

GEOMETRIC HERMITE INTERPOLATION FOR PLANAR PYTHAGOREAN-HODOGRAPH CUBICS

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ABSTRACT. We solve the geometric Hermite interpolation problem with planar Pythagorean-hodograph cubics. For every Hermite data, we determine the exact number of the geometric Hermite interpolants and represent the interpolants explicitly. We also present a simple criterion for determining whether the interpolants have a loop or not.

1. Introduction

In computer-aided geometric design, curves are usually represented by polynomial/rational parameterizations. But the derived objects, such as their offset curves, are not generally represented by rational parameterizations. To overcome this barrier, Farouki and Sakkalis (1990, 1994) introduced Pythagorean-hodograph (PH) curves, which are a special class of polynomial curves with polynomials as their speed functions. PH curves have many computationally attractive features, so that we can compute their arc lengths and bending energies and offset curves in an exact manner. For successively abundant results obtained by many researchers, see Farouki (2008) and references therein.

Hermite interpolation by PH curves are one of the main subjects in the society of these research. (For more details see for example Farouki and Neff, 1995; Albrecht and Farouki, 1996; Jüttler and Mäurer, 1999; Jüttler, 2001; Farouki et al., 2002; Pelosi et al., 2005; Šír et al., 2010.) In this paper, we present the G^1 Hermite interpolation with planar PH cubics. Such an interpolation was firstly studied by Meek and Walton (1997), depending upon the analysis of Farouki and Sakkalis (1990). Recently Byrtus and Bastl (2010) extended these results to more general data and presented a thorough analysis of the number and the quality of the interpolants; particularly if they contain a loop or not. To determine the shape of the interpolants, Byrtus and Bastl adapted the analysis of general cubic Bézier curves given by Stone and DeRose (1989).

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We improve the parallel results of Byrtus and Bastl. For each Hermite data, we determine the exact number of the G^1 Hermite interpolants and also explicitly represent the desired interpolants. We also present a simple and self-contained criterion for determining whether the interpolants contain a loop or not. Our method does not rely upon the results of general cubic curves, and have advantage in computations in comparison with the previous results.

This paper is organized as follows. Section 2 gives basic properties about PH cubics. In Section 3, we present the G^1 Hermite interpolations for every possible Hermite data and determine whether the interpolants have a loop or not with a simple criterion. In Section 4, we conclude this paper.

2. Planar pythagorean-hodograph cubics

A polynomial plane curve $\mathbf{r}(t) = (x(t), y(t))$ is called a *Pythagorean-hodograph* (PH) curve (Farouki et al, 1990) if there exists a polynomial $\sigma(t)$ such that

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2.$$

For a polynomial plane curve $\mathbf{r}(t)$, $\mathbf{r}(t)$ is a PH curve (Kubota, 1972) if and only if there are polynomials $h(t), u(t), v(t)$, satisfying

$$x'(t) = h(t)[u(t)^2 - v(t)^2], \quad y'(t) = h(t)[2u(t)v(t)]. \quad (1)$$

Here we note that if $\gcd(u(t), v(t)) = 1$, then $\gcd(u(t)^2 - v(t)^2, 2u(t)v(t)) = 1$. In this paper, we will assume that $h(t)$ is monic, i.e., the leading coefficient of $h(t)$ is 1.

Let $\mathbf{r}(t) = (x(t), y(t))$ be a PH cubic such that (1) for a monic polynomial $h(t)$ and polynomials $u(t), v(t)$ with $\gcd(u(t), v(t)) = 1$. Then the following are equivalent:

- (a) $\mathbf{r}(t)$ is a line;
- (b) $\deg(h(t)) = 2$;

Thus for a PH cubic $\mathbf{r}(t) = (x(t), y(t))$ which is a line, its hodograph $\mathbf{r}'(t)$ is expressed as $\mathbf{r}'(t) = h(t)(x_0, y_0)$ for a quadratic monic polynomial $h(t)$ and a nonzero point (x_0, y_0) . In this case, only one of the following is true:

- (a) $h(t) = (t - c)^2$ for some real number c ;
- (b) $h(t) = (t - c_1)(t - c_2)$ for some distinct real numbers c_1 and c_2 ;
- (c) $h(t) = (t - c_1)(t - \bar{c}_1)$ for some non-real complex number c_1 .

If we identify a point $\mathbf{p} = (a, b)$ in the plane \mathbb{R}^2 with the complex number $\mathbf{p}_{\mathbb{C}} = a + \sqrt{-1}b$ in the complex plane \mathbb{C} , then a polynomial curve $\mathbf{r}_{\mathbb{C}}(t) = x(t) + \sqrt{-1}y(t)$ is a PH curves (Farouki, 1994) if and only if there exists a polynomial $h(t)$ and a polynomial curve $\mathbf{w}(t) = u(t) + \sqrt{-1}v(t)$ such that

$$\mathbf{r}'_{\mathbb{C}}(t) = h(t)\mathbf{w}(t)^2. \quad (2)$$

Let $\mathbf{r}(t)$ be a PH cubic with $\mathbf{r}'_{\mathbb{C}}(t) = \mathbf{w}(t)^2$. Since $\mathbf{w}(t)$ is linear, we can write $\mathbf{w}(t)$ as

$$\mathbf{w}(t) = \mathbf{w}_0(1 - t) + \mathbf{w}_1t$$

in the Bernstein-Bézier form. The hodograph $\mathbf{r}'_{\mathbb{C}}(t)$ then is expressed as

$$\mathbf{r}'_{\mathbb{C}}(t) = \mathbf{w}_0^2(1-t)^2 + \mathbf{w}_0\mathbf{w}_12(1-t)t + \mathbf{w}_1^2t^2. \quad (3)$$

Representing the PH cubic $\mathbf{r}_{\mathbb{C}}(t)$ as

$$\mathbf{r}_{\mathbb{C}}(t) = \mathbf{p}_0(1-t)^3 + \mathbf{p}_13(1-t)^2t + \mathbf{p}_23(1-t)t^2 + \mathbf{p}_3t^3 \quad (4)$$

in the Bernstein-Bézier form, we obtain

$$\mathbf{p}_1 = \mathbf{p}_0 + \frac{1}{3}\mathbf{w}_0^2, \quad \mathbf{p}_2 = \mathbf{p}_1 + \frac{1}{3}\mathbf{w}_0\mathbf{w}_1, \quad \mathbf{p}_3 = \mathbf{p}_2 + \frac{1}{3}\mathbf{w}_1^2$$

where \mathbf{p}_0 is arbitrary.

Theorem 2.1. ([7]) *Let*

$$\mathbf{r}_{\mathbb{C}}(t) = \mathbf{p}_0(1-t)^3 + \mathbf{p}_13(1-t)^2t + \mathbf{p}_23(1-t)t^2 + \mathbf{p}_3t^3$$

be a nonlinear planar polynomial cubic in the Bernstein-Bézier form. Let $\mathbf{L}_k = \mathbf{p}_k - \mathbf{p}_{k-1}$ for $k = 1, 2, 3$ be the direction legs of the Bézier control polygon for $k = 1, 2, 3$. Then the condition

$$\mathbf{L}_2^2 = \mathbf{L}_1 \cdot \mathbf{L}_3$$

is sufficient and necessary to ensure that $\mathbf{r}_{\mathbb{C}}(t)$ is a PH curve.

Remark 1. If $\mathbf{r}'_{\mathbb{C}}(t) = \mathbf{w}(t)^2$, then Theorem 2.1 holds. But if $\mathbf{r}'_{\mathbb{C}}(t) = h(t)\mathbf{w}_0^2$ for some constant \mathbf{w}_0 and a quadratic polynomial $h(t)$ having different two zeros, then Theorem 2.1 does not hold,

3. Geometric Hermite interpolation

In this section, we will solve the geometric Hermite interpolation problem for planar Pythagorean-hodograph cubics.

Let \mathbf{p}_i and \mathbf{p}_f be the initial and final points, respectively. Let $\mathbf{d}_i = e^{i\theta_i}$ and $\mathbf{d}_f = e^{i\theta_f}$ be the directional vectors at \mathbf{p}_i and \mathbf{p}_f respectively. For a given \mathbf{p}_i , \mathbf{p}_f , \mathbf{d}_i and \mathbf{d}_f , the geometric Hermite interpolation problem is to find PH cubics $\mathbf{r}_{\mathbb{C}}(t)$, which satisfy

$$\mathbf{r}_{\mathbb{C}}(0) = \mathbf{p}_i, \quad \mathbf{r}_{\mathbb{C}}(1) = \mathbf{p}_f, \quad \frac{\mathbf{r}'_{\mathbb{C}}(0)}{|\mathbf{r}'_{\mathbb{C}}(0)|} = \mathbf{d}_i, \quad \frac{\mathbf{r}'_{\mathbb{C}}(1)}{|\mathbf{r}'_{\mathbb{C}}(1)|} = \mathbf{d}_f. \quad (5)$$

These equations are equivalent to

$$\int_0^1 \mathbf{r}'_{\mathbb{C}}(t) dt = \mathbf{p}_f - \mathbf{p}_i, \quad \frac{\mathbf{r}'_{\mathbb{C}}(0)}{|\mathbf{r}'_{\mathbb{C}}(0)|} = \mathbf{d}_i, \quad \frac{\mathbf{r}'_{\mathbb{C}}(1)}{|\mathbf{r}'_{\mathbb{C}}(1)|} = \mathbf{d}_f. \quad (6)$$

Moreover, considering translation and rotation, we may assume that $\mathbf{p}_i = \mathbf{0}$, $\mathbf{p}_f = k \in \mathbb{R}$, $-\pi < \theta_i \leq 0$ and $\theta_i \leq \theta_f < \theta_i + 2\pi$.

Suppose that $\theta_i = 0$ and $\theta_f = 0$ or π . Then the Hermite interpolations $\mathbf{r}_{\mathbb{C}}(t)$ are given by

$$\mathbf{r}_{\mathbb{C}}(t) = a3(1-t)^2t + (k-b)3(1-t)t^2 + kt^3 \quad (7)$$

for any positive real numbers a and

$$b = \begin{cases} \text{any positive real number} & \text{if } \theta_f = 0; \\ \text{any negative real number} & \text{if } \theta_f = \pi. \end{cases}$$

Now suppose that $\theta_i \neq 0$ or ($\theta_f \neq 0$ and $\theta_f \neq \pi$). Then the Hermite interpolants $\mathbf{r}_\mathbb{C}(t)$ must satisfy $\mathbf{r}'_\mathbb{C}(t) = \mathbf{w}(t)^2$. From (3) and (4), the equations (6) then becomes

$$\mathbf{w}_0^2 + \mathbf{w}_1^2 + \mathbf{w}_0\mathbf{w}_1 = 3k, \quad \frac{\mathbf{w}_0^2}{|\mathbf{w}_0|^2} = \mathbf{d}_i, \quad \frac{\mathbf{w}_1^2}{|\mathbf{w}_1|^2} = \mathbf{d}_f. \quad (8)$$

We here note that if two complex numbers \mathbf{u}_0 and \mathbf{u}_1 satisfy

$$\mathbf{u}_0^2 + \mathbf{u}_1^2 + \mathbf{u}_0\mathbf{u}_1 = 3K, \quad \frac{\mathbf{u}_0^2}{|\mathbf{u}_0|^2} = \mathbf{d}_i, \quad \frac{\mathbf{u}_1^2}{|\mathbf{u}_1|^2} = \mathbf{d}_f, \quad (9)$$

for some real number K such that

$$\begin{cases} k \cdot K > 0 & \text{if } k \neq 0; \\ K = 0 & \text{if } k = 0; \end{cases}$$

then $\delta\mathbf{u}_0$ and $\delta\mathbf{u}_1$ become solutions for the equations (8) of unknown \mathbf{w}_0 and \mathbf{w}_1 where

$$\delta = \begin{cases} \sqrt{\frac{k}{K}} & \text{if } k \neq 0; \\ 1 & \text{if } k = 0. \end{cases} \quad (10)$$

Thus we can summarize the geometric Hermite interpolation problem as follows:

- (a) Given data: a real number k , directional vectors $\mathbf{d}_i = e^{i\theta_i}$ and $\mathbf{d}_f = e^{i\theta_f}$ where $\theta_i \neq 0$ or ($\theta_f \neq 0$ and $\theta_f \neq \pi$);
- (b) Find out \mathbf{u}_0 and \mathbf{u}_1 , which satisfy $|\mathbf{u}_0| = 1$ and (9) for some real number K ;
- (c) For such δ in (10), $\mathbf{w}_0 = \delta\mathbf{u}_0$ and $\mathbf{w}_1 = \delta\mathbf{u}_1$ become solutions for the equations (8).
- (d) For $\mathbf{p}_0 = \mathbf{0}$, $\mathbf{p}_1 = \frac{1}{3}\mathbf{w}_0^2$, $\mathbf{p}_2 = \frac{1}{3}(\mathbf{w}_0^2 + \mathbf{w}_0\mathbf{w}_1)$, and $\mathbf{p}_3 = \frac{1}{3}(\mathbf{w}_0^2 + \mathbf{w}_0\mathbf{w}_1 + \mathbf{w}_1^2)$, the cubic $\mathbf{r}_\mathbb{C}(t)$, which are given by (4), are the solutions.

Let $\mathbf{d}_i = e^{i\theta_i}$ and $\mathbf{d}_f = e^{i\theta_f}$ be directional vectors. From now on, we will find \mathbf{u}_0 and \mathbf{u}_1 , which satisfy $|\mathbf{u}_0| = 1$ and (9) for some real number K . Then from the second and the third equations in (9), the complex numbers \mathbf{u}_0 and \mathbf{u}_1 with $|\mathbf{u}_0| = 1$ is given by

$$\mathbf{u}_0 = \pm e^{i\theta_i/2}, \quad \mathbf{u}_1 = \pm r e^{i\theta_f/2}$$

for some positive real number r . Let

$$f_\sigma(r) = e^{i\theta_i} + r^2 e^{i\theta_f} + \sigma \cdot r e^{i(\theta_i + \theta_f)/2}, \quad (\sigma = -1, +1)$$

for the positive real number r . Then to solve the first equation in (9) for some real number K is to find positive real numbers r which make $f_{+1}(r)$ or $f_{-1}(r)$

real numbers. Here we note that the real part and the imaginary part of $f_\sigma(r)$ are

$$\operatorname{Re} f_\sigma(r) = r^2 \cos \theta_f + \sigma \cdot r \cos \frac{\theta_i + \theta_f}{2} + \cos \theta_i,$$

$$\operatorname{Im} f_\sigma(r) = r^2 \sin \theta_f + \sigma \cdot r \sin \frac{\theta_i + \theta_f}{2} + \sin \theta_i,$$

respectively.

3.1. Case of $\theta_i = 0$.

Since $\sin \theta_i = 0$,

$$\operatorname{Im} f_\sigma(r_1) = r_1^2 \sin \theta_f + \sigma \cdot r_1 \sin \frac{\theta_f}{2} = r_1 \sin \frac{\theta_f}{2} \left(r_1 2 \cos \frac{\theta_f}{2} + \sigma \right)$$

has the positive zero

$$s_\sigma = \frac{-\sigma}{2 \cos \frac{\theta_f}{2}} \tag{11}$$

only when $\theta_f \neq \pi$, where

$$\sigma = \begin{cases} -1 & \text{if } 0 \leq \theta_f < \pi; \\ +1 & \text{if } \pi < \theta_f < 2\pi. \end{cases} \tag{12}$$

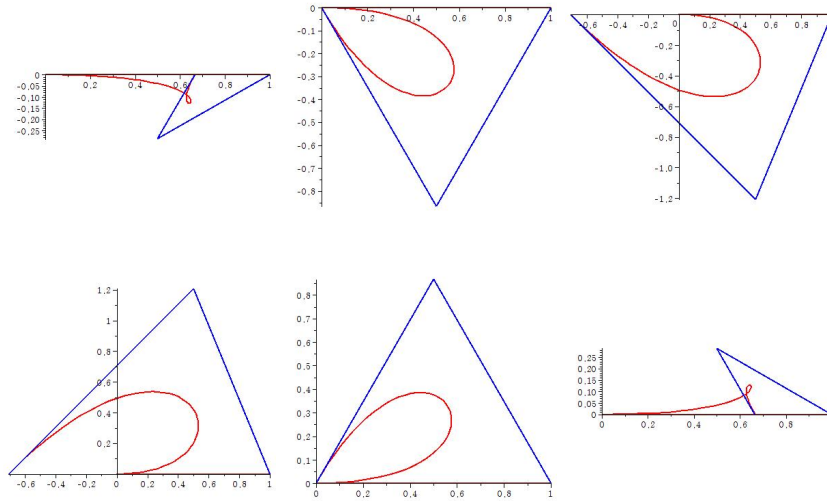


FIGURE 1. $\mathbf{r}_\sigma(t)$ in Theorem 3.1 with $\theta_i = 0$ and $\theta_f = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{5\pi}{3}$, respectively

Theorem 3.1. Let θ_i and θ_f be constants with $\theta_i = 0$, $0 < \theta_f < 2\pi$ and $\theta_f \neq \pi$.

(a) There is a (unique) PH cubic $\mathbf{r}_\sigma(t)$, which satisfies

$$\mathbf{r}_\sigma(0) = 0, \quad \mathbf{r}_\sigma(1) \in \mathbb{R}, \quad \frac{\mathbf{r}'_\sigma(0)}{|\mathbf{r}'_\sigma(0)|} = e^{\theta_i}, \quad \frac{\mathbf{r}'_\sigma(1)}{|\mathbf{r}'_\sigma(1)|} = e^{\theta_f}, \quad (13)$$

for σ in (12), and which have the first directional leg \mathbf{L}_1 of length 1 in the Bézier control polygon.

(b) The PH cubic $\mathbf{r}_\sigma(t)$ is expressed by

$$\mathbf{r}_\sigma(t) = \mathbf{p}_1 3(1-t)^2 t + \mathbf{p}_2 3(1-t)t^2 + \mathbf{p}_3 t^3,$$

where

$$\mathbf{p}_1 = e^{i\theta_i}, \quad \mathbf{p}_2 = e^{i\theta_i} + \sigma s_\sigma e^{i(\theta_i + \theta_f)/2}, \quad \mathbf{p}_3 = e^{i\theta_i} + \sigma s_\sigma e^{i(\theta_i + \theta_f)/2} + s_\sigma^2 e^{i\theta_f},$$

$$\text{with } s_\sigma = \frac{-\sigma}{2 \cos \frac{\theta_f}{2}}.$$

(c) The table shows the shapes of the PH cubic $\mathbf{r}_\sigma(t)$:

θ_f	$\mathbf{r}_{+1}(1)$	$\mathbf{r}_{-1}(1)$	$\mathbf{r}_{+1}(t)$ or $\mathbf{r}_{-1}(t)$
$0 < \theta_f < \frac{2\pi}{3}$	None	+	loop
$\theta_f = \frac{2\pi}{3}$	None	0	closed
$\frac{2\pi}{3} < \theta_f < \pi$	None	-	simple
$\pi < \theta_f < \frac{4\pi}{3}$	-	None	simple
$\theta_f = \frac{4\pi}{3}$	0	None	closed
$\frac{4\pi}{3} < \theta_f < 2\pi$	+	None	loop

Remark 2. For the case of $\theta_i = 0$ and $\theta_f = 0$ or π , we can have such cubics $\mathbf{r}(t)$ in Theorem 3.1, which are given by (7) with $a = 1$.

3.2. Case of $-\pi < \theta_i < 0$ and $\theta_f = 0$.

Since $\sin \theta_f = 0$,

$$\text{Im } f_\sigma(r) = \sigma \cdot r \sin \frac{\theta_i}{2} + \sin \theta_i = \sin \frac{\theta_i}{2} \left(\sigma \cdot r + 2 \cos \frac{\theta_i}{2} \right)$$

has the positive zero

$$s_\sigma = -2\sigma \cos \frac{\theta_i}{2}$$

only when $\sigma = -1$.

Theorem 3.2. Let θ_i and θ_f be constants with $-\pi < \theta_i < 0$ and $\theta_f = 0$.

(a) There is a (unique) PH cubic $\mathbf{r}_{-1}(t)$, which satisfies

$$\mathbf{r}_{-1}(0) = 0, \quad \mathbf{r}_{-1}(1) \in \mathbb{R}, \quad \frac{\mathbf{r}'_{-1}(0)}{|\mathbf{r}'_{-1}(0)|} = e^{\theta_i}, \quad \frac{\mathbf{r}'_{-1}(1)}{|\mathbf{r}'_{-1}(1)|} = e^{\theta_f},$$

and which have the first directional leg \mathbf{L}_1 of length 1 in the Bézier control polygon.

(b) The PH cubic $\mathbf{r}_{-1}(t)$ is expressed by

$$\mathbf{r}_{-1}(t) = \mathbf{p}_1 3(1-t)^2 t + \mathbf{p}_2 3(1-t)t^2 + \mathbf{p}_3 t^3,$$

where

$$\mathbf{p}_1 = e^{i\theta_i}, \quad \mathbf{p}_2 = e^{i\theta_i} - s_\sigma e^{i(\theta_i+\theta_f)/2}, \quad \mathbf{p}_3 = e^{i\theta_i} - s_\sigma e^{i(\theta_i+\theta_f)/2} + s_\sigma^2 e^{i\theta_f},$$

$$\text{with } s_{-1} = 2 \cos \frac{\theta_i}{2}.$$

(c) The table shows the shapes of the PH cubic $\mathbf{r}_\sigma(t)$:

θ_i	$\mathbf{r}_{-1}(1)$	$\mathbf{r}_{-1}(t)$
$-\pi < \theta_i < -\frac{2\pi}{3}$	–	simple
$\theta_i = -\frac{2\pi}{3}$	0	closed
$-\frac{2\pi}{3} < \theta_i < 0$	+	loop

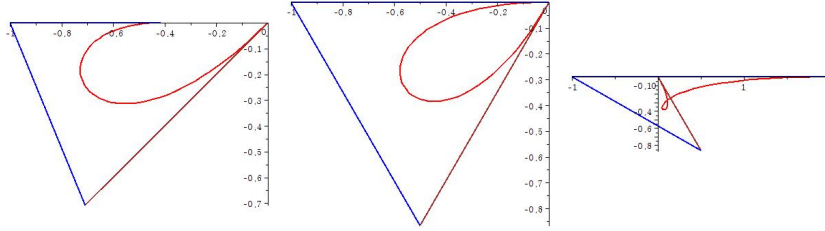


FIGURE 2. $\mathbf{r}_{-1}(t)$ in Theorem 3.2 with $\theta_f = 0$ and $\theta_i = -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}$, respectively

3.3. Case of $-\pi < \theta_i < 0$ and $\theta_f = \pi$.

Since $\sin \theta_f = 0$,

$$\text{Im } f_\sigma(r) = \sigma \cdot r \sin \frac{\theta_i + \pi}{2} + \sin \theta_i = \cos \frac{\theta_i}{2} \left(\sigma \cdot r + 2 \sin \frac{\theta_i}{2} \right)$$

has the positive root

$$s_\sigma = -2\sigma \sin \frac{\theta_i}{2}$$

only when $\sigma = +1$.

Theorem 3.3. Let θ_i and θ_f be constants with $-\pi < \theta_i < 0$ and $\theta_f = \pi$.

(a) There is a (unique) PH cubic $\mathbf{r}_{+1}(t)$, which satisfies

$$\mathbf{r}_{+1}(0) = 0, \quad \mathbf{r}_{+1}(1) \in \mathbb{R}, \quad \frac{\mathbf{r}'_{+1}(0)}{|\mathbf{r}'_{+1}(0)|} = e^{i\theta_i}, \quad \frac{\mathbf{r}'_{+1}(1)}{|\mathbf{r}'_{+1}(1)|} = e^{i\theta_f},$$

and which have the first directional leg \mathbf{L}_1 of length 1 in the Bézier control polygon.

(b) The PH cubic $\mathbf{r}_{+1}(t)$ is expressed by

$$\mathbf{r}_{+1}(t) = \mathbf{p}_1 3(1-t)^2 t + \mathbf{p}_2 3(1-t)t^2 + \mathbf{p}_3 t^3,$$

where

$$\mathbf{p}_1 = e^{i\theta_i}, \quad \mathbf{p}_2 = e^{i\theta_i} + s_{+1} e^{i(\theta_i + \theta_f)/2}, \quad \mathbf{p}_3 = e^{i\theta_i} + s_{+1} e^{i(\theta_i + \theta_f)/2} + s_{+1}^2 e^{i\theta_f},$$

$$\text{with } s_{+1} = -2 \sin \frac{\theta_i}{2}.$$

(c) The table shows the shapes of the PH cubic $\mathbf{r}_\sigma(t)$:

θ_i	$\mathbf{r}_{+1}(1)$	$\mathbf{r}_{+1}(t)$
$-\pi < \theta_i < -\frac{\pi}{3}$	-	loop
$\theta_i = -\frac{\pi}{3}$	0	closed
$-\frac{\pi}{3} < \theta_i < 0$	+	simple

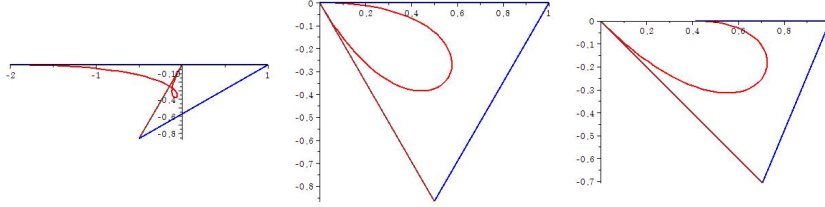


FIGURE 3. $\mathbf{r}_{+1}(t)$ in Theorem 3.3 with $\theta_f = \pi$ and $\theta_i = -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}$, respectively

3.4. Case of $-\pi < \theta_i < 0$ and $0 < \theta_f < \pi$.

Since $\theta_f > 0$ and $\sin \theta_i < 0$, $\text{Im } f_\sigma(r)$ has the positive zero

$$s_\sigma = -\sigma \frac{1 \cdot \sin \frac{\theta_i + \theta_f}{2}}{\sin \theta_f} + \sqrt{\frac{\sin^2 \frac{\theta_i + \theta_f}{2} - 4 \sin \theta_f \sin \theta_i}{4 \sin^2 \theta_f}} \quad (14)$$

When $\theta_i = 0$ and $\sigma = -1$, equation (14) equals (11).

Theorem 3.4. Let θ_i and θ_f be constants with $-\pi < \theta_i < 0$ and $0 < \theta_f < \pi$.

(a) There are exactly two PH cubics $\mathbf{r}_{+1}(t)$ and $\mathbf{r}_{-1}(t)$, which satisfies

$$\mathbf{r}_\sigma(0) = 0, \quad \mathbf{r}_\sigma(1) \in \mathbb{R}, \quad \frac{\mathbf{r}'_\sigma(0)}{|\mathbf{r}'_\sigma(0)|} = e^{i\theta_i}, \quad \frac{\mathbf{r}'_\sigma(1)}{|\mathbf{r}'_\sigma(1)|} = e^{i\theta_f}, \quad (15)$$

for $\sigma = +1, -1$, and which have the first directional leg \mathbf{L}_1 of length 1 in the Bézier control polygon.

(b) The PH cubics $\mathbf{r}_\sigma(t)$ are expressed by

$$\mathbf{r}_\sigma(t) = \mathbf{p}_1 3(1-t)^2 t + \mathbf{p}_2 3(1-t)t^2 + \mathbf{p}_3 t^3,$$

where

$$\mathbf{p}_1 = e^{i\theta_i}, \quad \mathbf{p}_2 = e^{i\theta_i} + \sigma s_\sigma e^{i(\theta_i+\theta_f)/2}, \quad \mathbf{p}_3 = e^{i\theta_i} + \sigma s_\sigma e^{i(\theta_i+\theta_f)/2} + s_\sigma^2 e^{i\theta_f},$$

with $s_\sigma = -\sigma \cdot A + \sqrt{B}$ where

$$A = \frac{\sin \frac{\theta_i+\theta_f}{2}}{2 \sin \theta_f}, \quad B = \frac{\sin^2 \frac{\theta_i+\theta_f}{2} - 4 \sin \theta_f \sin \theta_i}{4 \sin^2 \theta_f}.$$

(c) The table shows the shapes of the PH cubics $\mathbf{r}_\sigma(t)$:

$\eta = \theta_f - \theta_i$	$\mathbf{r}_{+1}(1)$	$\mathbf{r}_{+1}(t)$	$\mathbf{r}_{-1}(1)$	$\mathbf{r}_{-1}(t)$
$\eta < \frac{2\pi}{3}$	+	simple	+	loop
$\eta = \frac{2\pi}{3}$	+	simple	0	closed
$\frac{2\pi}{3} < \eta < \frac{4\pi}{3}$	+	simple	-	simple
$\eta = \frac{4\pi}{3}$	0	closed	-	simple
$\frac{4\pi}{3} < \eta$	-	loop	-	simple

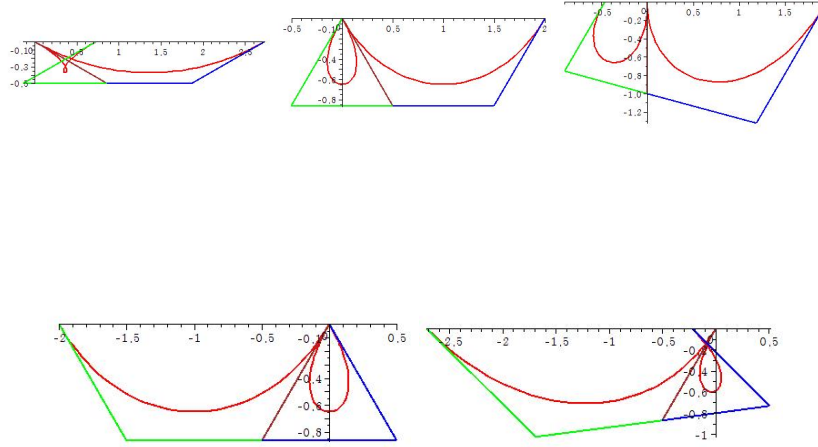


FIGURE 4. $\mathbf{r}_{+1}(t)$ and $\mathbf{r}_{-1}(t)$ in Theorem 3.4 with $(\theta_i, \theta_f) = (-\frac{\pi}{6}, \frac{\pi}{6}), (-\frac{\pi}{3}, -\frac{\pi}{3}), (-\frac{\pi}{2}, \frac{\pi}{3}), (-\frac{2\pi}{3}, \frac{2\pi}{3}), (-\frac{2\pi}{3}, \frac{4\pi}{3})$, respectively

3.5. Case of $-\pi < \theta_i < 0$ and ($-\pi < \theta_i \leq \theta_f < 0$ or $\pi < \theta_f < \theta_i + 2\pi$)

From the conditions, we have

$$\sin \theta_i < 0, \quad \sin \theta_f < 0, \quad \sigma \cdot \sin \frac{\theta_i + \theta_f}{2} > 0$$

where

$$\sigma = \begin{cases} -1 & \text{if } -\pi < \theta_i \leq \theta_f < 0, \\ +1 & \text{if } \pi < \theta_f < \theta_i + 2\pi. \end{cases}$$

Since

$$\begin{aligned} \operatorname{Im} f_\sigma(r) &= r^2 \sin \theta_f + \sigma \cdot r \sin \frac{\theta_i + \theta_f}{2} + \sin \theta_i \\ &= \sin \theta_f \left[\left(r + \sigma \cdot \frac{1}{2} \frac{\sin \frac{\theta_i + \theta_f}{2}}{\sin \theta_f} \right)^2 - \frac{\sin^2 \frac{\theta_i + \theta_f}{2} - 4 \sin \theta_i \sin \theta_f}{4 \sin^2 \theta_f} \right], \end{aligned}$$

$\operatorname{Im} f_{-\sigma}(r)$ has no positive real root.

3.5.1. Case of $\sin^2 \frac{\theta_i + \theta_f}{2} - 4 \sin \theta_i \sin \theta_f \geq 0$. In this case, $\operatorname{Im} f_\sigma(r)$ has positive real roots

$$s_{\sigma,1} = \sigma \cdot (-A) + \sigma \sqrt{B} \quad \text{and} \quad s_{\sigma,2} = \sigma \cdot (-A) - \sigma \sqrt{B}$$

where

$$A = \frac{\sin \frac{\theta_i + \theta_f}{2}}{2 \sin \theta_f}, \quad B = \frac{\sin^2 \frac{\theta_i + \theta_f}{2} - 4 \sin \theta_i \sin \theta_f}{4 \sin^2 \theta_f}.$$

We note that $s_{\sigma,1} = s_{\sigma,2}$ if and only if $\sin^2 \frac{\theta_i + \theta_f}{2} - 4 \sin \theta_i \sin \theta_f = 0$. Thus we obtain the PH cubics $\mathbf{r}_{\sigma,\tau}(t)$, which are expressed by

$$\begin{aligned} \mathbf{r}_{\sigma,\tau}(t) &= \mathbf{p}_1 3(1-t)^2 t + \mathbf{p}_2 3(1-t)t^2 + \mathbf{p}_3 t^3 \\ &= e^{\theta_i} [3(1-t)^2 t + 3(1-t)t^2 + t^3] + e^{\theta_f} (s_{\sigma,\tau}^2 t^3) \\ &\quad - e^{(\theta_i + \theta_f)/2} s_{\sigma,\tau} [3(1-t)t^2 + t^3], \end{aligned}$$

with $\tau = 1$ and 2.

We can determine whether or not the PH cubic $\mathbf{r}_{\sigma,\tau}(t)$ has a loop, as follows:

- (a) If $\mathbf{r}_{\sigma,\tau}(1) = 0$, the PH cubic $\mathbf{r}_{\sigma,\tau}(t)$ is a closed curve, thus has a loop.
- (b) Suppose that $\sigma \cdot \mathbf{r}_{\sigma,\tau}(1) < 0$. Then the PH cubic $\mathbf{r}_{\sigma,\tau}(t)$ has a loop if and only if $\mathbf{r}_{\sigma,\tau}(t)$ lies on the ray from 0 to the direction $(-\sigma)e^{(\theta_i + \theta_f)/2}$ for some $0 < t < 1$. This condition is equivalent to

$$\begin{cases} 3(1-t)^2 t + 3(1-t)t^2 + t^3 = s_{\sigma,\tau}^2 t^3, \\ (-\sigma) 2s_{\sigma,\tau}^2 t^3 \cdot \cos \frac{\theta_f - \theta_i}{2} \geq s_{\sigma,\tau} [3(1-t)t^2 + t^3] \end{cases} \quad (16)$$

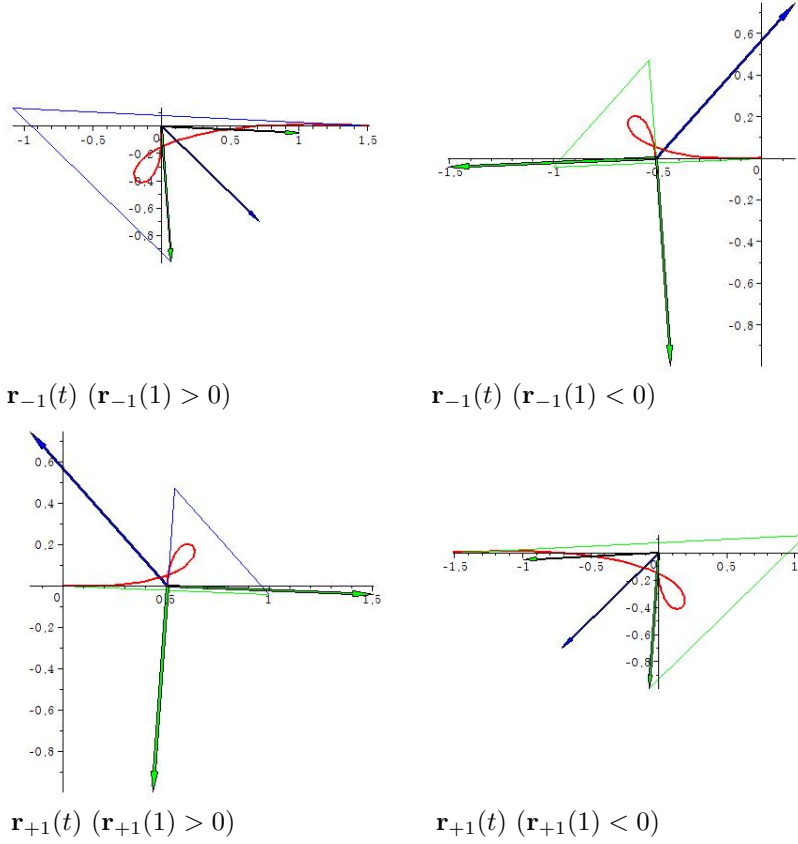


FIGURE 5. A criterion for determining whether the interpolant has a loop or not

for some $0 < t < 1$. We simplify (16) as

$$\begin{cases} (s_{\sigma,\tau} - 1)t^2 + 3t - 3 = 0, \\ (-\sigma) \cdot 2s_{\sigma,\tau}t \cdot \cos \frac{\theta_f - \theta_i}{2} \geq 3 - 2t \end{cases} \quad (17)$$

for some $0 < t < 1$. Since $s_{\sigma,\tau} > 1$, from the first equation in (16), we obtain

$$T_{\sigma\tau,1} = \frac{-3 + \sqrt{9 + 12(s_{\sigma,\tau}^2 - 1)}}{2(s_{\sigma,\tau}^2 - 1)}.$$

Finally we see that the PH cubic $\mathbf{r}_{\sigma,\tau}(t)$ has a loop if and only if $M_{\sigma\tau,1} \geq 0$, where

$$M_{\sigma\tau,1} = (-\sigma) \cdot 2s_{\sigma,\tau}T_{\sigma\tau,1} \cdot \cos \frac{\theta_f - \theta_i}{2} - 3 + 2T_{\sigma\tau,1}.$$

(c) Suppose that $\sigma \cdot \mathbf{r}_{\sigma,\tau}(1) > 0$. Then the PH cubic $\mathbf{r}_{\sigma,\tau}(t)$ has a loop if and only if $\mathbf{r}_{\sigma,\tau}(t) - \mathbf{r}_{\sigma,\tau}(1)$ lies on the ray from $\mathbf{r}_{\sigma,\tau}(1)$ to the direction $\sigma e^{(\theta_i + \theta_f)/2}$ for some $0 < t < 1$. This condition is equivalent to

$$\begin{cases} 3(1-t)^2 t + 3(1-t)t^2 + t^3 - 1 = s_{\sigma,\tau}^2(t^3 - 1), \\ (-\sigma) \cdot 2s_{\sigma,\tau}^2(t^3 - 1) \cdot \cos \frac{\theta_f - \theta_i}{2} \leq s_{\sigma,\tau}[3(1-t)t^2 + t^3 - 1] \end{cases} \quad (18)$$

for some $0 < t < 1$. We simplify (18) as

$$\begin{cases} (t-1)[(1-s_{\sigma,\tau}^2)t^2 - (2+s_{\sigma,\tau}^2)t + (1-s_{\sigma,\tau}^2)] = 0, \\ (-\sigma) \cdot 2s_{\sigma,\tau}^2(t^3 - 1) \cdot \cos \frac{\theta_f - \theta_i}{2} \leq s_{\sigma,\tau}(-2t^3 + 3t^2 - 1) \end{cases} \quad (19)$$

for some $0 < t < 1$. Since $s_{\sigma,\tau} < 1$, from the first equation in (18), we obtain

$$T_{\sigma\tau,2} = \frac{2 + s_{\sigma,\tau}^2 - \sqrt{3s_{\sigma,\tau}^2(4 - s_{\sigma,\tau}^2)}}{2(1 - s_{\sigma,\tau}^2)}.$$

Finally we see that the PH cubic $\mathbf{r}_{\sigma,\tau}(t)$ has a loop if and only if $M_{\sigma\tau,2} \leq 0$, where

$$M_{\sigma\tau,2} = (-\sigma) \cdot 2s_{\sigma,\tau}^2(T_{\sigma\tau,1}^3 - 1) \cdot \cos \frac{\theta_f - \theta_i}{2} - s_{\sigma,\tau}(-2T_{\sigma\tau,1}^3 + 3T_{\sigma\tau,1}^2 - 1).$$

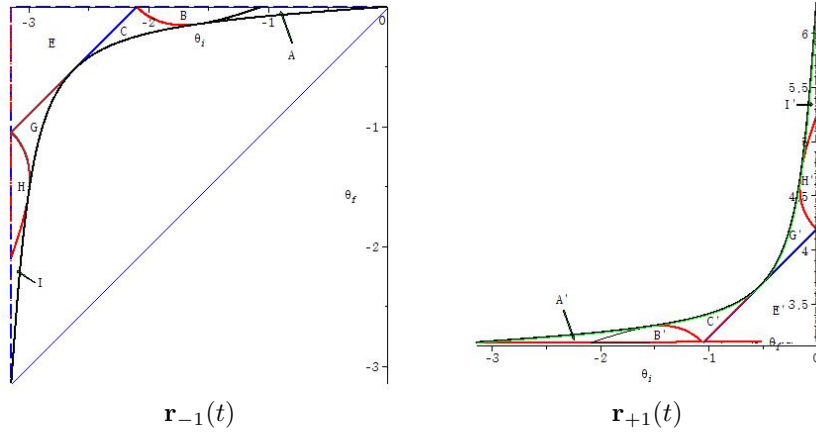
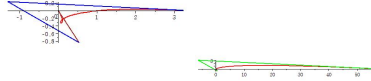
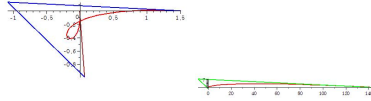


FIGURE 6. Domain of (θ_i, θ_f) for the Hermite interpolants $\mathbf{r}_\sigma(t)$ in Theorem 3.5

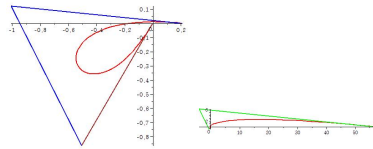
Theorem 3.5. Let θ_i and θ_f be constants with $-\pi < \theta_i < 0$ and $(-\pi < \theta_i \leq \theta_f < 0$ or $\pi < \theta_f < \theta_i + 2\pi)$.



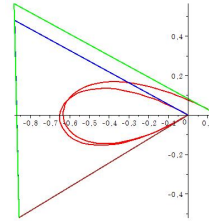
A: $\theta_i = -1, \theta_f = -0.05$



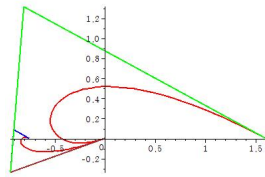
B: $\theta_i = -1.5, \theta_f = -0.05$



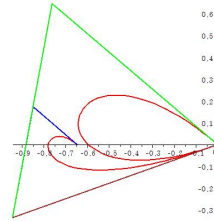
C: $\theta_i = -2.1, \theta_f = -0.1$



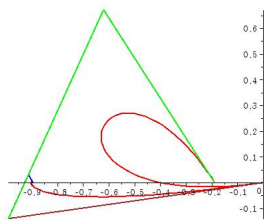
D: $\theta_i = -0.5 - \frac{2\pi}{3}, \theta_f = -0.5$



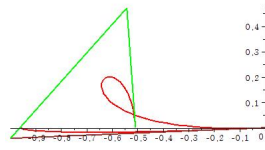
E: $\theta_i = -2.8, \theta_f = -0.5$



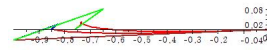
F: $\theta_i = -2.8, \theta_f = -2.8 + \frac{2\pi}{3}$



G: $\theta_i = -3.0, \theta_f = -1.0$



H: $\theta_i = -3.1, \theta_f = -1.5$



I: $\theta_i = -3.1, \theta_f = -2.5$

FIGURE 7. $\mathbf{r}_{-1,1}(t)$ and $\mathbf{r}_{-1,2}(t)$ in Theorem 3.5

(a) There are PH cubics $\mathbf{r}_{\sigma,\tau}(t)$, which satisfies

$$\mathbf{r}_{\sigma,\tau}(0) = \mathbf{0}, \quad \mathbf{r}_{\sigma,\tau}(1) \in \mathbb{R}, \quad \frac{\mathbf{r}'_{\sigma,\tau}(0)}{|\mathbf{r}'_{\sigma,\tau}(0)|} = e^{\theta_i}, \quad \frac{\mathbf{r}'_{\sigma,\tau}(1)}{|\mathbf{r}'_{\sigma,\tau}(1)|} = e^{\theta_f}, \quad (20)$$

for $\tau = 1$ and 2 ,

$$\sigma = \begin{cases} -1 & \text{if } -\pi < \theta_i \leq \theta_f < 0; \\ +1 & \text{if } \pi < \theta_f < \theta_i + 2\pi, \end{cases}$$

and which have the first directional leg \mathbf{L}_1 of length 1 in the Bézier control polygon, if and only if, $\sin^2 \frac{\theta_i + \theta_f}{2} - 4 \sin \theta_i \sin \theta_f \geq 0$. Moreover, $\mathbf{r}_{\sigma,1} = \mathbf{r}_{\sigma,2}$ if and only if $\sin^2 \frac{\theta_i + \theta_f}{2} - 4 \sin \theta_i \sin \theta_f = 0$.

(b) The PH cubics $\mathbf{r}_{\sigma,\tau}(t)$ are expressed by

$$\mathbf{r}_{\sigma,\tau}(t) = \mathbf{p}_1 3(1-t)^2 t + \mathbf{p}_2 3(1-t)t^2 + \mathbf{p}_3 t^3,$$

where

$$\mathbf{p}_1 = e^{i\theta_i}, \quad \mathbf{p}_2 = e^{i\theta_i} + \sigma s_{\sigma,\tau} e^{i(\theta_i + \theta_f)/2}, \quad \mathbf{p}_3 = e^{i\theta_f} + \sigma s_{\sigma,\tau} e^{i(\theta_i + \theta_f)/2} + s_{\sigma,\tau}^2 e^{i\theta_f},$$

with $s_{\sigma,1} = \sigma \cdot (-A) + \sigma\sqrt{B}$ and $s_{\sigma,2} = \sigma \cdot (-A) - \sigma\sqrt{B}$, where

$$A = \frac{\sin \frac{\theta_i + \theta_f}{2}}{2 \sin \theta_f}, \quad B = \frac{\sin^2 \frac{\theta_i + \theta_f}{2} - 4 \sin \theta_f \sin \theta_i}{4 \sin^2 \theta_f}.$$

(c) The table shows the shapes of the PH cubics $\mathbf{r}_{-1,\tau}(t)$:

$\eta = \theta_f - \theta_i, \psi = \theta_i + \theta_f$	$\mathbf{r}_{-1,1}(1)$	$\mathbf{r}_{-1,1}(t)$	$\mathbf{r}_{-1,2}(1)$	$\mathbf{r}_{-1,2}(t)$
A: $-\pi < \psi, \eta < \frac{2\pi}{3}, M_{-2,1} \geq 0$	+	loop	+	loop
B: $-\pi < \psi, \eta < \frac{2\pi}{3}, M_{-2,1} < 0, M_{-1,1} \geq 0$	+	loop	+	simple
C: $-\pi < \psi, \eta < \frac{2\pi}{3}, M_{-1,1} < 0$	+	simple	+	simple
D: $-\pi < \psi, \eta = \frac{2\pi}{3}$	0	closed	+	simple
E: $\eta > \frac{2\pi}{3}$	-	simple	+	simple
F: $\psi < -\pi, \eta = \frac{2\pi}{3}$	-	simple	0	closed
G: $\psi < -\pi, \eta < \frac{2\pi}{3}, M_{-2,2} > 0$	-	simple	-	simple
H: $\psi < -\pi, \eta < \frac{2\pi}{3}, M_{-2,2} \leq 0, M_{-1,2} > 0$	-	simple	-	loop
I: $\psi < -\pi, \eta < \frac{2\pi}{3}, M_{-1,2} \leq 0$	-	loop	-	loop

(d) The table shows the shapes of the PH cubics $\mathbf{r}_{+1,\tau}(t)$:

$\eta = \theta_f - \theta_i, \psi = \theta_i + \theta_f$	$\mathbf{r}_{+1,1}(1)$	$\mathbf{r}_{+1,1}(t)$	$\mathbf{r}_{+1,2}(1)$	$\mathbf{r}_{+1,2}(t)$
$A': \psi < \pi, \eta > \frac{4\pi}{3}, M_{+2,1} \geq 0$	–	loop	–	loop
$B': \psi < \pi, \eta > \frac{4\pi}{3}, M_{+2,1} < 0, M_{+1,1} \geq 0$	–	loop	–	simple
$C': \psi < \pi, \eta > \frac{4\pi}{3}, M_{+1,1} < 0$	–	simple	–	simple
$D': \psi < \pi, \eta = \frac{4\pi}{3}$	0	closed	–	simple
$E': \eta < \frac{4\pi}{3}$	+	simple	–	simple
$F': \psi > \pi, \eta = \frac{4\pi}{3}$	+	simple	0	closed
$G': \psi > \pi, \eta > \frac{4\pi}{3}, M_{+2,2} > 0$	+	simple	+	simple
$H': \psi > \pi, \eta > \frac{4\pi}{3}, M_{+2,2} \leq 0, M_{+1,2} > 0$	+	simple	+	loop
$I': \psi > \pi, \eta > \frac{4\pi}{3}, M_{+1,2} \leq 0$	+	loop	+	loop

3.5.2. Case of $\sin^2 \frac{\theta_i + \theta_f}{2} - 4 \sin \theta_i \sin \theta_f < 0$. In this case, $\text{Im } f_\sigma(r)$ has no positive real root.

4. Conclusion

In this paper, we present the geometric Hermite interpolation for planar PH cubics. For every Hermite data, we determine the exact number of the G^1 Hermite interpolants and also explicitly represent the desired interpolants. Moreover we also find a simple and self-contained criterion for determining whether the interpolants contain a loop or not. We want to apply the methodology to the spatial case.

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