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PRIMITIVE/SEIFERT KNOTS WHICH ARE NOT TWISTED TORUS KNOT POSITION

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Abstract. The twisted torus knots and the primitive/Seifert knots both lie on a genus 2 Heegaard surface of S^3. In [5], J. Dean used the twisted torus knots to provide an abundance of examples of primitive/Seifert knots. Also he showed that not all twisted torus knots are primitive/Seifert knots. In this paper, we study the other inclusion. In other words, it shows that not all primitive/Seifert knots are twisted torus knot position. In fact, we give infinitely many primitive/Seifert knots that are not twisted torus knot position.

1. Introduction

Two types of knots in S^3, the twisted torus knots and the primitive/Seifert knots, both lie on a genus 2 Heegaard surface of S^3. In [5], J. Dean defined primitive/Seifert knots, and to find primitive/Seifert knots he used the twisted torus knots. Furthermore he gave the criteria for twisted torus knots to be primitive/Seifert knots. Also he showed that not all twisted torus knots are primitive/Seifert knots. On the other hand, in [3] Berge and the author give the complete list of hyperbolic primitive/Seifert knots in S^3. Thus one natural question is whether or not all hyperbolic primitive/Seifert knots belong to the twisted torus knots. This paper gives an implication for a negative answer. In other words, we provide infinitely many primitive/Seifert knots which are not the twisted torus knot position.

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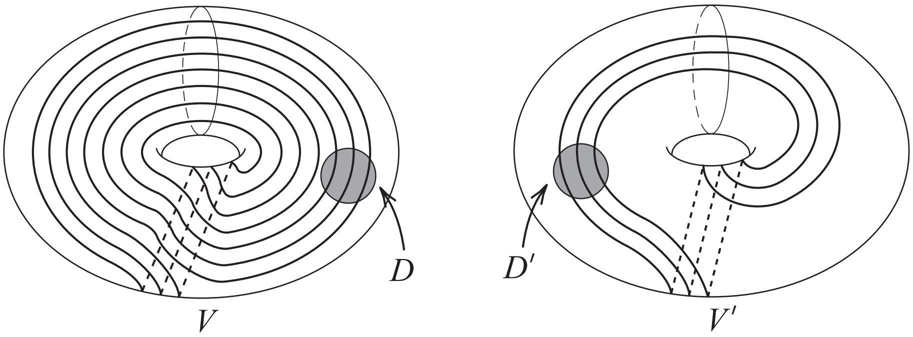

Figure 1. The $(7,3)$ torus knot $T(7,3)$ and 3 parallel copies $3T(1,1)$ of the $(1,1)$ torus knot.

Since the precise definitions of the twisted torus knots and the primitive/Seifert knots can both be found in [5], we give brief explanation on how to construct them.

First regarding the twisted torus knots, let $T(p,q)$ be the (p,q)-torus knot which lies in the boundary of the standardly embedded solid torus V in S^3. Let $rT(m,n)$ be the r parallel copies of $T(m,n)$ which lies in the boundary of another standardly embedded solid torus V' in S^3. Let D be the disk in ∂V so that $T(p,q)$ intersects D in r disjoint parallel arcs, where $0 \le r \le p+q$, and D' the disk in $\partial V'$ so that $rT(m,n)$ intersects D' in r disjoint parallel arcs, one for each component of $rT(m,n)$. See Figure 1. We excise the disks D and D' from their respective tori and glue the punctured tori together along their boundaries so that the orientations of $T(p,q)$ and $rT(m,n)$ align correctly. The resulting one must result in a knot and is called a twisted torus knot, which is denoted by $K(p,q,r,m,n)$. Figure 2 shows $K(7,3,3,1,1)$. It is obvious from the construction that the twisted torus knots lie on a standard genus 2 Heegaard surface of S^3. Let K be a simple closed curve in a genus 2 Heegaard surface Σ of S^3. Then we say that K is *a twisted torus knot position* if there exists a homeomorphism of S^3 sending (Σ, K) to (F, K'), where K' is a twisted torus knot lying in a standard genus 2 Heegaard surface F of S^3.

Now we describe primitive/Seifert knots. If H is a genus two handlebody and c is an essential simple closed curve in ∂H, $H[c]$ will denote the 3-manifold obtained by adding a 2-handle to H along c. The curve c in ∂H is *primitive* in H if $H[c]$ is a solid torus. We say c is *Seifert* in H if $H[c]$ is a Seifert-fibered space and not a solid torus. Note that

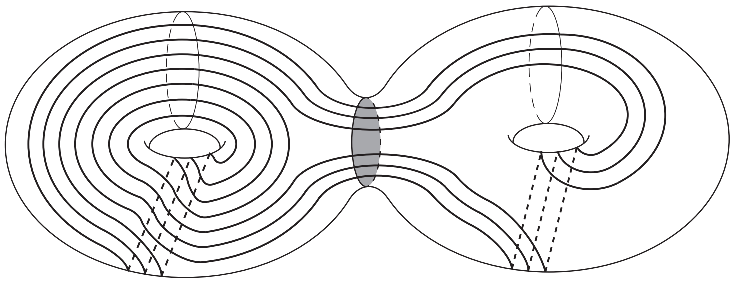

FIGURE 2. The twisted torus knot $K(7, 3, 3, 1, 1)$.

since H is a genus two handlebody, that c is Seifert in H implies that $H[c]$ is an orientable Seifert-fibered space over D^2 with two exceptional fibers, or an orientable Seifert-fibered space over the Möbius band with at most one exceptional fiber.

Suppose K is a simple closed curve in a genus two Heegaard surface Σ of S^3 bounding handlebodies H and H'. K in Σ is *primitive/Seifert* if it is primitive with respect to one of H or H', say H', and Seifert with respect to H.

2. Backgrounds on R-R diagrams

R-R diagrams are a type of planar diagram related to Heegaard diagrams. These diagrams were originally introduced by Osborne and Stevens in [6]. They enable us to analyze curves which lie very complicatedly on the boundary of a genus two handlebody. They are particularly useful for describing embeddings of simple closed curves in the boundary of a handlebody so that the embedded curves represent certain conjugacy classes in π_1 of the handlebody.

The basics of Heegaard diagrams and R-R diagrams of simple closed curves in the boundary of a genus two handlebody are well explained in the paper [1] of Berge. Here, we describe briefly terminologies related to R-R diagrams and show how to transform simple closed curves in the boundary of a genus two handlebody into R-R diagrams.

We start with a genus two handlebody H with a complete set of cutting disks $\{D_A, D_B\}$. Suppose $\mathcal{C}$ is a set of pairwise disjoint simple closed curves in the boundary Σ of H. Figure 3 shows two simple closed curves in the boundary of a genus two handlebody. Consider two parallel

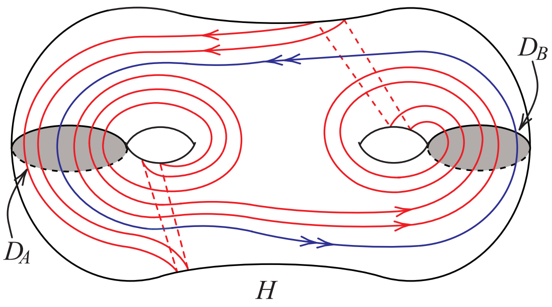

FIGURE 3. Two simple closed curves in the boundary of a genus two handlebody.

simple closed curves which separate the two cutting disks D_A and D_B. These two curves decompose Σ into two once-punctured tori F_A and F_B, and one annulus $\mathcal{A}$. These two curves were originally introduced by Zieschang [7] as *belt curves*, and the two once-punctured tori F_A and F_B are called *handles*. See Figures 4 and 5 for this decomposition.

With this decomposition of Σ set, we analyze each curve in $\mathcal{C}$ in each component of the decomposition as follows. We may assume after isotopy each curve $c \in \mathcal{C}$ is either disjoint from $\partial F_A \cup \partial F_B$, or c is cut by its intersections with $\partial F_A \cup \partial F_B$ into arcs, each properly embedded and essential in one of $\mathcal{A}, F_A, F_B$. A properly embedded essential arc in F_A or F_B is called a *connection*. Two connections in F_A or F_B are *parallel* if they are isotopic in F_A or F_B via an isotopy keeping their endpoints in ∂F_A or ∂F_B. A collection of pairwise disjoint connections on a given handle can be partitioned into *bands* of pairwise parallel connections. Note that since each handle is a once-punctured torus, there can be at most three nonparallel bands of connections on a given handle. Figure 5 shows that there are three nonparallel bands of connections in F_A and there are two in F_B.

In each handle, we merge parallel connections in one band into a single connection. This also merges the endpoints of connections on the boundary of each handle. With these endpoints merged in each handle, now we merge properly embedded essential arcs in $\mathcal{A}$ such that if n parallel arcs in $\mathcal{A}$ have the same endpoints in each handle after merging connections, then we merge these arcs into one edge and label this edge by n indicating n parallel arcs and call this edge with label n as *a band of width n*.

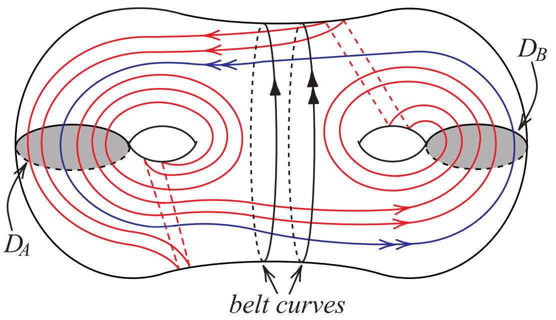

FIGURE 4. Belt curves bounding an annulus in a genus two surface.

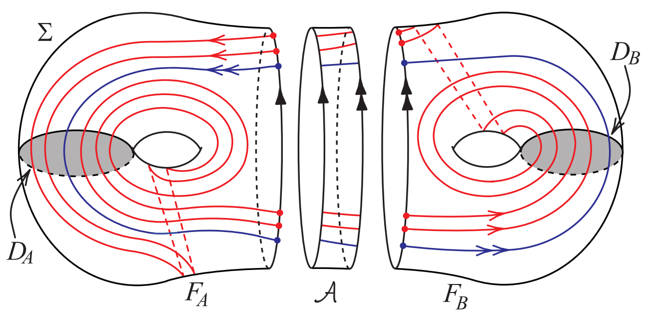

FIGURE 5. An annulus $\mathcal{A}$ and two handles F_A and F_B.

With all of the above information provided, we are ready to transform simple closed curves in the boundary of a genus two handlebody into R-R diagrams, which are type of a planar graph in S^2. First, we embed the annulus $\mathcal{A}$ in S^2 by deleting two disks from S^2 as shown in Figure 6. Next we immerse arc components of curves lying in the two handles F_A and F_B into S^2 as follows. Since each boundary of $\mathcal{A}$ is also the boundary of the handles F_A and F_B, and there are at most three connections after merger, put the six endpoints of the three connections in each boundary circle of $\mathcal{A}$ and connect two endpoints of a connection by a diameter. Figure 6 shows this transformation. We put the capital letters **A** and **B** to indicate correspondence to the two handles F_A and F_B respectively and we call them as $A-$handle and $B-$handle respectively.

We encode the endpoints of each band of connections by integers as follows. Orient the boundaries of the cutting disks D_A and D_B and each simple closed curve in $\mathcal{C}$. The orientation of a simple closed curve gives

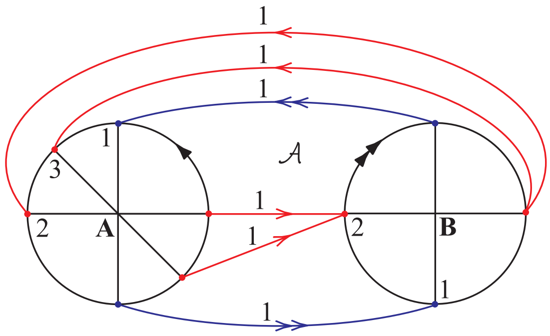

FIGURE 6. Immersion of curves of $\mathcal{C}$ into S^2 which becomes a corresponding R-R diagram.

the orientation of connections in a handle distinguishing its endpoints as an initial point and a terminal point. If a connection intersects a cutting disk s times positively, we label an initial point by s and a terminal point by $-s$, and we say the band of connections with endpoints labeled by s and $-s$ as s-connection or $(-s)$-connection. Note that the labels of endpoints satisfy the following conditions:

(1) If s and t are consecutive labels of endpoints of two connections, then $\gcd(s, t) = 1$.
(2) If s, t, and u are consecutive labels of endpoints of three connections, then $t = s + u$.

By disregarding the boundary circles of F_A and F_B in Figure 6, we finally obtain the corresponding R-R diagram. Figure 7 shows the transformation of two curves c_1 and c_2 in the boundary of a genus two handlebody into R-R diagram.

As mentioned at the beginning in this section, R-R diagram gives sufficient information about conjugacy classes of the element represented by c in $\pi_1(H)$. $\pi_1(H)$ is a free group $F(A, B)$ which is generated by A and B dual to the cutting disks D_A and D_B respectively. In Figure 7, c_1 and c_2 represent the conjugacy classes of AB and $A^3B^2A^2B^2$ respectively in $\pi_1(H)$.

3. R-R diagrams of twisted torus knots and their properties

In this section, we describe the R-R diagram of twisted torus knots. From the construction or Figures 1 and 2, a twisted torus knot lies on

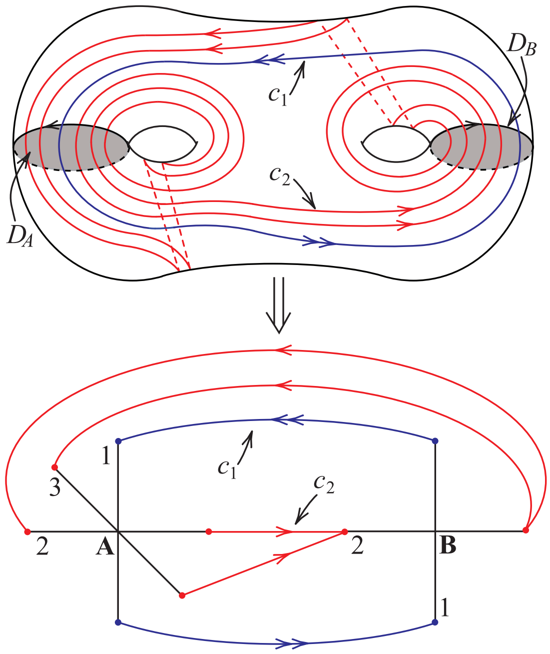

FIGURE 7. Transformation into R-R diagram.

the boundary of a standardly embedded genus two handlebody H in S^3, which is obtained from two solid tori by gluing the disks D and D'. Let H' be the closure of $S^3 - H$, which is also a genus two handlebody, and Σ the common boundary of H and H'. Then $(\Sigma; H, H')$ is a standard genus two Heegaard splitting of S^3.

Let Γ be a separating simple closed curve in Σ which bounds in both H and H' (in the construction of twisted torus knots, Γ can be chosen as the boundary of the disk D or D'). Let $\{D_A, D_B\}$ and $\{D_X, D_Y\}$ be complete sets of cutting disks of H and H' respectively, disjoint from Γ, with ∂D_A and ∂D_X on one side of Γ, and ∂D_B and ∂D_Y on the other side of Γ. See Figure 8. With this setup given, we can make R-R diagrams of twisted torus knots with respect to both H and H'. First, we regard Γ as the belt curve separating the boundary of H (H', respectively) into the two handles $A-$handle and $B-$handle ($X-$handle and $Y-$handle, respectively). According to the construction of twisted torus knots, since the intersection of a twisted torus knot $K(p, q, r, m, n)$ and the $B-$handle in H consists of r parallel arcs, there is only one band of connections on the $B-$handle. Similarly, there is only one band of

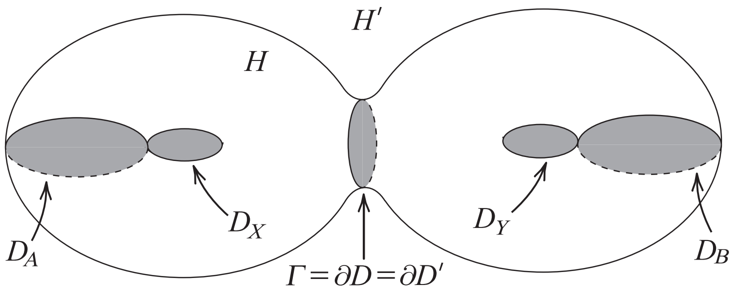

FIGURE 8. The $A-$ and $B-$handles in H and the $X-$ and $Y-$handles in H'.

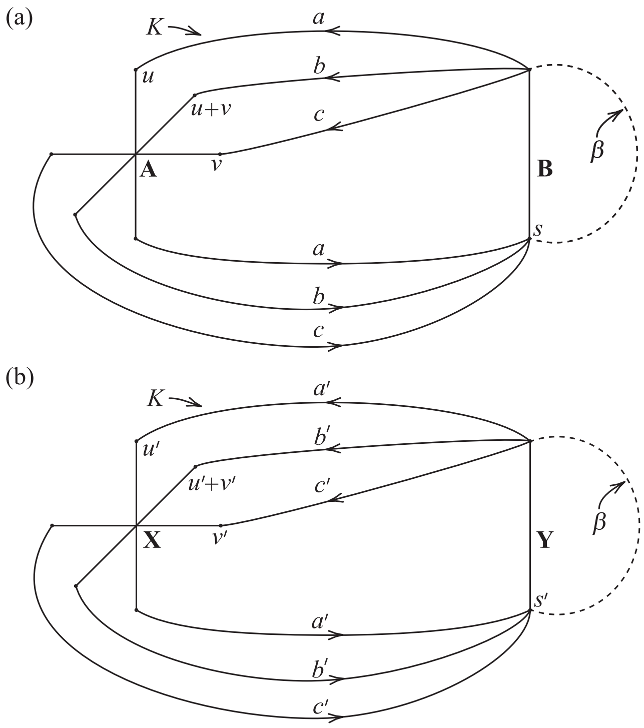

FIGURE 9. R-R diagrams of a twisted torus knot $K(p, q, r, m, n)$ with respect to H(in (a)) and H'(in (b)).

connections on the $Y-$handle. Therefore we have the R-R diagram of twisted torus knots as shown in Figure 9.

Proposition 3.1. *If K is a twisted torus knot which lies on a standard genus two Heegaard surface Σ of S^3 bounding two handlebodies H and H', then there exists a simple closed curve β on Σ which either bounds a disk, is primitive, or is a proper power in each handlebody.*

Proof. In Figure 9(a), the dotted curve β bounds a disk if $s = 0$, is primitive if $s = 1$, and is a proper power if $s > 1$ in H. Similarly in Figure 9(b), the curve β bounds a disk, is primitive, or is a proper power in H'. □

That β is a *proper power* curve in H means that β is disjoint from a separating disk in H, does not bound a disk in H, and is not primitive in H. Equivalently, $[\beta]$ is conjugate to $w^n,\ n \geq 2$, of $\pi_1(H)$, where w is a free generator of $\pi_1(H)$.

Remark 3.2. If the curve β in Proposition 3.1 bounds a disk in one of the handlebody, then the twisted torus knot K is a torus knot.

The existence of the curve β in Proposition 3.1 plays a crucial role in determining if a hyperbolic primitive/Seifert knot is not a twisted torus knot. In other words, due to Berge [2] the existence of the curve β enables us to make the following procedure for the determination;

(1) Find all candidates for β which is primitive or a proper power in H.
(2) Check if β is also primitive or a proper power in H'. If not, it is not a twisted torus knot.
(3) Locate the unique cutting disks D_{A^*} and D_{X^*} disjoint from β of H and H' respectively.
(4) Check if ∂D_{A^*} and ∂D_{X^*} intersect exactly once. If not, it is not a twisted torus knot.

4. Primitive/Seifert knots which are not twisted torus knot position

In this section, we give an infinite family of primitive/Seifert knots which are not a twisted torus knot position. According to the procedure, the first two steps are to find all candidates for β which is primitive or a proper power in both H and H'. The following theorem due to Cohen, Metzler, and Zimmerman provides the necessary condition for a simple closed curve being a primitive or a proper power curve in a genus two handlebody once one knows a cyclically reduced word of the curve in a free group of rank two $F(A, B)$, which is $\pi_1(H)$.

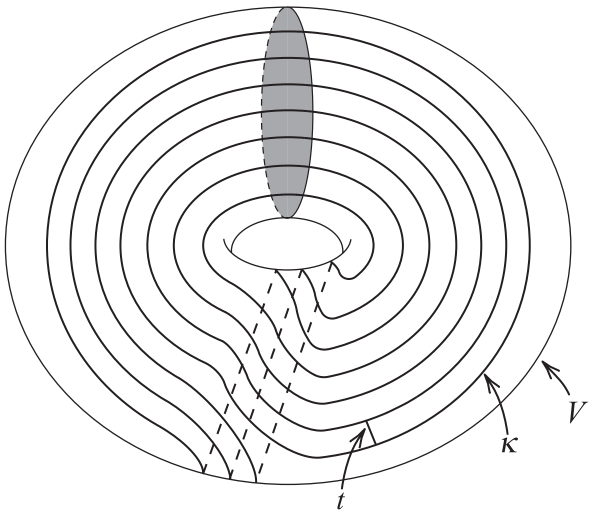

FIGURE 10. The $(7, 3)$-torus knot and one unknotting tunnel t which lie in the boundary of a standardly embedded solid torus V in S^3.

Theorem 4.1. ([4]) *Suppose a cyclic conjugate of*

$$W = A^{n_1} B^{m_1} \cdots A^{n_p} B^{m_p}$$

is a member of a basis of $F(A, B)$ *or a proper power of a member of a basis of* $F(A, B)$*, where* $p \geq 1$ *and each indicated exponent is nonzero. Then, after perhaps replacing* A *by* A^{-1} *or* B *by* B^{-1}*, there exists* $e > 0$ *such that:*

$$n_1 = \cdots = n_p = 1, \quad \textit{and} \quad \{m_1, \ldots, m_p\} \subseteq \{e, e+1\},$$

or

$$\{n_1, \ldots, n_p\} \subseteq \{e, e+1\}, \quad \textit{and} \quad m_1 = \cdots = m_p = 1.$$

We construct an infinite family of primtivie/Seifert knots as follows. Let κ be a (h, k)-torus knot in the boundary of a standardly embedded torus V in S^3 where $h, k > 1$, and t be an unknotting tunnel of κ as shown in Figure 10. Let $W = \overline{S^3 - V}$ and $T = V \cap W$. Also we let $H' = \overline{N(\kappa \cup t)}$, $H = \overline{S^3 - H'}$ and $\Sigma = H \cap H'$. Then the triple of $(\Sigma; H, H')$ is a genus 2 Heegaard splitting of S^3. With this Heegaard splitting given, consider the curves α_0, τ_1, and τ_2 in $\partial H' (= \Sigma)$ as shown in Figure 11. Here H' has been cut open along two disks to yield a 3-cell in which the disk D_M of H' is dual to the unknotting tunnel t of κ. Let

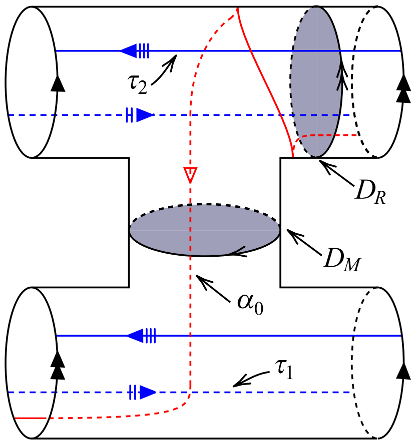

FIGURE 11. An infinite family of primitve/Seifert knots K in the genus 2 Heegaard splitting (Σ, H, H') of S^3, where $K = \alpha_0\mathbb{T}(\tau_1^{J_1}, \tau_2^{J_2})$ with $J_1, J_2 > 1$ and $J_1 \neq J_2$ and $J_1 \neq J_2 + 1$. Here the handlebody H' is the closure of a regular neighborhood of an (h, k) torus knot κ with $h, k > 1$ and an unknotting tunnel t of κ embedded in the boundary T of a standardly embedded solid torus V in S^3 as in Figure 10, and $H = \overline{S^3 - H'}$.

D_R be the meridian of κ, which appears as in Figure 11. Note that D_M and D_R form a complete set of cutting disks of H'. Let M and R be the boundaries of D_M and D_R respectively with the orientation given as in Figure 11. Finally we let $K = \alpha_0\mathbb{T}(\tau_1^{J_1}, \tau_2^{J_2})$ with $J_1, J_2 > 1$, $J_1 \neq J_2$ and $J_1 \neq J_2 + 1$, i.e. K is the curve obtained from α_0 twisted about τ_1 and τ_2 J_1 and J_2 times respectively.

In the following theorem, we will show that the curve K is primitive in H' and Seifert in H so that K is a primitive/Seifert knot in S^3.

Theorem 4.2. *The knot K described in Figure 11 is a hyperbolic primitive/Seifert knot in S^3.*

Proof. It follows from Figure 11 that since K intersects D_M in a single point, K is primitive in H'. In order to show that K is Seifert in H, we use R-R diagram of K with respect to H.

To find R-R diagram of K, we need to determine a complete set of cutting disks $\{D_A, D_B\}$ of H for the R-R diagram. Let $A = \overline{T - N(\kappa)}$, $D' = \overline{A \cap N(t)}$ and $D = \overline{A - D'}$. Note that ∂A consists of two copies of an

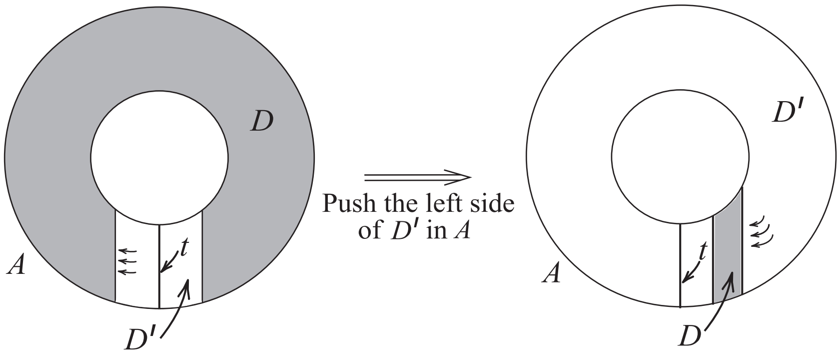

FIGURE 12. Isotopy on A.

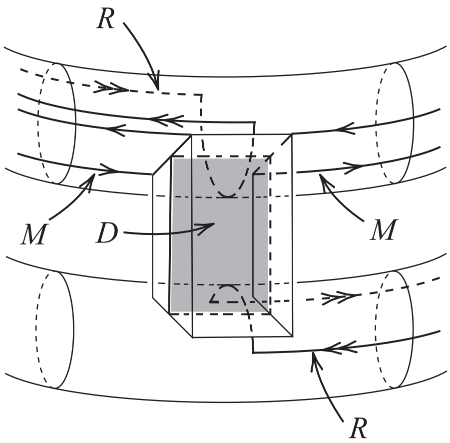

FIGURE 13. Local picture near D.

hk-curve on $\partial N(\kappa)$, and $t \subset D'$. To find a complete set of cutting disks of H, we perform an isotopy on A which switches the positions of D' and D. See Figure 12 for the isotopy performed. Figure 13 shows M and R after the isotopy. Here M is pushed onto $\partial N(\kappa)$.

We consider $V \cup_T W$ as $V' \cup T \times [-\frac{1}{2}, \frac{1}{2}] \cup W'$, where V' and W' are solid tori inside V and W respectively such that they have the same cores. Then it follows that $H \cong V' \cup D \times [-\frac{1}{2}, \frac{1}{2}] \cup W'$, where $D = D \times \{0\}$. Let $D_{V'}$ and $D_{W'}$ be meridians of V' and W', which are induced from the meridians D_V and D_W of V and W respectively. See Figure 14.

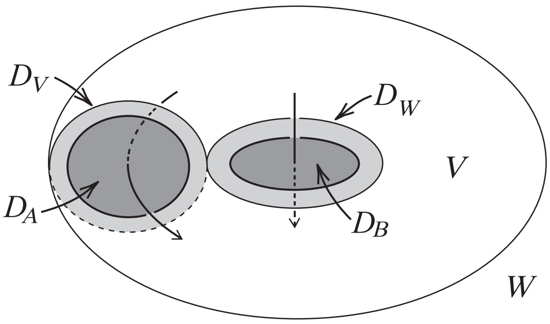

FIGURE 14. Meridians D_A and D_B in V' and W' respectively.

$D_{V'}$ and $D_{W'}$ form a complete set of cutting disks of H and thus we take $(D_{V'}, D_{W'})$ as (D_A, D_B) of H. Now we can consider R-R diagram where A and B–handles are associated with the two once-punctured tori which are created by cutting ∂H along the separating curve ∂D and the labels on connections on both handles are induced from ∂D_A and ∂D_B with the orientation given as in Figure 14.

Now it is easy to find corresponding R-R diagrams of the curves α_0, M, R, τ_1, and τ_2. First for M, since M intersects D_A h times and then D_B k times, M has exactly the same R-R diagram as in Figure 15. Similarly, R intersects D_A h' times and then D_B k' times for some positive integers h' and k' with $0 < h' < h$ and $0 < k' < k$, which satisfy the equation $hk' - h'k = 1$. Thus, R has R-R diagram as in Figure 15. For the curves τ_1 and τ_2, since they do not intersect the separating disk D of H, and intersect the cutting disks D_A and D_B h and k times respectively, their R-R diagrams appear as in Figure 15. By figuring out the intersections of α_0 with the other curves M, R, τ_1, and τ_2 we can obtain R-R diagram of α_0 as in Figure 15.

Let $\pi_1(H) = \langle A, B \rangle$ and $\pi_1(H') = \langle X, Y \rangle$, where A and B are dual to the cutting disks D_A and D_B respectively, and X and Y are dual to the cutting disks D_M and D_R respectively. From the R-R diagram in Figure 15 one has, in $\pi_1(H)$ and $\pi_1(H')$ respectively:

$$K = A^{(2J_1-1)h+2h'} B^{J_2k+k'},$$
$$K = XY^{-2J_1+J_2+1},$$

which shows that K is Seifert in H by Lemma 2.2 in [5] and primitive in H'. Thus K represents primitive/Seifert knots in S^3. Furthermore, by [3] these knots are hyperbolic unless $J_1 = J_2$ or $J_1 = J_2 + 1$. □

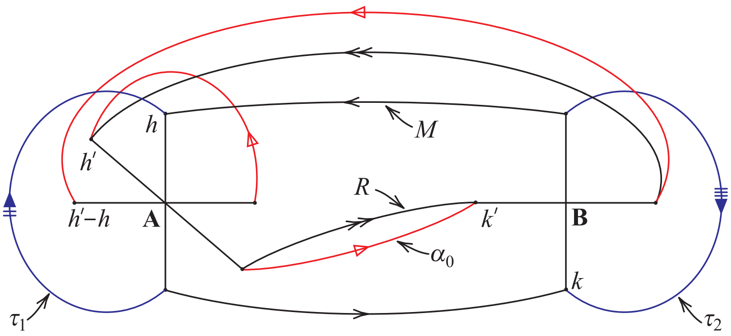

FIGURE 15. The R-R diagrams of the curves α_0, M, R, τ_1, and τ_2.

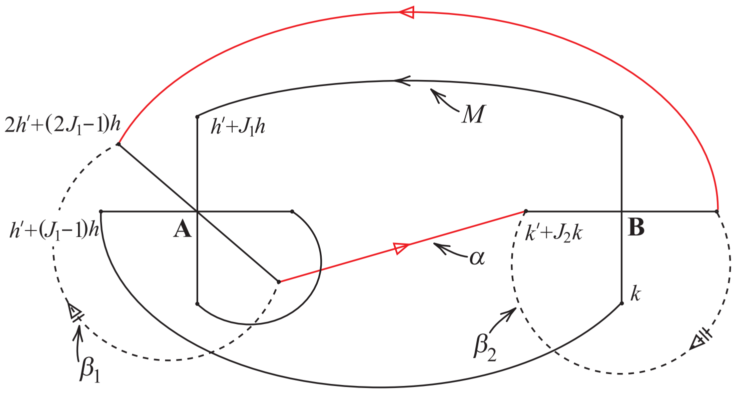

FIGURE 16. An alternative diagram of α and M obtained by performing the twists about τ_1 and τ_2 in Figure 15.

Theorem 4.3. *The knot K described in Figure 11 are not a twisted torus knot position.*

Proof. After applying for the twists about the curves τ_1 and τ_2 J_1 and J_2 times respectively, we obtain an alternative R-R diagram with the form of Figure 16, where the $A-$handle contains the bands of connections labeled by $h' + J_1h, 2h' + (2J_1 - 1)h, h' + (J_1 - 1)h$, and the $B-$handle contains the bands of connections labeled by $k' + J_2k, k$.

Now we apply for the procedure given at the last part in Section 3. First, we try to find a primitive or proper power curve β disjoint from K in H. If β has connections on both $A-$ and $B-$hanldes, then Theorem 4.1 implies that one of the labels of connections of β must be 1.

However, we will show that none of the labels of the possible band of connections of β is 1. In the $A-$handle, $h' + J_1 h, 2h' + (2J_1 - 1)h$, and $h' + J_1 h + n(2h' + (2J_1 - 1)h)$, where $n \in \mathbb{Z}$, are possible labels, and in the $B-$handle $k, k' + J_2 k$ and $k + m(k' + J_2 k)$, where $m \in \mathbb{Z}$, are possible labels for β. Since $0 < h' < h$, $0 < k' < k$, and $J_1, J_2 > 1$, none of these labels can be equal to 1.

Since there are no bands of connections labeled by 1 on both handles, only candidates of a primitive or proper power β disjoint from K are the two regular fiber curves β_1 and β_2 as shown in Figure 16. Algebraically, in $\pi_1(H)$ β_1 and β_2 are $A^{2h'+(2J_1-1)h}$ and $B^{k'+J_2 k}$ respectively.

Next, we need to check if β_1 and β_2 are primitive or a proper power in H'. However it follows from Figures 15 and 16 that $\beta_1 = Y^{-J_1} X Y^{1-J_1} X$ and $\beta_2 = X^{-1} Y^{J_2}$. Thus they are primitive in H'.

According to the procedure, next test is to locate the unique cutting disks of H and H' disjoint from β_i, $i = 1, 2$, and check if the boundaries of the two unique cutting disks intersect exactly in a single point.

For the curve β_1, it follows immediately from the R-R diagram that the cutting disk D_B in H is disjoint from β_1 and thus is the unique cutting disk in H disjoint from β_1. In order to find the unique cutting disk in H', since $\beta_1 = Y^{-J_1} X Y^{1-J_1} X$, we perform change of cutting disks twice, the first of which induces the automorphism $X \mapsto Y^{J_1} X$ and the second $Y \mapsto Y X^{-2}$. Then after performing, $\beta_1 = Y$. So there are a set of cutting disks of H', one of which is dual to $\beta_1 = Y$ and the other, say D_{X^*}, is the unique cutting disk in H' disjoint form β_1.

To see how many times ∂D_B intersects ∂D_{X^*}, we need to figure out to which element in $H_1(H') = \langle Y, X^* \rangle$ ∂D_B is carried under the changes of cutting disks performed above. The following explains this;

$$\begin{aligned}\partial D_B &= kX + k'Y \\ &\xrightarrow{X \mapsto J_1 Y + X} k(J_1 Y + X) + k'Y = (kJ_1 + k')Y + kX \\ &\xrightarrow{Y \mapsto -2X + Y} (kJ_1 + k')(-2X^* + Y) + kX^* \\ &= -((2J_1 - 1)k + 2k')X^* + (kJ_1 + k')Y.\end{aligned}$$

Therefore ∂D_B intersects ∂D_{X^*} $((2J_1 - 1)k + 2k')$ times, which is greater than 1.

For the curve β_2, the similar argument applies. The cutting disk D_A in H is disjoint from β_2 and thus is the unique cutting disk in H disjoint from β_2. In order to find the unique cutting disk in H', we perform change of cutting disks inducing the automorphism $X^{-1} \mapsto X^{-1} Y^{-J_2}$. Then after performing, $\beta_2 = X^{-1}$ and there are a set of cutting disks of H', one of which is dual to $\beta_2 = X^{-1}$ and the other, say D_{Y^*}, is the unique cutting disk in H' disjoint form β_2.

The following shows that ∂D_A intersects ∂D_{Y^*} (J_2h+h') times, which is greater than 1.

$$\begin{aligned}\partial D_A &= hX + h'Y \\ &\xrightarrow{X\mapsto J_2Y+X} h(J_2Y^* + X) + h'Y^* \\ &= hX + (J_2h + h')Y^*.\end{aligned}$$

This completes the proof. □

From Theorems 4.2 and 4.3, we have the following.

Corollary 4.4. *There exists an infinite family of primitive/Seifert knots which are not a twisted torus knot position.*

Theorem 4.3 shows that the knot K described in Figure 11 are not a twisted torus knot position. However it does not imply that the knot K described in Figure 11 are not a twisted torus knot because they might lie in a different genus 2 Heegaard splitting of S^3 which is a twisted torus knot position. Thus it is worth to work on the following question whose answer is expected be affirmative.

Question: *Are the knot K described in Figure* 11 *not a twisted torus knot?*

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