{s_2}[x])$ which has a right inverse and $\sigma_2(0) = id_{P_{s_2}}$ such that $\sigma(x) = \sigma_2(x)_{s_1}\ \sigma_1(x)_{s_2}$.*

*Proof.* (1) Let $\tau(x)$ be a left inverse of $\sigma(x)$. Consider the following diagram:




Since $\sigma(0) = id_{P_{s_1s_2}}$, it follows from [3, Lemma 4.3.5] that there exists $\sigma_0(x) \in End_{A_{s_1s_2}[x]}(P_{s_1s_2}[x])$ such that $\sigma(x) = id_{P_{s_1s_2}} + x\sigma_0(x)$. Since $\tau(x)\sigma(x) = id_{P_{s_1s_2}}$, we have $\tau(0) = \tau(0)id_{P_{s_1s_2}} = \tau(0)\sigma(0) = id_{P_{s_1s_2}}$. So, there exists $\tau_0(x) \in End_{A_{s_1s_2}[x]}(P_{s_1s_2}[x])$ such that $\tau(x) = id_{P_{s_1s_2}} + x\tau_0(x)$. Hence

$$\begin{aligned} id_{P_{s_1s_2}} &= \tau(x)\sigma(x) \\ &= (id_{P_{s_1s_2}} + x\tau_0(x))(id_{P_{s_1s_2}} + x\sigma_0(x)) \\ &= id_{P_{s_1s_2}} + x(\tau_0(x) + \sigma_0(x)) + x^2\tau_0(x)\sigma_0(x) \end{aligned}$$

From this equation we get

$$\begin{cases} \tau_0(x) + \sigma_0(x) = 0, \\ \tau_0(x)\sigma_0(x) = 0 \end{cases}$$

There exists a positive integer $k_1$ such that both $\lambda s_1^k\sigma_0(x)$ and $\lambda s_1^k\tau_0(x)$ are in $End_{A_{s_2}[x]}(P_{s_2}[x])$ for all $\lambda \in A$ and for all $k \geq k_1$. There exists a positive integer $k_2$ such that both $\mu s_2^k\sigma_0(x)$ and $\mu s_2^k\tau_0(x)$ are in $End_{A_{s_1}[x]}(P_{s_1}[x])$ for all $\mu \in A$ and for all $k \geq k_2$. Take $m = max\{k_1, k_2\}$. Then $As_1^m + As_2^m = A$. There exist $\lambda_0, \mu_0 \in A$ such that $\lambda_0 s_1^m - \mu_0 s_2^m = 1$. Let

$$\begin{aligned} \sigma_{12}(x) &= id_{P_{s_2}} + \lambda_0 s_1^m x\sigma_0(x) \in End_{A_{s_2}[x]}(P_{s_2}[x]), \\ \sigma_{21}(x) &= id_{P_{s_1}} + \mu_0 s_2^m x\sigma_0(x) \in End_{A_{s_1}[x]}(P_{s_1}[x]), \\ \tau_{12}(x) &= id_{P_{s_2}} + \lambda_0 s_1^m x\tau_0(x) \in End_{A_{s_2}[x]}(P_{s_2}[x]), \\ \tau_{21}(x) &= id_{P_{s_1}} + \mu_0 s_2^m x\tau_0(x) \in End_{A_{s_1}[x]}(P_{s_1}[x]). \end{aligned}$$

Then

$$\begin{aligned} \frac{\sigma_{12}(x)}{s_1^0} &= id_{P_{s_1s_2}} + \frac{\lambda_0 s_1^m}{s_1^0} x\sigma_0(x), \\ \frac{\sigma_{21}(x)}{s_2^0} &= id_{P_{s_1s_2}} + \frac{\mu_0 s_2^m}{s_2^0} x\sigma_0(x), \\ \frac{\tau_{12}(x)}{s_1^0} &= id_{P_{s_1s_2}} + \frac{\lambda_0 s_1^m}{s_1^0} x\tau_0(x), \\ \frac{\tau_{21}(x)}{s_2^0} &= id_{P_{s_1s_2}} + \frac{\mu_0 s_2^m}{s_2^0} x\tau_0(x) \end{aligned}$$

and these are all in $End_{A_{s_1s_2}[x]}(P_{s_1s_2}[x])$. Let's write

$$\sigma_{12}(x)_{s_1} = \frac{\sigma_1(x)}{s_1^0},\ \sigma_{21}(x)_{s_2} = \frac{\sigma_2(x)}{s_2^0},\ \tau_{12}(x)_{s_1} = \frac{\tau_1(x)}{s_1^0},\ \tau_{21}(x)_{s_2} = \frac{\tau_2(x)}{s_2^0}.$$

Then we have

$$\begin{pmatrix} \tau_{12}(x)_{s_1} \\ \tau_{21}(x)_{s_2} \end{pmatrix} \begin{pmatrix} \sigma_{12}(x)_{s_1} & \sigma_{21}(x)_{s_2} \end{pmatrix} = \begin{pmatrix} id_{P_{s_1s_2}} & \tau(x) \\ \sigma(x) & id_{P_{s_1s_2}} \end{pmatrix}.$$

(Further $\tau(x) = id_{P_{s_1s_2}} - x\sigma_0(x)$ and $\sigma(x) = id_{P_{s_1s_2}} - x\tau_0(x)$.) Now if we take $\sigma_1(x) = \tau_{21}(x)$ and $\sigma_2(x) = \sigma_{12}(x)$, then we can get the result (1).

(2) If we change the roles of $\sigma(x)$ and $\tau(x)$ in the proof of (1), then we can get

$$\begin{pmatrix} \sigma_{12}(x)_{s_1} \\ \sigma_{21}(x)_{s_2} \end{pmatrix} \begin{pmatrix} \tau_{12}(x)_{s_1} & \tau_{21}(x)_{s_2} \end{pmatrix} = \begin{pmatrix} id_{P_{s_1s_2}} & \sigma(x) \\ \tau(x) & id_{P_{s_1s_2}} \end{pmatrix}.$$

If we take $\sigma_1(x) = \tau_{21}(x)$ and $\sigma_2(x) = \sigma_{12}(x)$, then we can get the result (2). □

Finally, we prove that Theorem 3.4 is a generalization of Theorem 3.2. In fact, under the same assumption as in Theorem 3.4, let

$$\sigma(x) \in Aut_{A_{s_1s_2}[x]}(P_{s_1s_2}[x])$$

such that $\sigma(0) = id_{P_{s_1s_2}}$. Then $\sigma(x)$ has an inverse $\tau$(x), so that $\tau(x)\sigma(x) = id_{P_{s_1s_2}}$ and $\sigma(x)\tau(x) = id_{P_{s_1s_2}}$. From these two equations, we can get the last two equations, of the proof of Theorem 3.4, which are in the matrix forms. Now, take $\sigma_1(x) = \tau_{21}(x)$ and $\sigma_2(x) = \sigma_{12}(x)$. Then it follows from the two matrices that $\sigma_1(x)$ has an inverse $\sigma_{21}(x)$ and $\sigma_2(x)$ has an inverse $\tau_{12}(x)$. So, $\sigma_1(x) \in Aut_{A_{s_1}[x]}(P_{s_1}[x])$ and

$\sigma_2(x) \in Aut_{A_{s_2}[x]}(P_{s_2}[x])$ with $\sigma_1(0) = id_{P_{s_1}}$ and $\sigma_2(0) = id_{P_{s_2}}$, respectively, such that

$$\sigma(x) = \sigma_1(x)_{s_2}\sigma_2(x)_{s_1}.$$

This shows that Theorem 3.4 (1) holds. The remainder of the proof is similar.

Let's summarize the results. Let $s_1, s_2 \in A$ be such that $As_1 + As_2 = A$ and let $P$ be a projective $A$-module. Then we used the splitting lemma of Quillen to show that every element of $(id_{P_{s_1 s_2}} + xEnd_{A_{s_1 s_2}[x]}(P_{s_1 s_2}[x]))^*$ has two decompositions. And then we generalized the result to Theorem 3.4. Consequently, we sharpened a local version of the Quillen splitting lemma to generalize it.

## References

[1] S. M. Bhatwadekar and Raja Sridharan, *The Euler Class Group of a Noetherian Ring*, Composito Mathematica, **122** (2000), 183-222.

[2] R. Basu and R. Sridharan, *On Forster's Conjecture and Related Results*, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai-400005, India, 2002.

[3] F. Ischebeck and Ravi Rao, *Ideals and Reality, Projective Modules and Number of Generators of Ideals*, Springer, 2005.

[4] S. Mandal, *Projective Modules and Complete Intersections*, Springer, 1997.

[5] D. Quillen, *Projective Modules over Polynomial Rings*, Inv. Math. **36** (1976), 167-171.

[6] S. Raghavan Balwant Singh and R. Sridharan, *Homological Methods in Commutative Algebra*, Tata Institute of Fundamental Research, Bombay, 1975.

Sang Cheol Lee
Department of Mathematics Education and Institute of Pure and Applied Mathematics, Chonbuk National University,
Jeonju, Jeonbuk 561-756, South Korea.
E-mail: scl@jbnu.ac.kr