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CONVERGENCE TO COMMON FIXED POINTS FOR A FINITE FAMILY OF GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to study an implicit iteration process with errors and establish weak and strong convergence theorems to converge to common fixed points for a finite family of generalized asymptotically quasi-nonexpansive mappings in the framework of uniformly convex Banach spaces. Our results extend, improve and generalize some known results from the existing literature.

1. Introduction

Let C be a nonempty subset of a real Banach space E. Let $T\colon C \to C$ be a mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Recall the following concepts.

(1) T is said to be nonexpansive if

$$\|Tx - Ty\| \quad \leq \quad \|x - y\|, \tag{1}$$

for all $x, y \in C$.

(2) T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \quad \leq \quad \|x - p\|, \tag{2}$$

for all $x \in C$ and $p \in F(T)$.

(3) T is said to be asymptotically nonexpansive if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $r_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \quad \leq \quad (1 + r_n)\, \|x - y\|, \tag{3}$$

for all $x, y \in C$ and $n \geq 1$.

(4) T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad L\, \|x - y\|, \tag{4}$$

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for all $x, y \in C$ and $n \geq 1$.

(5) T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $r_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \quad \leq \quad (1 + r_n) \|x - p\|, \tag{5}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

(6) T is said to be asymptotically nonexpansive type [13] if

$$\limsup_{n\to\infty} \left\{ \sup_{x,\, y \in C} \Big(\|T^n x - T^n y\| - \|x - y\| \Big) \right\} \quad \leq \quad 0 \tag{6}$$

(7) T is said to be asymptotically quasi-nonexpansive type [21], if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \left\{ \sup_{x \in C,\ p \in F(T)} \Big(\|T^n x - p\| - \|x - p\| \Big) \right\} \quad \leq \quad 0 \tag{7}$$

(8) T is said to be generalized asymptotically quasi-nonexpansive [7] if there exist sequences $\{r_n\}$, $\{s_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} s_n$ such that

$$\|T^n x - p\| \quad \leq \quad (1 + r_n) \|x - p\| + s_n, \tag{8}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

If $s_n = 0$ for all $n \geq 1$, then T is known as an asymptotically quasi-nonexpansive mapping.

(9) T is said to be semi-compact if any bounded sequence $\{x_n\}$ in C with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some x^* in C.

Remark 1. It is easy to see that if $F(T)$ is nonempty, then nonexpansive, quasi-nonexpansive, asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping, asymptotically nonexpansive type mapping and asymptotically quasi-nonexpansive type mapping all are the special cases of generalized asymptotically quasi-nonexpansive mappings.

Remark 2. Let T be asymptotically nonexpansive type mapping. Put $G_n = \sup_{x,\, y \in C} \Big(\|T^n x - T^n y\| - \|x - y\| \Big) \vee 0, \forall n \geq 1$.

If $F(T) \neq \emptyset$, we obtain that $\|T^n x - p\| \leq \|x - p\| + G_n$ for all $x \in C$ and all $p \in F(T)$. Since $\lim_{n\to\infty} G_n = 0$, therefore T is generalized asymptotically quasi-nonexpansive mapping.

The class of asymptotically nonexpansive mappings which is an important generalization of that of nonexpansive mappings was introduced by Goebel and Kirk [5] who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. In [6], they extended this result to the broader class of uniformly $(L, 1)$- Lipschitzian mappings with $L < \lambda$, where λ is sufficiently near 1.

Iterative techniques for approximating fixed points of nonexpansive mappings and their generalizations (asymptotically nonexpansive mappings, etc.) have been studied by a number of authors (see, e.g., [1, 14, 20, 22, 25, 28, 29] and references cited therein), using the Mann iteration method [15] or the Ishikawa-type iteration method [8].

In [14], Liu established strong convergence of the Ishikawa-type iterative sequence with error member for a uniformly (L, α)-Lipschitzian asymptotically quasi-nonexpansive self-mappings of a nonempty compact convex subset of a uniformly convex Banach space. In [12], Khan and Takahashi studied the problem of approximating common fixed points of two asymptotically nonexpansive mappings and established a strong convergence theorem to converge to common fixed points for two asymptotically nonexpansive mappings on a nonempty compact convex subset of a uniformly convex Banach space.

In 2003, Zhou et al. [33] introduced a new class of generalized asymptotically nonexpansive mapping and gave a necessary and sufficient condition for the modified Ishikawa and Mann iterative sequences to converge to fixed points for the said class of mappings.

Recently, Imnang and Suantai [7] have studied multi-step iteration process for a finite family of generalized asymptotically quasi-nonexpansive mappings and gave a necessary and sufficient condition for the said scheme and mappings to converge to the common fixed points and also they established some strong convergence theorems in the frame work of uniformly convex Banach spaces.

In 2001, Xu and Ori [31] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H. Let C be a nonempty subset of H. Let $T_1, T_2, \dots, T_N$ be self-mappings of C and suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \dots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0, 1)$, $x_0 \in C$:

$$\begin{aligned} x_1 &= t_1 x_0 + (1 - t_1) T_1 x_1, \\ x_2 &= t_2 x_1 + (1 - t_2) T_2 x_2, \\ &\vdots \\ x_N &= t_N x_{N-1} + (1 - t_N) T_N x_N, \\ x_{N+1} &= t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1 \tag{9}$$

where $T_k = T_{k \bmod N}$. (Here the mod N function takes values in $\{1, 2, \dots, N\}$). And they proved the weak convergence of the process (9).

The objective of this paper is to study an implicit iteration process with errors for a finite family of generalized asymptotically quasi-nonexpansive mappings and to establish some strong convergence theorems for the said iteration process to a common fixed point for a finite family of above said mappings in a uniformly convex Banach spaces. The results presented in this paper extend and improve the corresponding results of [3]-[4], [9]-[11], [17]-[23], [25], [30] and many others.

2. Preliminaries

Let C be a closed convex subset of a real Banach space E. Let $T\colon C \to C$ be a mapping.

Definition 1. The modified Mann iteration scheme $\{x_n\}$ is defined by

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1 \end{aligned} \tag{10}$$

where $\{\alpha_n\}$ is a suitable sequence in $[0, 1]$.

Definition 2. The modified Ishikawa iteration scheme $\{x_n\}$ is defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n, \quad n \geq 1 \end{aligned} \tag{11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are some suitable sequences in $[0, 1]$.

Definition 3. Let $\{T_1, T_2, \dots, T_N\}$ be a family of asymptotically quasi-nonexpansive in the intermediate sense self mappings of C into itself. Let $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$ is the set of all fixed points of T_i for each $i \in \{1, 2, \dots, N\}$. Let $\{\alpha_n\}$ a real sequence in $(0, 1)$ and $x_0 \in C$, then the iterative sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1}, \\ &\vdots \\ x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N}, \\ x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1}, \\ &\vdots \end{aligned} \tag{12}$$

is called the modified implicit iteration process for a finite family of asymptotically quasi-nonexpansive in the intermediate sense mappings $\{T_1, T_2, \dots, T_N\}$.

Since each $n \geq 1$ can be written as $n = (k-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Hence the above iteration process can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{k(n)} x_n, \quad \forall n \geq 1. \tag{13}$$

Now, we define an implicit iteration process with errors in the sense of Xu [32] as follows:

Definition 4. Let $\{T_i : i \in I\}$, where $I = \{1, 2, \ldots, N\}$ be a finite family of generalized asymptotically quasi-nonexpansive self-mappings on a convex subset C of a Banach space E. The implicit iterative process with an error term, in the sense of Xu [32], and with an initial value $x_0 \in C$, is defined as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\
x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1^2 x_{N+1} + \gamma_{N+1} u_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 x_{2N} + \gamma_{2N} u_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1^3 x_{2N+1} + \gamma_{2N+1} u_{2N+1}, \\
&\vdots
\end{aligned} \tag{14}$$

where $\{u_n\}$ is a bounded sequence in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$.

The above sequence can be written in compact form as

$$x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n \tag{15}$$

with $n \geq 1$, $n = (k-1)N + i$, $i \in I$ and $T_n = T_i(mod\ N) = T_i$.

In the sequel we need the following concepts and lemmas to prove our main results:

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demi-closed at 0 if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to 0, we have $Tx = 0$.

A Banach space E is said to satisfy Opial's [16] property if for each x in E and each sequence $\{x_n\}$ weakly convergent to x, the following condition holds for $x \neq y$:

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|. \tag{16}$$

It is well known that all Hilbert spaces and $l_p(1 < p < \infty)$ spaces have Opial's [16] property while L_p spaces $(p \neq 2)$ have not.

Recall that a mapping $T\colon C \to C$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [24] if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

A family $\{T_i : i \in I\}$ of self-mappings of C with $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ is said to satisfy the following conditions.

(i) Condition $(\overline{A})$ [2]. If there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\frac{1}{N}\sum_{i=1}^{N} \|x - T_i x\| \geq f(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

(ii) Condition $(\overline{B})$ [2]. If there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\max_{1\leq i\leq N}\{\|x - T_i x\|\} \geq f(d(x, F))$ for all $x \in C$.

(iii) Condition $(\overline{C})$ [2]. If there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - T_l x\| \geq f(d(x, F))$ for all $x \in C$ and for at least one T_l, $l = 1, 2, \ldots, N$.

Note that $(\overline{B})$ and $(\overline{C})$ are equivalent, $(\overline{B})$ reduces to condition (A) when all but one of $T_i's$ are identities, and in addition, it also condition $(\overline{A})$.

It is well known that every continuous and demicompact mapping must satisfy condition (A) (see [24]). Since every completely continuous mapping $T\colon C \to C$ is continuous and demicompact so that it satisfies condition (A). Thus we will use condition $(\overline{C})$ instead of the demicompactness and complete continuity of a family of mappings $\{T_i : i \in I\}$.

Lemma 2.1. (See [28]) *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \qquad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. (See [23]) *Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$, for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.3. (See [27], Lemma 2.7) *Let E be a Banach space which satisfies Opial's property and let $\{x_n\}$ be a sequence in E. Let $u, v \in E$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exists. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converges weakly to u and v, respectively, then $u = v$.*

3. Convergence theorems in Banach spaces

Our first result is the strong convergence theorems of the implicit iteration process with errors (15) for a finite family of generalized asymptotically quasi-nonexpansive mappings in a Banach space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Banach space E. Let $T_i\colon C \to C$ $(i \in I = \{1, 2, \ldots, N\})$ be N generalized asymptotically quasi-nonexpansive mappings with $\{r_{in}\}, \{s_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (15) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Proof. We only prove the sufficiency because the necessity is obvious. For any $p \in F = \cap_{i=1}^{N} F(T_i)$, using (8) and (15), we have

$$\begin{aligned}
\|x_n - p\| &= \left\|\alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n - p\right\| \\
&= \left\|\alpha_n (x_{n-1} - p) + \beta_n (T_i^k x_n - p) + \gamma_n (u_n - p)\right\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n \left\|T_i^k x_n - p\right\| + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n [(1 + r_{ik}) \|x_n - p\| \\
&\quad + s_{ik}] + \gamma_n \|u_n - p\| \\
&= \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n - \gamma_n)[(1 + r_{ik}) \|x_n - p\| \\
&\quad + s_{ik}] + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)[(1 + r_{ik}) \|x_n - p\| \\
&\quad + s_{ik}] + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)(1 + r_{ik}) \|x_n - p\| \\
&\quad + (1 - \alpha_n) s_{ik} + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n + r_{ik}) \|x_n - p\| \\
&\quad + (1 - \alpha_n) s_{ik} + \gamma_n \|u_n - p\| . \qquad (17)
\end{aligned}$$

Since $0 < \delta \leq \alpha_n \leq 1 - \delta < 1$. Thus, we have from (17) that

$$\begin{aligned}
\alpha_n \|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + r_{ik} \|x_n - p\| \\
&\quad + (1 - \alpha_n) s_{ik} + \gamma_n \|u_n - p\| , \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
\|x_n - p\| &\leq \|x_{n-1} - p\| + \frac{r_{ik}}{\alpha_n} \|x_n - p\| \\
&\quad + \Big(\frac{1}{\alpha_n} - 1\Big) s_{ik} + \frac{\gamma_n}{\alpha_n} \|u_n - p\|
\end{aligned}$$

$$\leq \|x_{n-1}-p\| + \frac{r_{ik}}{\delta}\|x_n-p\| + \Big(\frac{1}{\delta}-1\Big)s_{ik} + \frac{\gamma_n}{\delta}\|u_n-p\|. \tag{19}$$

That is,

$$\begin{aligned}\|x_n-p\| &\leq \Big(\frac{\delta}{\delta-r_{ik}}\Big)\|x_{n-1}-p\| + \Big(\frac{1}{\delta}-1\Big)\Big(\frac{\delta}{\delta-r_{ik}}\Big)s_{ik} \\ &\quad + \Big(\frac{\delta}{\delta-r_{ik}}\Big)\frac{M}{\delta}\gamma_n \\ &= \Big(1+\frac{r_{ik}}{\delta-r_{ik}}\Big)\|x_{n-1}-p\| + \Big(\frac{1-\delta}{\delta-r_{ik}}\Big)s_{ik} \\ &\quad + \Big(\frac{M}{\delta-r_{ik}}\Big)\gamma_n \\ &= (1+u_{ik})\|x_{n-1}-p\| + Qs_{ik} + R\gamma_n \end{aligned} \tag{20}$$

for some $Q > 0$ and $R > 0$, where $u_{ik} = \frac{r_{ik}}{\delta-r_{ik}}$, $Q = \Big(\frac{1-\delta}{\delta-r_{ik}}\Big)$, $R = \Big(\frac{M}{\delta-r_{ik}}\Big)$ and $M = \sup_{n\geq 1}\{\|u_n-p\|\}$, since $\{u_n\}$ is a bounded sequence in C. Since $\sum_{k=1}^{\infty} r_{ik} < \infty$ for all $i \in I$, it follows that $\sum_{k=1}^{\infty} u_{ik} < \infty$. This implies that

$$d(x_n, F) \leq (1+u_{ik})d(x_{n-1}, F) + Qs_{ik} + R\gamma_n. \tag{21}$$

Since by assumptions, $\sum_{k=1}^{\infty} u_{ik} < \infty$, $\sum_{k=1}^{\infty} s_{ik} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Therefore, applying Lemma 2.1 to the inequalities (20) and (21), we conclude that both $\lim_{n\to\infty}\|x_n-p\|$ and $\lim_{n\to\infty} d(x_n, F)$ exists. Since by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, so by Lemma 2.1, we have

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{22}$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in C. Note that if $x \geq 0$, then $1 + x \leq e^x$. Thus, from (20), it follows that

$$\begin{aligned}\|x_{n+m}-p\| &\leq exp\Big[\sum_{i=1}^{N}\sum_{k=1}^{\infty} u_{ik}\Big]\|x_n-p\| + Q\sum_{i=1}^{N}\sum_{k=1}^{\infty} s_{ik} + R\sum_{n=1}^{\infty}\gamma_n \\ &< K\|x_n-p\| + Q\sum_{i=1}^{N}\sum_{k=1}^{\infty} s_{ik} + R\sum_{n=1}^{\infty}\gamma_n \end{aligned} \tag{23}$$

where

$$K = exp\Big[\sum_{i=1}^{N}\sum_{k=1}^{\infty} u_{ik}\Big] + 1 < \infty.$$

Let $\varepsilon > 0$. As $\lim_{n\to\infty}\|x_n-p\|$ exists, therefore for $\varepsilon > 0$, there exists a natural number n_1 such that $\|x_n-p\| < \varepsilon/6(1+K)$, $\sum_{i=1}^{N}\sum_{k=n_1}^{\infty} s_{ik} < \varepsilon/3Q$ and $\sum_{n=n_1}^{\infty}\gamma_n < \varepsilon/3R$ for all $n \geq n_1$. So we can find $p^* \in F$ such that

$\|x_{n_1} - p^*\| < \varepsilon/3(1+K)$. Hence, for all $n \geq n_1$ and $m \geq 1$, we have that

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\
&\leq K\|x_{n_1} - p^*\| + Q\sum_{i=1}^{N}\sum_{k=n_1}^{\infty} s_{ik} \\
&\quad + R\sum_{n=n_1}^{\infty}\gamma_n + \|x_{n_1} - p^*\| \\
&= (1+K)\|x_{n_1} - p^*\| + Q\sum_{i=1}^{N}\sum_{k=n_1}^{\infty} s_{ik} \\
&\quad + R\sum_{n=n_1}^{\infty}\gamma_n \\
&< (1+K).\frac{\varepsilon}{3(1+K)} + Q.\frac{\varepsilon}{3Q} + R.\frac{\varepsilon}{3R} = \varepsilon. \qquad (24)
\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in C. Thus, the completeness of E implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = z \in C$. It remains to show that $z \in F$. Notice that

$$|d(z,F) - d(x_n,F)| \leq \|z - x_n\|, \ n \geq 1. \qquad (25)$$

Since $\lim_{n\to\infty} d(x_n,F) = 0$ and $\lim_{n\to\infty}\|z - x_n\| = 0$, we obtain that $z \in F$. Thus, the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$. This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a Banach space E. Let $T_i\colon C \to C$ $(i \in I = \{1,2,\ldots,N\})$ be N generalized asymptotically quasi-nonexpansive mappings with $\{r_{in}\},\{s_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (15) with the restrictions $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.*

Proof. The proof of Theorem 3.2 follows from Lemma 2.1 and Theorem 3.1. This completes the proof. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a Banach space E. Let $T_i\colon C \to C$ $(i \in I = \{1,2,\ldots,N\})$ be N generalized asymptotically quasi-nonexpansive mappings with $\{r_{in}\},\{s_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (15) with the restrictions $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i \in I = \{1,2,\ldots,N\}$ and $\{T_i : i \in I\}$ satisfies condition $(\overline{C})$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$.*

Proof. From $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i \in I = \{1, 2, \ldots, N\}$ and $\{T_i : i \in I\}$ satisfying condition $(\overline{C})$, there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x_n - T_{i_0} x_n\| \geq f(d(x_n, F))$ for some $i_0 \in I = \{1, 2, \ldots, N\}$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. From Theorem 3.1, we obtain that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$. This completes the proof. □

4. Convergence theorems in uniformly convex Banach spaces

In this section, we establish weak and strong convergence theorems of the implicit iteration process with errors (15) for a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings in the setting of uniformly convex Banach spaces.

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T_i\colon C \to C$ $(i \in I = \{1, 2, \ldots, N\})$ be N uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with $\{r_{in}\}, \{s_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (15) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$.*

Proof. It follows from (20) and (21), we have

$$\|x_n - p\| \quad \leq \quad (1 + u_{ik})\,\|x_{n-1} - p\| + Q s_{ik} + R\gamma_n, \tag{26}$$

and

$$d(x_n, F) \quad \leq \quad (1 + u_{ik}) d(x_{n-1}, F) + Q s_{ik} + R\gamma_n. \tag{27}$$

Since by assumptions, $\sum_{k=1}^{\infty} u_{ik} < \infty$, $\sum_{k=1}^{\infty} s_{ik} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Therefore, by Lemma 2.1, we conclude that both $\lim_{n\to\infty}\|x_n - p\|$ and $\lim_{n\to\infty} d(x_n, F)$ exist.

Without loss of generality, we can assume that

$$\lim_{n\to\infty}\|x_n - p\| = a, \tag{28}$$

where $a \geq 0$ is some number. Since $\{\|x_n - p\|\}$ is a convergent sequence and so $\{x_n\}$ is a bounded sequence in C.

Observe that

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n - p\| \\ &= \|\beta_n [T_i^k x_n - p + \gamma_n (u_n - x_{n-1})] \\ &\quad + (1 - \beta_n)[x_{n-1} - p + \gamma_n (u_n - x_{n-1})]\|. \end{aligned} \tag{29}$$

From $\sum\limits_{n=1}^{\infty} \gamma_n < \infty$ and (28), it follows that

$$\limsup_{n\to\infty}\|x_{n-1} - p + \gamma_n (u_n - x_{n-1})\|$$

$$\begin{aligned} &\leq \limsup_{n\to\infty}\Big(\|x_{n-1}-p\| + \gamma_n \|u_n - x_{n-1}\| \Big) \\ &\leq a, \end{aligned} \tag{30}$$

and hence

$$\begin{aligned} &\limsup_{n\to\infty} \left\|T_n^k x_n - p + \gamma_n(u_n - x_{n-1})\right\| \\ \leq\ &\limsup_{n\to\infty}\Big(\left\|T_n^k x_n - p\right\| + \gamma_n \|u_n - x_{n-1}\| \Big) \\ \leq\ &\limsup_{n\to\infty}\Big((1+r_{nk})\|x_n - p\| + s_{nk} + \gamma_n \|u_n - x_{n-1}\| \Big) \\ \leq\ & a, \end{aligned} \tag{31}$$

where $n = (k-1)N + i$, $i = i(n) \in \{1, 2, \ldots, N\}$ and $k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$.
Therefore from (29) - (31) and Lemma 2.2, we have that

$$\lim_{n\to\infty} \left\|T_n^k x_n - x_{n-1}\right\| = 0. \tag{32}$$

Moreover, since

$$\begin{aligned} \|x_n - x_{n-1}\| &= \left\|\alpha_n x_{n-1} + \beta_n T_n^k x_n + \gamma_n u_n - x_{n-1}\right\| \\ &= \left\|\beta_n[T_n^k x_n - x_{n-1}] + \gamma_n[u_n - x_{n-1}]\right\| \\ &\leq \beta_n \left\|T_n^k x_n - x_{n-1}\right\| + \gamma_n \|u_n - x_{n-1}\|, \end{aligned} \tag{33}$$

hence by (32), we obtain

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0, \tag{34}$$

and hence $\|x_n - x_{n+l}\| \to 0$ as $n \to \infty$ and $l < 2N$. Now, for $n > N$, we have

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq \left\|x_{n-1} - T_n^k x_n\right\| + \left\|T_n^k x_n - T_n x_n\right\| \\ &\leq \left\|x_{n-1} - T_n^k x_n\right\| + L\left\|T_n^{k-1} x_n - x_n\right\| \\ &\leq \left\|x_{n-1} - T_n^k x_n\right\| + L\left\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\right\| \\ &\quad + L\Big[\left\|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)-1}\right\| \\ &\quad + \left\|x_{(n-N)-1} - x_n\right\| \Big]. \end{aligned} \tag{35}$$

Since for each $n > N$, $n \equiv (n-N)\ (mod\ N)$. Thus $T_n = T_{n-N}$, therefore

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq \left\|x_{n-1} - T_n^k x_n\right\| + L^2\left\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\right\| \\ &\quad + L\left\|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)-1}\right\| \\ &\quad + L\left\|x_{(n-N)-1} - x_n\right\|. \end{aligned} \tag{36}$$

Using (32) and (34), we obtain

$$\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0. \tag{37}$$

Now,

$$\|x_n - T_n x_n\| \quad \leq \quad \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\| \,. \tag{38}$$

From (34), (37) and (38), we have

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{39}$$

This completes the proof. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E satisfying the Opial's property. Let $T_i\colon C \to C$ $(i \in I = \{1, 2, \ldots, N\})$ be N uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with $\{r_{in}\}, \{s_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (15) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. If $I - T_i$ for all $i \in I = \{1, 2, \ldots, N\}$, is demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_i : i \in I\}$.*

Proof. By Theorem 4.1, we have $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$, for all $i \in I = \{1, 2, \ldots, N\}$. Since E is uniformly convex and $\{x_n\}$ is bounded, without loss of generality we may assume that $x_n \to u$ weakly as $n \to \infty$ for some $u \in C$. Since $I - T_i$ for all $i \in I = \{1, 2, \ldots, N\}$, is demiclosed at 0, we have $u \in F$. Suppose that there are subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ that converges weakly to u and v, respectively. Again, as above, we can prove that $u, v \in F$. By Theorem 3.1, $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.3 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_i : i \in I\}$. This completes the proof. □

Theorem 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T_i\colon C \to C$ $(i \in I = \{1, 2, \ldots, N\})$ be N uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings with $\{r_{in}\}, \{s_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (15) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Assume that the mappings $\{T_i : i \in I\}$ satisfies condition $(\overline{C})$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$.*

Proof. From (20), we have

$$\|x_n - p\| \quad \leq \quad (1 + u_{ik}) \, \|x_{n-1} - p\| + Q s_{ik} + R\gamma_n. \tag{40}$$

This implies that

$$d(x_n, F) \quad \leq \quad (1 + u_{ik}) d(x_{n-1}, F) + Q s_{ik} + R\gamma_n. \tag{41}$$

Since by assumptions, $\sum_{k=1}^{\infty} u_{ik} < \infty$, $\sum_{k=1}^{\infty} s_{ik} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Therefore, by Lemma 2.1, we conclude that $\lim_{n\to\infty} d(x_n, F)$ exists. By Theorem 4.1, we have $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$, for all $i \in I = \{1, 2, \ldots, N\}$.

Since $\{T_i : i \in I\}$ satisfies condition $(\overline{C})$, there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x_n - T_{i_0}x_n\| \geq f(d(x_n, F))$ for some $i_0 \in I = \{1, 2, \ldots, N\}$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 3.1, we can conclude that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$. This completes the proof. □

Remark 3. The family of generalized asymptotically quasi-nonexpansive mappings in Theorem 4.1 and 4.2 can be replaced by a family of asymptotically quasi-nonexpansive mappings. Theorem 4.2 generalizes and improves Theorem 3.2 and 4.2 of [10], Theorem 2.9 of [19], Theorem 3.1 of [9], Theorem 2.1 of [23] and Theorem 1 of [17] to the more general class of a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings and implicit iteration process with errors considered in this paper. Theorem 4.3 generalizes and improves Theorem 3.3 of [10], Theorem 4.3 of [18], Theorem 2 of [4], Theorem 2.4 of [19], Theorem 4.2 of [3], Theorem 3.2 of [9], Theorem 3.4 of [25], Theorem 2 of [11], Theorem 3 of [20], Theorem 2.1-2.3 of [30], Theorem 1.5 of [22], Theorem 2.2 of [23], Theorem 3.3 of [26] and Theorem 2 of [17] by using condition $(\overline{C})$ instead of condition $(\overline{A})$ or semi-compactness or completely continuous or compactness to the more general class of a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mappings and implicit iteration process with errors considered in this paper.

Example 1. Let E be the real line with the usual norm $|.|$ and $K = [0,1]$. Define $T\colon K \to K$ by

$$T(x) = \begin{cases} \frac{x}{2}\, sin\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

for all $x \in K = [0,1]$. Then T is an asymptotically quasi-nonexpansive mapping with the constant sequence $\{k_n\} = \{1\}$ for each $n \geq 1$ and therefore it is asymptotically quasi-nonexpansive type mapping. By Remark 2.2, T is generalized asymptotically quasi-nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$ and $\{s_n\} = \{0\}$ for each $n \geq 1$. But the converse is not true in general.

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