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COMMUTATIVITY OF ASSOCIATION SCHEMES OF ORDER pq

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Abstract. Let (X, S) be an association scheme where X is a finite set and S is a partition of $X \times X$. The size of X is called the *order* of (X, S). We define $\mathcal{C}$ to be the set of positive integers m such that each association scheme of order m is commutative. It is known that each prime is belonged to $\mathcal{C}$ and it is conjectured that each prime square is belonged to $\mathcal{C}$. In this article we give a sufficient condition for a scheme of order pq to be commutative where p and q are primes, and obtain a partial answer for the conjecture in case where $p = q$.

1. Introduction

Let (X, S) be an association scheme (or shortly, scheme) where X is a finite set and S is a partition of $X \times X$ (see Section 2 for definition and [2], [3], [14] for basic concepts). The size of X is called the *order* of (X, S).

Following [14] we shall write the adjacency matrix of $s \in S$ as σ_s, i.e., σ_s is the $\{0, 1\}$-matrix whose rows and columns are indexed by the elements of X and its (x, y)-entry is equal to one if and only if $(x, y) \in s$. Then the subspace spanned by $\{\sigma_s \mid s \in S\}$ is a subalgebra of the full matrix algebra over a field F. We call it the *adjacency algebra* of (X, S) over F, and denote it by FS. We say that (X, S) is *commutative* if $\mathbb{C}S$ is commutative where $\mathbb{C}$ is the complex number field.

In group theory it is well-known that any group of prime or prime square order is abelian (see [13, Thm.A] for the relationship between groups and schemes). On the other hand, any scheme of prime order is commutative (see [9]), and any schurian scheme of prime square order is also commutative as follows (see Section 2 for terminologies):

Theorem 1.1. ([6, Thm.5.9]) *Let p be a prime and (X, S) a scheme of order p^2. Then (X, S) is commutative if one of the following conditions holds:*

(i) *(X, S) is schurian;*

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(ii) *There exists a thin closed subset T of S with $n_T \geq p$.*
(iii) *There exists a strongly-normal closed subset T of S with $n_T \leq p$.*

However, it is still open whether or not any scheme of prime square order is commutative.

Let p and q be primes. It is also known that any group of order pq is abelian if $p < q$ and $p \nmid q - 1$. Though it is natural to expect the similar result for schemes, there exists a non-commutative scheme of order 15 (see [8] and [10]). In this article we deal with schemes of order pq where p and q are primes, and show a sufficient condition for them to be commutative.

The following is our main result (see Section 2, 3 for terminologies):

Theorem 1.2. *Let (X, S) be a scheme of order pq where p, q are primes. Suppose that S has a nice closed subset T which satisfies the following:*
(i) *$n_T = p$;*
(ii) *$S/\!\!/T$ has the same intersection numbers as a schurian scheme;*
(iii) *$1 < \frac{q-1}{|S/\!\!/T|-1} \leq p$.*

Then (X, S) is commutative.

It is still open whether any scheme of prime order has the same intersection numbers as a schurian scheme ([9] and [12]). In other words (ii) as in Theorem 1.2 might be removed. The following theorem deals with the case of $p = q$ in Theorem 1.2:

Theorem 1.3. *Let (X, S) be a scheme of prime square order. If S has a non-trivial nice closed subset, then (X, S) is commutative.*

The following gives a sufficient condition for a non-trivial closed subset to be nice:

Corollary 1.4. *Let (X, S) be a scheme of prime square order and T a non-trivial closed subset of S. Then T is nice and S is commutative if T is flat and $\gcd(n_t, n_{s^T}) = 1$ for some $t \in T^\sharp$ and $s \in S \setminus T$.*

Under the same notation as Corollary 1.4 it can be easily checked that T is flat if $\frac{n_T-1}{|T|-1} \leq 3$ (see [10] for the detail).

In Section 2 we prepare necessary notation to make the paper as self-contained as possible. In Section 3 we prepare several lemmas from combinatorics. In Section 4 we prepare some lemmas to generalize arguments given in [7]. In Section 5 we prove our main results and corollary.

2. Preliminaries

We use the same notation on association schemes as in [14].

Let X be a finite set and S a partition of $X \times X$. For $s \in S$ we denote by s^* the set of $(x, y) \in X \times X$ with $(y, x) \in s$. For $x \in X$ and $s \in S$ we denote by xs the set of $y \in X$ with $(x, y) \in s$. We denote by 1_X the set of (x, x) with $x \in X$.

We say that (X, S) is an *association scheme* (or shortly, *scheme*) if it satisfies the following conditions:

(i) $1_X \in S$;
(ii) For each $s \in S$ we have $s^* \in S$;
(iii) For all $s, t, u \in S$ the size of $xs \cap yt^*$ is constant whenever $(x, y) \in u$. We denote the constant by a_{stu}.

The numbers $(a_{stu} \mid s, t, u \in S)$ are called the *intersection numbers* of (X, S). For $s \in S$ the number $a_{ss^*1_X}$ is called the *valency* of s, and denoted by n_s.

For the remainder of this section we assume that (X, S) is a scheme.

For $x, y \in X$ we denote by $r(x, y)$ the unique element of S containing (x, y).

Recall that we define σ_s to be the adjacency matrix of $s \in S$ in Section 1. For a subset T of S we shall write the sum of σ_t with $t \in T$ as σ_T, the sum of n_t with $t \in T$ as n_T, and the subspace spanned by $\{\sigma_t \mid t \in T\}$ over a field F as FT.

For $T, U \subseteq S$ and $s \in S$ we denote the coefficient of σ_s in $\sigma_T\sigma_U$ by a_{TUs}, and we define the *complex product* of T and U, denoted by TU, to be

$$\{s \in S \mid a_{TUs} > 0\}.$$

The complex product is an associative binary operation on the power set of S (see [13] or [14]). In this article we shall write a singleton $\{t\}$ with $t \in S$ in the complex product as t like the notation for a coset in group theory.

We have the following equations on intersection numbers (see [1], [2], [13] or [14]):

Lemma 2.1. *We have the following:*

(i) *For all $s, t \in S$ we have $a_{s1_Xt} = \delta_{s,t}$;*
(ii) *For all $s, t \in S$ we have $n_s n_t = \sum_{u \in S} a_{stu} n_u$;*
(iii) *For all $s, t, u \in S$ we have $a_{stu} = a_{t^*s^*u^*}$;*
(iv) *For all $s, t, u \in S$ we have $n_u a_{stu} = n_s a_{ut^*s} = n_t a_{s^*ut}$;*
(v) *For all $s, t \in S$ we have $|st| \leq \gcd(n_s, n_t)$.*

For a non-empty subset T of S we say that T is *closed* if $TT^* \subseteq T$ where we denote by T^* the set of t^* with $t \in T$.

Lemma 2.2. ([14, p.17]) *Let T be a non-empty subset of S. Then the following are equivalent:*

(i) *T is closed;*
(ii) *$TT \subseteq T$;*
(iii) *$\bigcup_{t \in T} t$ is an equivalence on X.*

It is easy to check that $\{1_X\}$ and S are closed. They are called *trivial*.

Lemma 2.3. ([14, Lem.2.3.4]) *For each closed subset T of S and $s \in S$ we have the following:*

(i) *$n_T n_s = a_{Tss} n_{Ts}$;*
(ii) *$n_{Ts} n_T = a_{(Ts)Ts} n_{TsT}$.*

(iii) *If* $n_T = a_{Tss}$, *then* $Ts = s$ *and* $n_T \mid n_s$*;*
(iv) *If* $n_T = a_{(Ts)Ts}$, *then* $Ts = sT$*.*

For the remainder of this section we assume that T is a closed subset of S. We shall write $T \setminus \{1_X\}$ as $T^\sharp$.

For $x \in X$ we denote by xT the equivalence class of $\bigcup_{t\in T} t$ containing x. Then it is known (see [13] or [14]) that $(xT, \{t \cap (xT \times xT)\}_{t\in T})$ is a scheme, which is called the *subscheme* of (X, S) *with respect to* x and T, and denoted by $(X, S)_{xT}$.

We call $(a_{stu} \mid s, t, u \in T)$ the *intersection numbers* of T, which coincide with those of $(X, S)_{xT}$ for $x \in X$.

Notice that $\mathbb{C}T$ is not only a subspace but also a subalgebra of the full matrix algebra over $\mathbb{C}$, which is isomorphic to the adjacency algebra of $(X, S)_{xT}$ via $\sigma_t \mapsto \sigma_{t\cap(xT\times xT)}$.

We denote the set of equivalence classes of $\bigcup_{t\in T} t$ by X/T. For $s \in S$ we define s^T to be

$$\{(xT, yT) \mid r(x, y) \in TsT\}.$$

We denote by $S/\!\!/T$ the set of s^T with $s \in S$. Then it is known (see [13] or [14]) that $(X/T, S/\!\!/T)$ is a scheme, called the *factor* scheme of (X, S) over T.

We say that T is *thin* if $n_t = 1$ for each $t \in T$.

We say that T is *commutative* (*symmetric*) if $\mathbb{C}T$ is commutative ($t = t^*$ for each $t \in T$, respectively). Note that any symmetric closed subset is commutative by Lemma 2.1(ii).

Lemma 2.4. ([13]) *We have the following:*

(i) $n_S = n_T n_{S/\!\!/T}$*;*
(ii) $n_{s^T} = n_{TsT}/n_T$*;*
(iii) *For each* $s \in S$ *we have* $\sigma_T\sigma_s = a_{Tss}\sigma_{Ts}$ *and* $\sigma_s\sigma_T = a_{sTs}\sigma_{sT}$*.*

We say that T is *normal* in S if $Ts = sT$ for each $s \in S$, or equivalently σ_T is central in $\mathbb{C}S$ by Lemma 2.4(iii). We say that T is *strongly-normal* if $sTs^* \subseteq T$ for each $s \in S$, equivalently, $S/\!\!/T$ is thin (see [13, Thm. 2.2.3]). For a positive integer k we say that T is *k-equivalenced* if $n_t = k$ for each $t \in T^\sharp$. We say that T is *flat* if it is k-equivalenced for some k and $a_{tt^*s} \le 1$ for all $s, t \in T^\sharp$.

We say that (X, S) is *schurian* if S is the set of orbitals of a transitive permutation group of X, or equivalently, (X, S) is a factor scheme of a thin scheme (see [2]).

Let Π be a partition of X. We say that Π is *equitable* if, for each $s \in S$ and $C, D \in \Pi$, $|xs \cap D|$ is constant whenever $x \in C$. It is easy to check that $\{\{x\} \mid x \in X\}$ and $\{X\}$ are equitable, and they are called *discrete* and *trivial*, respectively.

The following theorems will be used later in this article:

Theorem 2.5. ([9]) *Let* (X, S) *be a scheme of prime order. Then we have the following:*

(i) *All non-principal irreducible characters of* $\mathbb{C}T$ *are algebraic conjugate;*
(ii) S *is* k*-equivalenced where* $k = \frac{n_S - 1}{|S|-1}$*;*
(iii) S *is commutative;*
(iv) *If* $k > 1$*, then* $a_{stu} < k$ *for all* $s, t, u \in T^\sharp$*.*

Theorem 2.6. ([7]) *If* $|X|$ *is a prime square, then any closed subset of* S *is normal in* S*.*

Theorem 2.7. ([11]) *If* S *is flat and* $|X|$ *is a prime, then* $\{X\}$ *is a unique equitable partition without any singleton.*

Lemma 2.8. *For* $x, y \in X$*,* $\Pi_{x,y} := \{xT \cap ys \mid s \in S\}$ *is an equitable partition of* $(X, S)_{xT}$*.*

Proof. Let $s_1, s_2 \in S$ such that $xT \cap ys_i \neq \emptyset$ for $i = 1, 2$ and $t \in T$. It suffices to show that $|zt \cap (xT \cap ys_2)|$ does not depend on the choice of $z \in xT \cap ys_1$. Since $zt \subseteq zT = xT$ and $(z, y) \in s_1^*$, it is equal to $|zt \cap ys_2| = a_{ts_2^* s_1^*}$. □

Lemma 2.9. *Let* (X, S) *be a scheme of prime order. Then, for all non-negative integers* a_s *with* $s \in S$*,* $\sum_{s\in S} a_s\sigma_s$ *is singular if and only if* $a_t = a_u$ *for all* $t, u \in S$*.*

Proof. "If" part is obvious.

Since $\mathbb{C}S$ is semisimple and commutative, $\mathbb{C}S$ has a basis $\{e_\chi \mid \chi \in \mathrm{Irr}(S)\}$ where $\mathrm{Irr}(S)$ is the set of irreducible characters of (X, S) and e_χ is the central primitive idempotent affording χ. Thus,

$$\sum_{s\in S} a_s\sigma_s = \sum_{\chi\in \mathrm{Irr}(S)} b_\chi e_\chi \quad \text{for some } b_\chi \in \mathbb{C}$$

Note that $\{b_\chi \mid \chi \in \mathrm{Irr}(S)\}$ are the eigenvalues of $\sum_{s\in S} a_s\sigma_s$.

Suppose that $\sum_{s\in S} a_s\sigma_s$ is singular. Then

$$\mu(\sum_{s\in S} a_s\sigma_s) = \mu(\sum_{\chi\in \mathrm{Irr}(S)} b_\chi e_\chi) = b_\mu = 0$$

for some $\mu \in \mathrm{Irr}(S)$.

If μ is principal, then

$$0 = \mu(\sum_{s\in S} a_s n_s) = \sum_{s\in S} a_s n_s.$$

This implies that $a_s = 0$ for each $s \in S$ as desired, since a_s are non-negative.

If μ is non-principal, then, by Theorem 2.5(i), $b_\tau = 0$ for all non-principal $\tau \in \mathrm{Irr}(S)$, and, hence, $\sum_{s\in S} a_s\sigma_s$ is a scalar multiple of the all-one matrix as desired. □

3. From combinatorics

Throughout this section we assume that (X, S) is a scheme and T is a closed subset of S.

Lemma 3.1. *For $s \in S$ we have the following:*

(i) $a_{Tss} = 1$ *if and only if* $ss^* \cap T = \{1_X\}$*;*

(ii) *If* $a_{Tss} = 1$*, then* $|T| \le |Ts|$*, and the equality holds if and only if the complex product* ts *is a singleton for each* $t \in T$*.*

Proof. (i) Suppose $a_{Tss} = 1$. Let $t \in ss^* \cap T$. Then we can take $x, y, z \in X$ with $(x, y) \in t$, $(x, z) \in s$ and $(z, y) \in s^*$. This implies that $x, y \in xT \cap zs^*$. Since $a_{Tss} = |xT \cap zs^*|$, it follows from $a_{Tss} = 1$ that $x = y$, and, hence $t = 1_X$. Thus, we have $ss^* \cap T \subseteq \{1_X\}$. Since $1_X \in T$ and $1_X \in ss^*$ by Lemma 2.1(i),(iv), we conclude that $ss^* \cap T = \{1_X\}$.

Suppose $ss^* \cap T = \{1_X\}$ and $(x, y) \in s$. It is clear that $a_{Tss} \ge 1$ since $x \in x1_X \cap ys^*$. Let $z, w \in xT \cap ys^*$. Then $r(z, w) \in T \cap ss^*$. Since $ss^* \cap T = \{1_X\}$, it follows that $r(z, w) = 1_X$, and, hence $z = w$. This implies that $a_{Tss} \le 1$.

(ii) Suppose $a_{Tss} = 1$ and $t, u \in T$. If $ts \cap us \neq \emptyset$, then $t^*u \cap ss^* \neq \emptyset$, and $t = u$ by (i) and Lemma 2.1(i),(iv). This implies that $\{ts \mid t \in T\}$ are disjoint, and

$$|Ts| = |\bigcup_{t\in T} ts| = \sum_{t \in T} |ts| \ge |T|.$$

From this equation it is clear that the equality holds if and only if $|ts| = 1$ for each $t \in T$ □

Lemma 3.2. *For each $s \in S$, if $Ts = sT$, then $n_s = n_{s^T} a_{Tss}$.*

Proof. By Lemma 2.4(iii),

$$n_T n_s = a_{Tss} n_{Ts} = a_{Tss} n_{sT}.$$

Since $n_{TsT} = n_T n_{s^T}$ by Lemma 2.4(ii) and $TsT = Ts$ by the assumption, $n_s = n_{s^T} a_{Tss}$. □

Lemma 3.3. *Suppose that $n_T = p$ is a prime and* $\max\{n_{s^T} \mid s \in S\} < p$*. Then, for each $s \in S$ the following are equivalent:* (i) $p \mid n_s$;(ii) $TsT = \{s\}$; (iii) $Ts = \{s\}$;(iv) $sT = \{s\}$.

Proof. By Lemma 2.4(ii), $n_{TsT} = n_{s^T} n_T$. By the assumption,

$$p \mid n_{TsT} \text{ and } p^2 \nmid n_{TsT}. \tag{1}$$

On the other hand, by Lemma 2.3(i),(ii),

$$n_{TsT} = \frac{n_{Ts} n_T}{a_{(Ts)Ts}} = \frac{n_T n_s n_T}{a_{Tss} a_{(Ts)Ts}}. \tag{2}$$

Suppose that $p \mid n_s$. Then, by (1) and (2), $p^2 \mid a_{Tss} a_{(Ts)Ts}$. Since

$$a_{Tss} \le n_T = p \text{ and } a_{(Ts)Ts} \le n_T = p,$$

it follows that

$$p = a_{Tss} = a_{(Ts)Ts}.$$

By Lemma 2.3(iii),(iv),

$$n_{TsT} = n_{Ts} = n_s.$$

Since $\{s\} \subseteq TsT$, it follows that $TsT = \{s\}$.

Suppose that $TsT = \{s\}$. Then $n_s = n_{TsT} = n_{s^T} n_T = pn_{s^T}$. Thus, (i) and (ii) are equivalent.

Suppose $Ts = \{s\}$. Then, by Lemma 2.3(i),(iii), $p \mid n_s$. Therefore, (iii) implies (i). Similarly, (iv) implies (i).

Clearly, (ii) implies (iii) and (iv). This completes the proof. □

Lemma 3.4. *If $n_T = p$ is a prime and $\max\{n_{s^T} \mid s \in S\} < p$, then T is normal in S and $\{s \in S \mid p \nmid n_s\}$ is closed.*

Proof. Suppose that T is not normal in S. Then $Ts \neq sT$ for some $s \in S$. By Lemma 3.3, $Ts \neq \{s\}$. By Lemma 2.3(ii),(iv), $a_{Tss} < n_T$ and $a_{(Ts)Ts} < n_T$. Since $n_{s^T} = n_{TsT}/n_T$ by Lemma 2.4(ii), it follows from Lemma 2.3(i),(ii) that

$$n_{s^T} = \frac{n_T n_s n_T}{n_T a_{Tss} a_{(Ts)Ts}} = \frac{n_s n_T}{a_{Tss} a_{(Ts)Ts}},$$

which is a positive integer divisible by $n_T = p$, a contradiction. Thus, T is normal in S.

Let $u, v \in \{s \in S \mid p \nmid n_s\}$ and $w \in uv$. It suffices to show that $p \nmid n_w$. Suppose the contrary, i.e., $p \mid n_w$. By Lemma 3.3, $TwT = \{w\}$. By Lemma 2.1(iv), $p \mid \operatorname{lcm}(n_{u^*}, n_w) \mid a_{u^*wv} n_v$.

We claim that

$$a_{u^*wv} = a_{u^*Tu^*} a_{(u^*)^T w^T v^T}.$$

Since $TwT = \{w\}$, $Tw^*T = \{w^*\}$. Let $(x, y) \in v$. Then

$$a_{u^*wv} = |xu^* \cap yw^*| = |xu^* \cap yTw^*T| = \sum_{i=1}^{m} |xu^* \cap y_i T|$$

where yTw^*T is a disjoint union of $\{y_i T \mid i = 1, 2, \ldots, n_{w^T}\}$. Note that $xu^* \cap y_i T \neq \emptyset$ if and only if $xTu^*T \cap y_i T \neq \emptyset$, since T is normal in S. Since $|xu^* \cap y_i T| = a_{u^*Tu^*}$ whenever $xu^* \cap y_i T \neq \emptyset$, it follows that

$$a_{u^*wv} = |\{i \mid xu^* \cap y_i T \neq \emptyset\}| a_{u^*Tu^*} = a_{(u^*)^T w^T v^T} a_{u^*Tu^*}.$$

Thus, the claim holds.

Since $p \nmid n_v$ and $a_{(u^*)^T w^T v^T} \leq n_{w^T} < p$, it follows from the claim that $p \mid a_{u^*Tu^*}$. Since $a_{Tuu} = a_{u^*Tu^*} \leq p$, it follows from Lemma 2.3(iii) that $p \mid n_{u^*} = n_u$, a contradiction. This completes the proof. □

We define $\mathcal{S}_T$ to be the set of $s \in S$ with $a_{Tss} = 1$.

Lemma 3.5. *Suppose that T is normal in S. Then*

$$\{\sigma_t\sigma_s \mid t \in T, s \in \mathcal{S}_T\} = \{\sigma_u \mid u \in S\}$$

if and only if $S = \bigcup\{Ts \mid s \in \mathcal{S}_T\}$ and $|Ts| = |T|$ for each $s \in \mathcal{S}_T$.

Proof. Suppose that the former condition holds. Let $u \in S$. Then $\sigma_u = \sigma_t\sigma_s$ for some $t \in T$ and $s \in \mathcal{S}_T$. This implies that $\{u\} = ts \subseteq Ts$. Thus, $S = \bigcup\{Ts \mid s \in \mathcal{S}_T\}$ holds. Since ts is a singleton for each $t \in T$ and $s \in \mathcal{S}_T$ by the assumption, it follows from Lemma 3.1(ii) that $|T| = |Ts|$.

Suppose that the latter condition holds. Let $u \in S$, $s \in \mathcal{S}_T$ and $t \in T$ with $u \in ts$. By Lemma 3.1(ii), ts is a singleton. Thus, $\{u\} = ts$.

We claim that $a_{tsu} = 1$. Otherwise, we can take $z, w \in xt \cap ys^*$ with $(x, y) \in u$ and $z \neq w$.This implies that $r(z, w) \in t^*t \cap ss^*$. It follows from Lemma 3.1(i) that $r(z, w) = 1_X$, and, hence, $z = w$, a contradiction.

By the claim, we have $\sigma_u = \sigma_t\sigma_s$.

Since u is arbitrarily taken and we can take $s \in \mathcal{S}_T$ and $t \in T$ with $u \in ts$ by the assumption, $\{\sigma_u \mid u \in S\} \subseteq \{\sigma_t\sigma_s \mid t \in T, s \in \mathcal{S}_T\}$ holds. The converse inclusion also holds since ts is a singleton and $a_{tsu} = 1$. □

We say that T is *nice* if T is normal in S and one of the properties as in Lemma 3.5 holds.

Remark that both $\{1_X\}$ and S are nice.

Lemma 3.6. *If T is nice and $s \in \mathcal{S}_T$, then we have the following:*

(i) *$s^* \in \mathcal{S}_T$;*
(ii) *For each $t \in T$ there exists a unique $t^s \in T$ such that $ts = st^s$;*
(iii) *For each $t \in T$ we have $\sigma_t(\sigma_s\sigma_{s^*}) = (\sigma_s\sigma_{s^*})\sigma_t$.*

Proof. (i) Since $n_s = n_{s^*}$ and $n_{s^T} = n_{(s^T)^*}$, (i) follows from Lemma 3.2.

(ii) Since T is normal and $a_{Tss} = a_{sTs} = 1$, the properties as in Lemma 3.5 are equivalent to $\{\sigma_s\sigma_t \mid t \in T, s \in \mathcal{S}_T\} = \{\sigma_u \mid u \in S\}$. This implies that there exists $t^s \in T$ such that $ts \in st^s$. Since st^s is also a singleton, $ts = st^s$. Suppose $ts = st'$ for $t' \in T$. Then $s^*s \cap t'(t^s)^* \neq \emptyset$. By (i) and Lemma 3.1(i), $t' = t^s$. Therefore, the uniqueness of t^s is proved.

(iii) By (ii),

$$t(ss^*) = (ts)s^* = (st^s)s^* = s(t^s s^*) = ss^*(t^s)^{s^*}.$$

Note that, by Lemma 3.1(i), t is a unique element in $t(ss^*) \cap T$ and $(t^s)^{s^*}$ is also a unique element in it. Thus, $t = (t^s)^{s^*}$. Since $\sigma_t\sigma_s = \sigma_{ts} = \sigma_{st^s} = \sigma_s\sigma_{t^s}$ by Lemma 3.5, it follows that

$$\sigma_t(\sigma_s\sigma_{s^*}) = \sigma_{ts}\sigma_{s^*} = \sigma_s\sigma t^s\sigma_{s^*} = \sigma_s\sigma t^s s^* = \sigma_s\sigma_{s^*}\sigma_{(t^s)^{s^*}} = \sigma_s\sigma_{s^*}\sigma_t.$$

□

Proposition 3.7. *Suppose that T is nice and n_T is a prime. Then, for all $u, s_1, s_2 \in \mathcal{S}_T \setminus T$ with $\emptyset \subsetneq Tu \cap s_1s_2 \subsetneq Tu$, if $\sigma_{s_1}\sigma_{s_2}$ centralizes $\mathbb{C}T$, then σ_u also does.*

Proof. Let $u, s_1, s_2 \in \mathcal{S}_T$ be as in the statement of the proposition. Note that

$$\sigma_{s_1}\sigma_{s_2} = \sum_{t\in T} a_{s_1s_2(tu)}\sigma_{tu} + \sum_{v\in S\setminus Tu} a_{s_1s_2v}\sigma_v.$$

Let $t_1 \in T$. Then, by the assumption,

$$\sigma_{t_1}\sigma_{s_1}\sigma_{s_2} = \sigma_{s_1}\sigma_{s_2}\sigma_{t_1}.$$

Since T is normal in S, the right or left multiplication of σ_{t_1} leaves each of $\mathbb{C}(Tu)$ and $\mathbb{C}(S \setminus Tu)$ invariant. This implies that σ_{t_1} commutes with $\sum_{t\in T} a_{s_1s_2(tu)}\sigma_{tu}$.

Note that $\sigma_{tu} = \sigma_t\sigma_u$ by Lemma 3.5 since T is assumed to be nice. Since T is commutative by Theorem 2.5(iii) and $t_1u = u(t_1)^u$ by Lemma 3.6(ii), it follows that

$$\sigma_{t_1}\sum_{t\in T} a_{s_1s_2(tu)}\sigma_{tu} = \sigma_{t_1}\sum_{t\in T} a_{s_1s_2(tu)}\sigma_t\sigma_u$$
$$= (\sum_{t\in T} a_{s_1s_2(tu)}\sigma_t)\sigma_{t_1}\sigma_u = (\sum_{t\in T} a_{s_1s_2(tu)}\sigma_t)\sigma_u\sigma_{(t_1)^u}$$

On the other hand,

$$(\sum_{t\in T} a_{s_1s_2(tu)}\sigma_{tu})\sigma_{t_1} = (\sum_{t\in T} a_{s_1s_2(tu)}\sigma_t)\sigma_u\sigma_{t_1}.$$

Note that $\sum_{t\in T} a_{s_1s_2(tu)}\sigma_t$ is a linear combination of the adjacency matrices in T with non-negative integral coefficients. Since $\emptyset \lneq Tu \cap ss^* \lneq Tu$, it is not a scalar of σ_T. Thus, we can apply Lemma 2.9 to obtain that $\sum_{t\in T} a_{s_1s_2(tu)}\sigma_{tu}$ is invertible.

Therefore,

$$\sigma_u\sigma_{t_1} = \sigma_u\sigma_{(t_1)^u}.$$

Let $(x, y) \in u$. Then the submatrix of σ_u induced by $xT \times yT$ is a permutation matrix since $u \in \mathcal{S}_T$ and T is normal in S, especially, it is invertible. This implies that the submatrix of σ_{t_1} induced by $yT \times yT$ coincides with that of $\sigma_{(t_1)^u}$. Since $(x, y) \in u$ is arbitrarily taken, $\sigma_{t_1} = \sigma_{(t_1)^u}$, implying that σ_u commutes with σ_{t_1}. Since t_1 is arbitrarily taken, the proposition holds. □

Lemma 3.8. *For all $s_1, s_2, u \in \mathcal{S}_T$, if $Tu \subseteq s_1s_2$, then $n_T \le n_{(s_1)^T}$. Moreover, if the equality holds, then $n_T = a_{(s_1)^T(s_2)^Tu^T}$.*

Proof. By the assumption, $(\sigma_{s_1}\sigma_{s_2})_{xT,yT}$ has no zero entry where $(x, y) \in u$. Since

$$(\sigma_{s_1}\sigma_{s_2})_{xT,yT} = \sum_{zT\in X/T} (\sigma_{s_1})_{xT,zT}(\sigma_{s_2})_{yT,zT}$$

and there are exactly $a_{(s_1)^T(s_2)^Tu^T}$ nonzero permutation matrices in the summation, it follows from $s_1, s_2, u \in \mathcal{S}_T$ that

$$n_T \le a_{(s_1)^T(s_2)^Tu^T} \le n_{(s_1)^T}.$$

This completes the proof. □

Proposition 3.9. *Suppose that T is nice and both n_T and $n_{S/\!/T}$ are primes with $1 < \frac{n_{S/\!/T}-1}{|S/\!/T|-1} \le n_T$. Then $\mathbb{C}T$ is in the center of $\mathbb{C}S$.*

Proof. For short we shall write n_T, $n_{S/\!/T}$ and $\frac{n_{S/\!/T}-1}{|S/\!/T|-1}$ as p, q and k, respectively.

By Theorem 2.5(ii), $S/\!/T$ is k-equivalenced.

Let $s_1, s_2 \in \mathcal{S}_T$ such that $\{s_1, s_2\}$ is not contained in T.

We claim $(s_1 s_2) \setminus T \neq \emptyset$. Assume the contrary, i.e., $s_1 s_2 \subseteq T$. Since T is normal in S,

$$(Ts_1T)(Ts_2T) = T(s_1s_2)T \subseteq T.$$

Applying Lemma 2.1(i),(ii) for $(X/T, S/\!/T)$ we obtain that

$$(s_2)^T = ((s_1)^T)^* \text{ and } n_{(s_1)^T} = n_{(s_2)^T} = 1,$$

which contradicts the assumption $k > 1$.

We claim that, for each $u \in \mathcal{S}_T$, $Tu \cap s_1s_2 \subsetneq Tu$. Suppose the contrary, i.e., $Tu \subseteq s_1s_2$. Then, by Lemma 3.8, $n_{(s_1)^T} \ge p$. By the assumption of $1 < k \le p$, $k = n_{(s_1)^T} = p$. It follows from Lemma 3.8 that $a_{(s_1)^T(s_2)^T u^T} = k$, which contradicts Theorem 2.5(iv). Therefore, the assumptions on s_1, s_2, u given in Proposition 3.7 are satisfied.

Lemma 3.6(iii) and the first claim show the existence of $u \in \mathcal{S}_T \setminus T$ such that σ_u centralizes $\mathbb{C}T$

Proposition 3.7 shows that, for all $s_1, s_2, u \in \mathcal{S}_T$ with $Tu \cap s_1s_2 \neq \emptyset$, if both of σ_{s_1} and σ_{s_2} centralizes $\mathbb{C}T$, then so u does. Since $\mathcal{S}_T \subseteq \bigcup_{i=0}^{\infty} Tu^i$, it follows that each element of $\{\sigma_s \mid s \in \mathcal{S}_T\}$ centralizes $\mathbb{C}T$. Since T is commutative by Theorem 2.5(iii), the proposition follows from Lemma 3.5. □

4. From modular representation

Throughout this section we assume that (X, S) is a scheme, T is a normal closed subset of S and F is a field.

Lemma 4.1. ([5]) *The F-linear map $\varphi : FS \to F(S/\!/T)$ defined by*

$$\sigma_s \mapsto \frac{n_s}{n_{s^T}} \sigma_{s^T}$$

is an F-algebra homomorphism.

Lemma 4.2. *If T is nice, then $\varphi : FS \to F(S/\!/T)$ as in Lemma 4.1 is onto.*

Proof. For each $s \in S$ we can take $u \in \mathcal{S}_T$ such that $TsT = TuT$ since T is nice. By the definition of φ and Lemma 3.2, we have

$$\varphi(\sigma_u) = \frac{n_s}{n_{s^T}} \sigma_{u^T} = \sigma_{u^T} = \sigma_{s^T}.$$

This implies that φ is onto. □

Lemma 4.3. ([7, Thm.3.5]) *If S has the same intersection numbers as a schurian scheme of prime order, then $FS = F[\sigma_s]$ for each $s \in S^\sharp$.*

Proof. Suppose that the characteristic of F is equal to n_S. Then, by [7, Thm. 3.5], $FS = F[\sigma_s]$ for some $s \in S^\sharp$. By the assumption, the group of permutations of S which preserve the intersection numbers acts transitively on $S^\sharp$. Thus, $FS = F[\sigma_s]$ for each $s \in S^\sharp$.

Suppose that the characteristic of F is zero or prime to n_S. Then FS is semisimple by [6, Thm. 5.4], and σ_s has $|S|$ distinct eigenvalues in a splitting field of F by the assumption and [6, Thm. 5.3]. This implies that $\{\sigma_s^i \mid i = 0, 1, \dots, |S|-1\}$ are linearly independent, and, hence, $FS = F[\sigma_s]$. □

For the remainder of this section we assume that

(i) F is of characteristic p where p is a prime;
(ii) T is a nice closed subset of valency p;
(iii) $\varphi : FS \to F(S/\!\!/T)$ is the F-algebra homomorphism as in Lemma 4.1.

Note that FT is a subalgebra of FS, and we shall denote by $\mathrm{Rad}(FT)$ is the Jacobson radical of FT. In [4, Cor. 3.5],

$$\mathrm{Rad}(FT) = \bigoplus_{t \in T^\sharp} (\sigma_t - n_t \sigma_{1_X}). \tag{3}$$

The following is something to generalize arguments given in [7].

Lemma 4.4. *We have the following:*

(i) $\ker\varphi = \mathrm{Rad}(FT)(FS)$*;*
(ii) *If $S/\!\!/T$ has the same intersection numbers as a schurian scheme of prime order, then $FS = (FT)F[\sigma_s]$ for $s \in S \setminus T$.*

Proof. (i) Since φ is an algebra homomorphism, we have, for each $t \in T$,

$$\varphi(\sigma_t - n_t\sigma_{1_X}) = \frac{n_t}{n_{t^T}}\sigma_{t^T} - n_t\left(\frac{n_{1_X}}{n_{(1_X)^T}}\right)\sigma_{(1_X)^T} = 0.$$

It follows from (3) that $\mathrm{Rad}(FT)(FS)$ is contained in $\ker\varphi$.

We claim that

$$|S/\!\!/T|(|T|-1) \le \dim(\mathrm{Rad}(FT))(FS).$$

We can choose a subset $\mathcal{U} \subseteq \mathcal{S}_T$ such that S is a disjoint union of Tu with $u \in \mathcal{U}$ since T is nice. It suffices to show that $\{(\sigma_t - n_t\sigma_{1_X})\sigma_s \mid s \in \mathcal{U}, t \in T^\sharp\}$ is linearly independent. Suppose that

$$\sum_{t \in T^\sharp, s \in \mathcal{U}} c_{ts}(\sigma_t - n_t\sigma_{1_X})\sigma_s = 0.$$

Thus,

$$\sum_{t \in T^\sharp, s \in \mathcal{U}} c_{ts}\sigma_t\sigma_s - \sum_{s \in \mathcal{S}}\left(\sum_{t \in T^\sharp} c_{ts}n_t\right)\sigma_s = 0.$$

Notice that $\{\sigma_t\sigma_s \mid t \in T, s \in \mathcal{U}\}$ are distinct and linearly independent by Lemma 3.5. Therefore, the coefficients c_{ts} are zero.

On the other hand, since φ is onto,

$$\dim(\ker\varphi) = |S| - |S/\!\!/T|.$$

Since T is nice, $|S| = |T||S/\!\!/T|$ by Lemma 3.5. It follows from the claim that

$$|S/\!\!/T|(|T|-1) \le \dim((FS)(\mathrm{Rad}(FT)) \le \dim(\ker\varphi) = |S/\!\!/T|(|T|-1).$$

This implies that the equality holds and the two spaces are equal.

(ii) Let $s \in S \setminus T$. Since $F(S/\!\!/T) = F[\sigma_{s^T}]$ by Lemma 4.3 and φ is onto by Lemma 4.2, $FS = (FT)F[\sigma_s] + \ker\varphi$. By (i),

$$FS = (FT)F[\sigma_s] + \mathrm{Rad}(FT)(FS)$$

Applying Nakayama's Lemma for FT-modules we conclude that $FS = (FT)F[\sigma_s]$. □

Proposition 4.5. *Suppose $FS = (FT)F[\sigma_s]$ for some $s \in S$ and $n_{S/\!\!/T}$ is a prime with $\frac{n_{S/\!\!/T}-1}{|S/\!\!/T|-1} \le p$. Then (X, S) is commutative if and only if $\mathbb{C}T$ is in the center of $\mathbb{C}S$.*

Proof. The "only if" part is obvious. Suppose that $\mathbb{C}T$ is in the center of $\mathbb{C}S$. By Lemma 4.4(ii), FS is commutative.

Let $s_i \in \mathcal{S}_T$ for $i = 1, 2$. Since T is assumed to be nice, it follows from Lemma 3.2 that $n_{s_i} = n_{(s_i)^T}$ for $i = 1, 2$. Since $n_{S/\!\!/T}$ is a prime, it follows from Theorem 2.5(ii) that $S/\!\!/T$ is k-equivalenced where $k = \frac{n_{S/\!\!/T}-1}{|S/\!\!/T|-1}$. Thus, $n_{s_i} \in \{1, k\}$ for $i = 1, 2$. Therefore, we conclude from the assumption and Theorem 2.5(iv) that each coefficient of any non-diagonal adjacency matrix in $\sigma_{s_i}\sigma_{s_j}$ is less than p, and hence,

$$\sigma_{s_i}\sigma_{s_j} \equiv \sigma_{s_j}\sigma_{s_i} \mod p \text{ if and only if } \sigma_{s_i}\sigma_{s_j} = \sigma_{s_j}\sigma_{s_i}.$$

Let $u_1, u_2 \in S$. By Lemma 3.5, $\sigma_{u_i} = \sigma_{t_i}\sigma_{s_i}$ for some $t_1, t_2 \in T$ and $s_1, s_2 \in \mathcal{S}_T$.

Since $\mathbb{C}T$ is in the center of $\mathbb{C}S$, t_i commutes with s_j for $i, j = 1, 2$. The above equation with the fact that FS is commutative shows that σ_{u_1} commutes with σ_{u_2}. This completes the proof. □

5. Proof of our main results

5.1. Proof of Theorem 1.2

It is a direct consequence of Proposition 3.9, Lemma 4.4 and Proposition 4.5.

5.2. Proof of Theorem 1.3

By Theorem 1.1, (X, S) is commutative if $S/\!\!/T$ is thin. Thus, we may assume that $S/\!\!/T$ is k-equivalenced for some $1 < k \le p-1$. By Proposition 3.9, $\mathbb{C}T$ is in the center of $\mathbb{C}S$.

Let F be a field of characteristic p. Then, by [7, Thm. 3.5], $F(S/\!\!/T) = F[\sigma_{s^T}]$ for some $s \in S \setminus T$. Applying Proposition 4.5 with the fact that $\mathbb{C}T$ is in the center of $\mathbb{C}S$ we obtain that S is commutative.

5.3. Proof of Corollary 1.4

Suppose that T is a flat non-trivial closed subset. Since T is non-trivial, it follows from Lemma 2.4 that $n_T = p$, and T is normal by Theorem 2.6. Applying Theorem 2.7 for $(X,S)_{xT}$ for $x \in X$ we obtain from Lemma 2.8 that $\Pi_{x,y} := \{ys \cap xT \mid s \in S\}$ is trivial or has at least one singleton where $y \in X$. We shall write $r(x,y)$ as r for short.

If $\Pi_{x,y}$ is trivial and $r \in S \setminus T$, then $a_{Trr} = n_T$. By Lemma 2.3(iii), $n_T \mid n_r$. By Lemma 3.4, $\{s \in S \mid p \nmid n_s\}$ is closed and $r \notin \{s \in S \mid p \nmid n_s\}$. This implies that each element of $S \setminus T$ has valency divisible by p. By Lemma 3.3, $TsT = \{s\}$ for each $s \in S \setminus T$. In this case it can be easily checked that (X,S) is commutative by a direct computation. Thus, we may assume that $\Pi_{x,y}$ has at least one singleton for all $x, y \in X$. This implies that $S = \bigcup_{s \in \mathcal{S}_T} Ts$.

Let $t \in T^\sharp$ and $s \in \mathcal{S}_T \setminus T$. Then, by Theorem 2.5(ii),

$$n_t = \frac{p-1}{|T|-1}, \;\; n_s = n_{s^T} = \frac{p-1}{|S/\!\!/T|-1}.$$

Since $\gcd(n_t, n_s) = 1$, it follows from Lemma 2.1(v) that ts is a singleton for each $t \in T$. By Lemma 3.1(ii), $|Ts| = |T|$ for each $s \in \mathcal{S}_T$. Thus, T is nice.

Therefore, we conclude from Theorem 1.3 that (X,S) is commutative.

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