Honam Mathematical J. **35** (2013), No. 4, pp. 757–764
http://dx.doi.org/10.5831/HMJ.2013.35.4.757

A MISCELLANY OF SELECTION THEOREMS WITHOUT CONVEXITY

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Abstract. In this paper, we give sufficient conditions for a map with nonconvex values to have a continuous selection and the selection extension property in LC-metric spaces under the one-point extension property. And we apply it to weakly lower semicontinuous maps and generalize previous results. We also get a continuous selection theorem for almost lower semicontinuous maps with closed sub-admissible values in $\mathbb{R}$-trees.

1. Introduction

In 1956, Michael [13] stated the important and well-known selection theorem;

Theorem 1.1. *Let X be a paracompact space, Y be a Banach space, and $F : X \multimap Y$ be a lower semicontinuous map with closed convex values. Then F has a continuous selection; that is, there is a continuous function $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.*

Michael's selection theorem has been extended to spaces with a generalized definition of convexity and maps with a weaker notion of lower semicontinuity. The following selection theorems are examples of such results in LC-metric spaces, hyperconvex spaces and $\mathbb{R}$-trees;

Theorem 1.2. (Gutev [5]) *Let X be a paracompact space, Y be a Banach space, and $F : X \multimap Y$ be a weakly lower semicontinuous map with closed convex values. then F has a continuous selection.*

Received October 14, 2013. Accepted November 20, 2013.
2010 Mathematics Subject Classification. 54C60, 54C65, 54E50.
Key words and phrases. selection, LC-metric space, weakly lower semicontinuous, quasi-lower semicontinuous, almost lower semicontinuous, one-point extension property, selection extension, hyperconvex, R-tree.

Theorem 1.3. (Ben-El-Mechaiekh and Oudadess [2]) *Let X be a paracompact space, $(Y;\Gamma)$ be a complete LC-metric space, $Z \subset X$ with $\dim_X Z \leq 0$ and $F : X \multimap Y$ be a lower semicontinuous map with closed values such that $F(x)$ is Γ-convex for $x \notin Z$. Then F admits a continuous selection.*

Theorem 1.4. (Markin [11]) *Let X be a paracompact topological space, (M,d) be a hyperconvex metric space and $F : X \multimap M$ be a quasi-lower semicontinuous map with closed sub-admissible values. Then F has a continuous selection.*

In Section 3, we give sufficient conditions for a map with nonconvex values to have a continuous selection and the selection extension property in *LC*-metric spaces under the one-point extension property. We apply it to weakly lower semicontinuous maps and obtain the generalization of Theorem 1.2 and 1.3. In Section 4, Theorem 1.4 also be extended to an almost lower semicontinuous map in an $\mathbb{R}$-tree.

2. Preliminaries

A *multimap* (or *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$. For $A \subset X$, let $F(A) = \bigcup\{F(x) : x \in A\}$. Let $\overline{\quad}$ and GrF denote the closure and graph of F, respectively.

Let X be a topological space and (Y,d) be a metric space. A multimap $F : X \multimap Y$ is called;

(1) *lower semicontinuous* (lsc) at $x \in X$, if for each open set W with $W \cap F(x) \neq \emptyset$, there is a neighborhood $U(x)$ of x such that $F(z) \cap W \neq \emptyset$ for all $z \in U(x)$.

(2) *weakly lower semicontinuous* (wlsc) [15] (or *quasi lower semicontinuous* in [5, 6]) at $x \in X$, every neighborhood $U(x)$ of x and $\epsilon > 0$, there is a point $x' \in U(x)$ such that for every point $y \in F(x')$ there is a neighborhood U_y of x for which $y \in \bigcap_{z \in U_y} B_\epsilon(F(z))$.

(3) *quasi-lower semicontinuous* (qlsc) at $x \in X$, if for each $\epsilon > 0$, there are $y \in F(x)$ and a neighborhood $U(x)$ of x such that $F(z) \cap B_\epsilon(y) \neq \emptyset$ for all $z \in U(x)$.

(4) *almost lower semicontinuous* (alsc) at $x \in X$, if for each $\epsilon > 0$, there is a neighborhood $U(x)$ of x such that $\bigcap_{z \in U(x)} B_\epsilon(F(z)) \neq \emptyset$.

If F is lsc [wlsc, qlsc, alsc, respectively] at each $x \in X$, F is called *lsc* [*wlsc, qlsc, alsc*, respectively]. Note that F is lsc at each $x \in X$ if and only if for each $y \in F(x)$ and $\epsilon > 0$, there is a neighborhood $U(x)$ of x such that $y \in \bigcap_{z \in U(x)} B_\epsilon(F(z))$ ([4], Lemma 3.1). Therefore (1) $\Longrightarrow$ (2). Clearly (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) also holds, see [9].

Let X be a topological space and $\langle X \rangle$ denote the set of all nonempty finite subsets of X. A *C-structure* on X is given by a map $\Gamma : \langle X \rangle \multimap X$ such that

(1) for all $A \in \langle X \rangle$, $\Gamma_A = \Gamma(A)$ is nonempty and contractible; and
(2) for all $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

A pair (X, Γ) is then called a *C-space* by Horvath [7] and an *H-space* by Bardaro and Ceppitelli [1].

Any convex subset X of a topological vector space is a C-space (X, Γ) by putting $\Gamma_A = \mathrm{co}A$, the convex hull of A. Other examples of (X, Γ) are any convex space, any pseudo-convex space, any homeomorphic image of a convex space, any contractible space, and so on. See [1, 7].

For an (X, Γ), a subset C of X is said to be Γ-*convex* (or a C-set) if $A \in \langle C \rangle$ implies $\Gamma_A \subset C$.

A C-space (X, Γ) is called an *LC-metric space* if X is equipped with a metric d such that for any $\epsilon > 0$, the set $B_\epsilon(C) = \{x \in X : d(x, C) < \epsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, and open balls are Γ-convex. For details, see Horvath [7].

Note that each singletons are Γ-convex in LC-metric spaces ([9], Lemma 5.4).

Let X be a topological space. If $Z \subset X$, then $\dim_Z X \leq 0$ means that $\dim E \leq 0$ for every set $E \subset Z$ which is closed in X [14].

Let X and Y be topological spaces. A map $F : X \multimap Y$ has the *selection extension property* provided that for each closed subset A of X, every continuous selection for $F|_A$ extends to a continuous selection for F. Here $F|_A$ denotes the restriction of F to A. A map $F : X \multimap Y$ has the *one-point extension property* provided that for each lsc map $L : X \multimap Y$ with $L(z) \subset F(z)$ for all $z \in X$ and for each $(x, y) \in \mathrm{Gr}F \backslash \mathrm{Gr}L$, there is an lsc map $L^* : X \multimap Y$ such that $(x, y) \in \mathrm{Gr}L^*$ and

$$L(z) \subset L^*(z) \subset F(z)$$

for all $z \in X$.

3. Selections for weakly lower semicontinuous maps in *LC*-metric spaces

We begin with the following basic facts about lsc maps in [13];

Proposition 3.1. *Let X be a topological space and (Y,d) be a complete metric space.*

(1) *If $F : X \multimap Y$ is lsc, so is $\overline{F}$.*
(2) *If $G_i : X \multimap Y$ is lsc for each $i \in I$, then $\bigcup_{i\in I} G_i$ is lsc, where $(\bigcup_{i\in I} G_i)(x) := \bigcup_{i\in I} G_i(x)$.*
(3) *Let $F : X \multimap Y$ be lsc, A be a nonempty closed subset of X, and $f : X \to Y$ be a continuous function with $f(x) \in F(x)$ for every $x \in A$. Then the map G defined on X by*

$$G(x) = \begin{cases} F(x) & \text{if } x \in X\backslash A, \\ \{f(x)\} & \text{if } x \in A \end{cases}$$

is lsc.

We give a sufficient condition for a map to have a continuous selection and the selection extension property;

Theorem 3.2. *Let X be a paracompact space, $(Y;\Gamma)$ be a complete LC-metric space, $Z \subset X$ with $\dim_X Z \leq 0$ and $F : X \multimap Y$ be a map with closed values such that $F(x)$ is Γ-convex for $x \notin Z$. Let F have an lsc selection and the one-point extension property. Then F has a continuous selection and the selection extension property.*

Proof. (1) Define a map $F_0 : X \multimap Y$ by $F_0(x) = \bigcup\{\phi(x) : \phi$ is an lsc selection for $F\}$. Since F has an lsc selection, $F_0(x)$ is nonempty for all $x \in X$. F has closed values, so $\overline{F}_0 \subset F$. By Proposition 3.1, $\overline{F}_0$ is an lsc selection of F. But F_0 is a maximal lsc selection of F, so $F_0 = \overline{F}_0$.

Now we verify that $F_0(x)$ is Γ-convex for all $x \notin Z$. Suppose $F_0(x)$ is not Γ-convex for some $x \notin Z$, i.e., $\Gamma_A \backslash F_0(x) \neq \emptyset$ for some $A \in \langle F_0(x)\rangle$. Then there exists a $y \in \Gamma_A$ such that $y \notin F_0(x)$. Since $A \in \langle F_0(x)\rangle \subset \langle F(x)\rangle$ and $F(x)$ is Γ-convex, $y \in F(x)$. Thus $(x,y) \in \mathrm{Gr}F \backslash \mathrm{Gr}F_0$. Since F has the one-point extension property, there is an lsc map $F_0^* : X \multimap Y$ such that $(x,y) \in \mathrm{Gr}F_0^*$, and $F_0(z) \subset F_0^*(z) \subset F(z)$ for all $z \in X$. This contradicts the maximality of F_0.

Therefore by Theorem 1.3, F_0 has a continuous selection $f : X \to Y$ which is a continuous selection of F.

(2) Let A be a closed subset of X and $g : A \to Y$ be a continuous selection for $F|_A$, then $g(x) \in F_0(x)$ for all $x \in A$. Otherwise $(x, g(x)) \notin \mathrm{Gr}(F_0)$ for some $x \in A$. Since F has the one-point extension property, there is an lsc map $F_0^* : X \multimap Y$ such that $(x,y) \in \mathrm{Gr}F_0^*$, and $F_0(z) \subset F_0^*(z) \subset F(z)$ for all $z \in X$. This contradicts the maximality of F_0.

Define $F_g : X \multimap Y$ by

$$F_g(x) = \begin{cases} F_0(x) & \text{if } x \in X \backslash A, \\ \{g(x)\} & \text{if } x \in A. \end{cases}$$

Then by Proposition 3.1, F_g is lsc with closed values and $F_g(x)$ is Γ-convex for $x \notin Z$. Therefore by Theorem 1.3, F_g has a continuous selection $f : X \to Y$ which becomes a continuous selection of F and extends $g : A \to Y$ to X. □

Remarks. 1. If F has a continuous selection f, then the map $F_0 : X \multimap Y$ defined above is nonempty for all $x \in X$ and a maximal lsc selection of F. Therefore the following statements are equivalent;

(a) F has a continuous selection;
(b) F has a maximal lsc selection;
(c) F has an lsc selection.

2. Let Y be a Banach space, $Z = \emptyset$ and F_0 be the lsc selection of F. Then coF_0 is also an lsc selection of F. Since F_0 is a maximal lsc selection of F, F_0 =coF_0. So the conclusion of Theorem 3.2 holds without assuming the one-point extension property in this case.

Proposition 3.3. (Gutev [6]) *Let X be a topological space, (Y, d) be a complete metric space and $F : X \multimap Y$ be a wlsc map with closed values. Then F has an lsc selection.*

Theorem 3.4. *Let X be a paracompact space, $(Y;\Gamma)$ be a complete LC-metric space and $Z \subset X$ with $\dim_X Z \leq 0$. Let $F : X \multimap Y$ be a wlsc map with closed values such that $F(x)$ is Γ-convex for $x \notin Z$ and F have the one-point extension property. Then F has a continuous selection and the selection extension property.*

Proof. Since $F : X \multimap Y$ a wlsc map with closed values, F admits an lsc selection, so by Theorem 3.2, the conclusion holds. □

Remarks. 1. Note that if $F : X \multimap Y$ is an lsc map with closed values such that $F(x)$ is Γ-convex for $x \notin Z$, then F has the one-point extension property. Indeed, for any lsc map $L : X \multimap Y$ with $L(z) \subset F(z)$ for all $z \in X$ and $(x, a) \in \mathrm{Gr}F \backslash \mathrm{Gr}L$, define $F^* : X \multimap Y$ by

$$F^*(z) = \begin{cases} F(x) & \text{if } z \neq x, \\ \{a\} & \text{if } z = x. \end{cases}$$

Then F^* is an lsc map with closed values such that $F(x)$ is Γ-convex for $x \notin Z$ by Proposition 3.1. By Theorem 1.3, F^* admits a continuous selection $g : X \to Y$. Define the map $L^* : X \multimap Y$ by $L^*(z) :=$

$L(z) \cup \{g(z)\}$ for all $x \in X$. Then L^* is lsc so, we have $(x, a) \in \mathrm{Gr}L^*$ and $L(z) \subset L^*(z) \subset F(z)$ for all $x \in X$. For details, see Chu and Huang [3].

Therefore Theorem 3.4 is a generalization of Theorem 1.3.

2. As we mentioned in Remarks after Theorem 3.2, if Y is a Banach space and $Z = \emptyset$, then Theorem 3.4 becomes Theorem 1.2.

If $Z = \emptyset$, we obtain the following;

Corollary 3.5. *Let X be a paracompact space, $(Y;\Gamma)$ be a complete LC-metric space, and $F : X \multimap Y$ be a wlsc map with closed Γ-convex values. If F has the one-point extension property, then F has a continuous selection and the selection extension property.*

Remark. Corollary 3.5 generalize and extend Theorem 3.3 in Horvath [7].

4. Selections for almost lower semicontinuous maps in R-trees

A metric space (M, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, \gamma_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, \gamma_\alpha)\}$ of closed balls in M for which $d(x_\alpha, x_\beta) \leq \gamma_\alpha + \gamma_\beta$.

Horvath [8] showed that any hyperconvex metric space M is a complete LC-metric space with $\Gamma_A = \bigcap\{B : B \text{ is a closed ball containing } A\}$ for each $A \in \langle M \rangle$. A Γ-convex subset of M is said to be *sub-admissible.*

An *R-tree* is a metric space (M, d) satisfying:

(i) There is a unique geodesic segment (denoted by $[x, y]$) joining each pair of points $x, y \in M$.

(ii) If $x, y, z \in M$, then $[x, y] \cap [x, z]=[x, w]$ for some $w \in M$.

(iii) If $x, y, z \in M$, then $[x, y] \cap [y, z]= \{y\}$, then $[x, y] \cup [y, z]=[x, z]$.

Note that a complete R-tree is hyperconvex [10].

Proposition 4.1. *Let X be a topological space and Y be a metric space. If a multimap $F : X \multimap Y$ is alsc at $x \in X$, then F is qlsc at $x \in X$.*

Proof. For $\epsilon > 0$, there is a neighborhood $U(x)$ of x such that

$$\bigcap_{z \in U(x)} B(F(z), \epsilon/2) \neq \emptyset.$$

Select any $y \in \bigcap_{z \in U(x)} B(F(z), \epsilon/2)$. For each $z \in U(x)$, choose $y_z \in F(z)$ such that $d(y, y_z) < \epsilon/2$. Note that $y_x \in F(x)$ and $d(y_x, y_z) \leq d(y_x, y) + d(y, y_z) < \epsilon$ for each $z \in U(x)$. Hence $y_z \in B(y_x, \epsilon) \cap F(z)$ for all $z \in U(x)$. □

Combining Theorem 1.4 and Proposition 4.1, we have the following result;

Theorem 4.2. *Let X be a paracompact topological space, (M, d) be a complete R-tree and $F : X \multimap M$ be an alsc map with closed sub-admissible values. Then F has a continuous selection.*

A subset X of M is said to be *convex* if X includes every geodesic segment joining any two of its points.

In an $\mathbb{R}$-tree M, every closed ball contains the segment joining any two of its points, so $[x, y] \subset \bigcap\{B : B$ is a closed ball containing $\{x, y\}\}$ for any $x, y \in M$. Therefore a sub-admissible set in an $\mathbb{R}$-tree is convex.

Note that F has sub-admissible but, not bounded values in Theorem 4.2. So Theorem 4.2 is comparable to the following Theorem in [12].

Theorem 4.3. *Let X be a paracompact topological space, (M, d) be a complete $\mathbb{R}$-tree, and $F : X \multimap M$ be an alsc map with closed bounded convex values. Then F has a continuous selection.*

References

[1] C. Bardaro and R. Ceppitelli, *Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities*, J. Math. Anal. Appl. **132** (1988), 484-490.

[2] H. Ben-El-Mechaiekh and M. Oudadess, *Some selection theorems without convexity*, J. Math. Anal. Appl. **195** (1995), 614-618.

[3] L. -J. Chu and C. -H. Huang, *Generalized selection theorems without convexity*, Nonlinear Anal. **73** (2010), 3224-3231.

[4] F. Deutsch and V. Indumathi and K. Schnatz, *Lower semicontinuity, almost lower semicontinuity, and continuous selections for set-valued mappings*, J. approx. theory **53** (1988), 266-294.

[5] V. G. Gutev, *Unified selection and factorization theorems*, Compt. Rend. Acad. Bulgur. Sci. **40** (1987), 13-15.

[6] V. G. Gutev, *Selection theorems under an assumption weaker than lower semicontinuity*, Topol. Appl. **50** (1993), 129-138.

[7] C. D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341-357.

[8] C. D. Horvath, *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse Math. (6) **2** (1993), 253-269.

[9] H. Kim and S. Lee, *Approximate selections of almost lower semicontinuous multimaps in C-spaces*, Nonlinear Anal. **64** (2006), 401-408.

[10] W. A. Kirk, *Hyperconvexity of R-trees*, Fund. Math. **156** (1998), 67-72.

[11] J.T. Markin, *A selection theorem for quasi-lower semicontinuous mappings in hyperconvex spaces*, J. Math. Anal. Appl. **321** (2006), 862-866.

[12] J.T. Markin, *Fixed points, selections and best approximation for multivalued mappings in R-trees*, Nonlinear Anal. **67** (2007), 2712-2716.

[13] E. Michael, *Continuous selections* I, Ann. of Math. **63** (1956), 361-382.

[14] E. Michael and C. Pixley, *A unified theorem on continuous selections*, Pacific J. Math. **87** (1980), 187-188.

[15] K. Przeslawski and L.E. Rybiński, *Michael selection theorem under weak lower semicontinuity assumption*, Proc. Amer. Math. Soc. **109** (1990), 537-543.

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