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CHARACTERIZATION OF BMO OR LIPSCHITZ FUNCTIONS BY GARSIA-TYPE NORMS ON A BOUNDED DOMAIN

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Abstract. In this paper, we prove that the BMO norm and the Garsia norm are equivalent on a bounded domain in $\mathbb{R}^N$. Also, we investigate the equivalent relation between the Lipschitz norm and the Garsia-type norm for harmonic functions.

1. Introduction and statement of results

Let D be a bounded domain with C^2 boundary in $\mathbb{R}^N$. This means that there is a C^2, real-valued function ρ such that

$$D = \{x \in \mathbb{R}^N : \rho(x) < 0\}$$

and $\nabla\rho \neq 0$ on ∂D. From now on, in this paper, we assume that D is a bounded domain in $\mathbb{R}^N$ with C^2 defining function ρ.

There exists the Poisson kernel $P : D \times \partial D \to \mathbb{R}^+$ satisfying reproducing property for harmonic functions. The Poisson transform of a continuous function f on ∂D is defined by

$$\mathcal{P}f(x) = \int_{\partial D} P(x, y) f(y) d\sigma(y), \quad x \in D,$$

where $d\sigma$ is the surface measure of the boundary of D.

For $r > 0$, we denote the Euclidean metric ball in the boundary by $Q = \{y \in \partial D : |y - \tilde{x}| < r\}$, where $\tilde{x}$ is a boundary point. The integral mean f_Q is defined by $f_Q = \frac{1}{\sigma(Q)} \int_Q f d\sigma$. We define the BMO norm as follwos:

$$\|f\|^2_{BMO} = \sup_Q \frac{1}{\sigma(Q)} \int_Q |f - f_Q|^2 d\sigma.$$

The space BMO of bounded mean oscillation is a set of all L^2 function on the boundary ∂D with finite norm $\|f\|_{BMO} < \infty$.

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Further, with $f \in L^2(\partial D)$ we associate the nonnegative function

$$\mathcal{G}_f(x) = \int_{y \in \partial D} |f(y) - \mathcal{P}f(x)|^2 P(x,y) d\sigma(y).$$

The Garsia norm is defined by

$$\|f\|_G^2 = \sup\{\mathcal{G}_f(x) : x \in D\}, \; f \in L^2(\partial D).$$

Theorem 1.1. *Let $f \in L^2(\partial D)$. Then*

$$\|f\|_{BMO} < \infty \text{ if and only if } \|f\|_G < \infty.$$

For the unit ball in $\mathbb{C}^n$ the BMO norm is defined by using the non-isotropic ball in the unit sphere. The same result as Theorem 1.1 on the unit ball in $\mathbb{C}^n$ was proved by Garsia (see [5], one-dimensional case) and by Axler-Shapiro (see [1], n-dimensional case).

Let E be a bounded subset of $\mathbb{R}^N$ and f is a function on E. For $0 < \alpha < 1$, we set

$$\Delta_{f,\alpha}(x,y) = \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

and

$$\Delta_{f,\alpha}(E) = \sup\{\Delta_{f,\alpha}(x,y) : x, y \in E, x \neq y\}.$$

We define the Lipschitz norm by $\|f\|_{\Lambda_\alpha(E)} = \Delta_{f,\alpha}(E) + \sup|f|$. Let $\Lambda_\alpha(E)$ be the set of all functions satisfying $\|f\|_{\Lambda_\alpha(E)} < \infty$. It is called the *Lipschitz space of order* α on E.

We see that the Poisson transform $\mathcal{P} : \Lambda_\alpha(\partial D) \mapsto \Lambda_\alpha(D)$ is bounded. Thus we can induce the following:

$$\|f\|_{\Lambda_\alpha(D)} \sim \|f\|_{\Lambda_\alpha(\partial D)} \sim \|f\|_{\Lambda_\alpha(\bar{D})}.$$

for all harmonic functions $f \in C(\bar{D})$. The inequality $\|f\|_{\Lambda_\alpha(\partial D)} \leq \|f\|_{\Lambda_\alpha(D)}$ can be easily checked without the harmonic condition. However, the converse is false if f is not harmonic by the following example.

Example 1.2. Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^N$. We define functions $f_n(x) = n|x|^\alpha - n$ on $\mathbb{B}$. Then $\|f_n\|_{\Lambda_\alpha(\partial\mathbb{B})} = 0$ and $\|f_n\|_{\Lambda_\alpha(\mathbb{B})} \geq n \to \infty$ as $n \to \infty$. □

Now we will introduce four quantities which are closely related with the Lipschitz norm. First, we define the Garsia-type norm by

$$\mathcal{G}_{f,\alpha} = \sup_{x \in D} \frac{1}{\delta(x)^\alpha} \int_{y \in \partial D} |f(y) - \mathcal{P}f(x)| P(x,y) d\sigma(y), \; f \in L^1(\partial D), \quad (1)$$

where $\delta(x) = \operatorname{dist}(x, \partial D)$.

Well-known Hardy-Littlewood lemma tells us that [7]

$$\Delta_{f,\alpha}(D) \lesssim \sup_{x \in D} \delta^{1-\alpha}(x)|\nabla f(x)| + \sup_{x \in D} |f(x)|.$$

Furthermore, for all harmonic functions f,

$$\sup_{x \in D} \delta^{1-\alpha}(x)|\nabla f(x)| \lesssim \Delta_{f,\alpha}(D).$$

We define $HL_\alpha(\nabla f)$ by

$$HL_\alpha(\nabla f) = \sup_{x\in D} \delta(x)^{1-\alpha}|\nabla f(x)|,$$

for any differentiable functions f, where HL stands for the Hardy-Littlewood quantity.

Next two theorems tell us the relation between four quantities $\Delta_{f,\alpha}(D)$, $\Delta_{f,\alpha}(\partial D)$, $HL_\alpha(\nabla f)$, and $\mathcal{G}_{f,\alpha}$.

Theorem 1.3.

$$\mathcal{G}_{f,\alpha} \lesssim \Delta_{f,\alpha}(\partial D) \lesssim \Delta_{f,\alpha}(D) \lesssim HL_\alpha(\nabla f) + \sup_D |f|.$$

We define several kinds of Lipschitz norms as following:

$$\begin{cases} \|f\|_{\Lambda_\alpha(D)} = \Delta_{f,\alpha}(D) + \sup_D |f| \\ \|f\|_{\Lambda_\alpha(\partial D)} = \Delta_{f,\alpha}(\partial D) + \sup_{\partial D} |f| \\ \|f\|_{HL_\alpha} = HL_\alpha(\nabla f) + \sup_D |f| \\ \|f\|_{G,\alpha} = \mathcal{G}_{f,\alpha} + \sup_D |f|. \end{cases} \tag{2}$$

We know that if h is harmonic, then $\|f\|_{\Lambda_\alpha(D)}, \|f\|_{\Lambda_\alpha(\partial D)}$, and $\|f\|_{HL_\alpha}$ are equnvalent.

Next theorem tell us that $\|f\|_{G,\alpha}$ is equivalent to the other Lipschitz norms.

Theorem 1.4. *For $0 < \alpha < 1$ the Garsia-type norm $\|f\|_{G,\alpha}$ is equivalent to other Lipschitz norms in (2) for the harmonic and $L^1(\partial D)$ function f.*

Further, we define the Garsia-type p-norm $\mathcal{G}_{f,\alpha,p}$ using L^p integral by

$$\mathcal{G}_{f,\alpha,p} = \sup_{x\in D} \frac{1}{\delta(x)^\alpha} \left\{ \int_{y\in\partial D} |f(y) - \mathcal{P}f(x)|^p P(x,y) d\sigma(y) \right\}^{1/p}.$$

We have the following equivalence between $G_{f,\alpha,p}$ and the Lipschitz norm. This can be achieved by the same technique as in Theorem 1.4 using the Hölder inequality.

Corollary 1.5. *Let $p \geq 1$ and $0 < \alpha < 1/p$. Then*

$$\mathcal{G}_{f,\alpha,p} + \sup_{x\in D} |f(x)| \sim \|f\|_{\Lambda_\alpha(D)}$$

for all harmonic functions f.

The following example shows that the condition $0 < \alpha < 1/p$ in Corollary 1.5 is essential.

Example 1.6. Let $p = 2$ and $\mathbb{B}^2$ is the unit ball in $\mathbb{R}^2$. Define a harmonic function f by $f(x_1, x_2) = x_1 + x_2$. Then $f \in \Lambda_\alpha(\mathbb{B}^2)$. Denote x by (x_1, x_2). By easy computation, we can compute the following integral

$$\begin{aligned} \int_{|y|=1} |f(y) - \mathcal{P}f(x)|^2 P(x,y) d\sigma(y) &= \int_{|y|=1} |f(y)|^2 P(x,y) d\sigma(y) - |f(x)|^2 \\ &= 1 - |x|^2. \end{aligned}$$

Hence
$$\mathcal{G}_{f,\alpha,2} = \sup_{x\in\mathbb{B}^2} (1-|x|^2)^{1/2-\alpha}.$$
This is unbounded if α is greater than $1/2$.

There are more general weighted Lipschitz spaces which are extensively studied in ([2], [3], [4]), and so on. Especially, in [3] they proved the same results in weighted Lipschitz spaces like as ours. However, they considered the holomorphic case on the unit ball in $\mathbb{C}^n$.

2. Integral estimates

We have the following size estimates for the Poisson kernel in ([6], [8]):
$$\left|\nabla_x^k P(x,y)\right| \sim \frac{\delta(x)}{|x-y|^{N+k}}, \quad \text{for} \quad k=0,1. \tag{3}$$
For $x \in D$ let $\tilde{x}$ be the boundary point with $\delta(x) = \operatorname{dist}(x,\tilde{x})$.

Lemma 2.1. *Let $0<\alpha<1$. Then we have*
$$\int_{y\in\partial D} |y-\tilde{x}|^\alpha P(\zeta,z)d\sigma(y) \lesssim \delta(x)^\alpha \quad \textit{for all} \quad x\in D.$$

Proof. Since $|y-\tilde{x}| \le |y-x| + |x-\tilde{x}| \le 2|y-x|$, we have
$$|y-\tilde{x}|^\alpha P(x,y) \lesssim \frac{\delta(x)}{|x-y|^{N-\alpha}}.$$
Thus it is sufficient only to prove that
$$\int_{y\in\partial D} \frac{1}{|x-y|^{N-\alpha}} d\sigma(y) \lesssim \frac{1}{\delta(x)^{1-\alpha}} \quad \text{for all} \quad x\in D.$$

Let $Q_k = \{y\in\partial D : |y-\tilde{x}| < 2^k\delta(x)\}$ for all $k=1,2,\cdots$. Then Q_k is a covering of the boundary of D. We can compute the above integral on Q_1. We obtain that
$$\begin{aligned}\int_{Q_1} \frac{1}{|x-y|^{N-\alpha}}\, d\sigma(y) &\le \int_{Q_1} \frac{1}{\delta(x)^{N-\alpha}}\, d\sigma(y)\\ &\lesssim \frac{1}{\delta(x)^{1-\alpha}}.\end{aligned}$$
For $y\in Q_k\setminus Q_{k-1} (k\ge 2)$, by the triangle inequality, we get the following
$$|x-y| \ge |y-\tilde{x}| - |\tilde{x}-x| = |y-\tilde{x}| - \delta(x) \ge 2^{k-1}\delta(x) - \delta(x) \ge 2^{k-2}\delta(x).$$
Thus we have
$$\begin{aligned}\int_{Q_k\setminus Q_{k-1}} \frac{1}{|x-y|^{N-\alpha}}\, d\sigma(y) &\le \int_{Q_k\setminus Q_{k-1}} \frac{1}{(2^{k-2}\delta(x))^{N-\alpha}}\, d\sigma(y)\\ &\lesssim \frac{1}{2^{k(1-\alpha)}}\frac{1}{\delta(x)^{1-\alpha}}.\end{aligned}$$

Since the series $\sum_{k=2}^{\infty} 1/2^{k(1-\alpha)}$ converges, we finally arrive that

$$\begin{aligned}\int_{y\in\partial D} \frac{1}{|x-y|^{N-\alpha}}\ d\sigma(y) &\le \int_{Q_1} + \sum_{k=2}^{\infty} \int_{Q_k\setminus Q_{k-1}} \frac{1}{|x-y|^{N-\alpha}} d\sigma(y)\\ &\lesssim \sum_{k=0}^{\infty} \frac{1}{2^{k(1-\alpha)}} \frac{1}{\delta(x)^{1-\alpha}}\\ &\lesssim \frac{1}{\delta(x)^{1-\alpha}}.\end{aligned}$$

This is the end of the proof. □

3. *BMO* functions

Proof of Theorem 1.1. We suppose $\|f\|_G < \infty$. Let $x \in D$ and $Q = Q(\tilde{x}, \delta(x)) = \{y \in \partial D : |y-\tilde{x}| < \delta(x)\}$. Then

$$\begin{aligned}\mathcal{G}_f(x) &= \int_{\partial D} |f(y) - \mathcal{P}f(x)|^2 P(x,y) d\sigma(y)\\ &\gtrsim \int_Q |f(y) - \mathcal{P}f(x)|^2 \frac{\delta(x)}{(2\delta(x))^N} d\sigma(y)\\ &\gtrsim \frac{1}{\sigma(Q)} \int_Q |f(y) - \mathcal{P}f(x)|^2 d\sigma(y).\end{aligned}$$

As x runs over $\{x \in D : \delta(x) < r_0\}$, the above Q runs all balls of radius less than r_0. By Lemma 5.1 in [9], we have $\|f\|_{BMO} \lesssim \|f\|_G$.

For the other implication, we suppose that $\|f\|_{BMO} < \infty$. Since $P(x,y)$ is smooth on ∂D, for $f \in L^2(\partial D)$, there exists $r_0 > 0$ such that

$$\sup_{\substack{x\in D,\\ \delta(x)\ge r_0}} \mathcal{P}(|f - \mathcal{P}f(x)|^2) < +\infty.$$

Fix $x \in D$ with $\delta(x) < r_0$. Let $\tilde{x}$ be the boundary point with $\delta(x) = \operatorname{dist}(x, \tilde{x})$. Let $Q_k = \{y \in \partial D : |y - \tilde{x}| \le 2^k \delta(x)\}$ for $k \ge 1$. Then Q_k is a covering of ∂D. In order to estimate $P(|f - \mathcal{P}f(x)|^2)$, we first compute $P(|f - f_{Q_1}|^2)$. By covering property,

$$\begin{aligned}\int_{\partial D} |f(y) - f_{Q_1}|^2 P(x,y) d\sigma(y) &= \int_{Q_1} |f(y) - f_{Q_1}|^2 P(x,y) d\sigma(y)\\ &\quad + \sum_{k=2}^{\infty} \int_{Q_k\setminus Q_{k-1}} |f(y) - f_{Q_1}|^2 P(x,y) d\sigma(y).\end{aligned} \tag{4}$$

Now, we will compute each term in the equation above. We have

$$\begin{aligned}\int_{Q_1}|f(y)-f_{Q_1}|^2P(x,y)d\sigma(y) &\lesssim \int_{Q_1}|f(y)-f_{Q_1}|^2\frac{\delta(x)}{|x-y|^N}d\sigma(y)\\ &\le \frac{1}{\delta(x)^{N-1}}\int_{Q_1}|f(y)-f_{Q_1}|^2d\sigma(y)\\ &\lesssim \frac{1}{\sigma(Q_1)}\int_{Q_1}|f(y)-f_{Q_1}|^2d\sigma(y)\lesssim \|f\|^2_{BMO}.\end{aligned} \tag{5}$$

For $k\ge 2$,

$$\begin{aligned}&\int_{Q_k\setminus Q_{k-1}}|f(y)-f_{Q_1}|^2P(x,y)d\sigma(y)\\ &\lesssim \int_{Q_k\setminus Q_{k-1}}|f(y)-f_{Q_1}|^2\frac{\delta(x)}{|x-y|^N}d\sigma(y)\\ &\le \int_{Q_k\setminus Q_{k-1}}|f(y)-f_{Q_1}|^2\frac{\delta(x)}{(2^{k-2}\delta(x))^N}d\sigma(y)\\ &\lesssim \frac{1}{2^k}\frac{1}{\sigma(Q_k)}\int_{Q_k}|f(y)-f_{Q_1}|^2d\sigma(y)\\ &\lesssim \frac{1}{2^k}\|f\|^2_{BMO}+\frac{1}{2^k}|f_{Q_k}-f_{Q_1}|^2.\end{aligned} \tag{6}$$

Note that $|f_{Q_k}-f_{Q_1}|^2\lesssim\sum_{j=2}^k k|f_{Q_j}-f_{Q_{j-1}}|^2$. For each $j=2,\dots,k$,

$$\begin{aligned}|f_{Q_k}-f_{Q_{k-1}}|^2 &= \left|f_{Q_k}-\frac{1}{\sigma(Q_{k-1})}\int_{Q_{k-1}}f\ d\sigma\right|^2\\ &\le \left(\frac{1}{\sigma(Q_{k-1})}\int_{Q_{k-1}}|f_{Q_k}-f|d\sigma\right)^2\\ &\lesssim \frac{1}{\sigma(Q_k)}\int_{Q_k}|f_{Q_k}-f|^2d\sigma\le\|f\|^2_{BMO}.\end{aligned} \tag{7}$$

Since series $\sum k/2^k$ converges, by (4), (5), (6), and (7), we have

$$\int_{\partial D}|f(y)-f_{Q_1}|^2P(x,y)d\sigma(y)\lesssim\|f\|^2_{BMO}.$$

We return to the estimate of $\mathcal{P}(|f-\mathcal{P}f(x)|^2)$. It follows that

$$\begin{aligned}&\int_{\partial D}|f(y)-f_{Q_1}|^2P(x,y)d\sigma(y)\\ &\quad=\int_{\partial D}|f(y)-\mathcal{P}f(x)+\mathcal{P}f(x)-f_{Q_1}|^2P(x,y)d\sigma(y)\\ &\quad=\int_{\partial D}|f(y)-\mathcal{P}f(x)|^2P(x,y)d\sigma(y)+|\mathcal{P}f(x)-f_{Q_1}|^2.\end{aligned}$$

The last equality is followed by reproducing property of the Poisson kernel. Therefore

$$\begin{aligned}\|f\|_G^2 &= \sup_{x\in D}\int_{\partial D}|f(y)-\mathcal{P}f(x)|^2P(x,y)d\sigma(y)\\ &< \sup_{x\in D}\int_{\partial D}|f(y)-f_{Q_1}|^2P(x,y)d\sigma(y)\\ &\lesssim \|f\|_{BMO}^2.\end{aligned}$$

□

4. Lipschitz functions

Proof of Theorem 1.3. It is enough to prove that $\mathcal{G}_{f,\alpha}\lesssim\Delta_{f,\alpha}(\partial D)$. Let $x\in D$ and $y\in\partial D$. Then

$$|f(y)-\mathcal{P}f(x)|\le|f(y)-f(\tilde{x})|+|f(\tilde{x})-\mathcal{P}f(x)|.$$

We have

$$\begin{aligned}|f(\tilde{x})-\mathcal{P}f(x)| &= \left|\int_{y\in\partial D}(f(\tilde{x})-f(y))P(x,y)d\sigma(y)\right|\\ &\lesssim \Delta_{f,\alpha}(\partial D)\int_{y\in\partial D}|\tilde{x}-y|^\alpha P(x,y)d\sigma(y)\\ &\lesssim \Delta_{f,\alpha}(\partial D)\delta(x)^\alpha\end{aligned}$$

by Lemma 2.1.

Thus we have

$$|f(y)-\mathcal{P}f(x)|\lesssim\Delta_{f,\alpha}(\partial D)(|y-\tilde{x}|^\alpha+\delta(x)^\alpha).$$

Therefore

$$\begin{aligned}&\int_{y\in\partial D}|f(y)-\mathcal{P}f(x)|P(x,y)d\sigma(y)\\ &\lesssim \Delta_{f,\alpha}(\partial D)\left\{\int_{y\in\partial D}|y-\tilde{x}|^\alpha P(x,y)d\sigma(y)+\delta(x)^\alpha\right\}\\ &\lesssim \Delta_{f,\alpha}(\partial D)\delta(x)^\alpha\end{aligned}$$

by Lemma 2.1. Hence we have $\mathcal{G}_{f,\alpha}\lesssim\Delta_{f,\alpha}(\partial D)$. □

Proof of Theorem 1.4. By Lemma 1.3, it is enough to prove that $HL_\alpha(\nabla f)\lesssim\|f\|_{G,\alpha}$.

Since f is harmonic, it follows that $f(x)=\int_{\partial D}f(y)P(x,y)d\sigma(y)$ for all x in D. If we differentiate the both side, we get

$$\begin{aligned}\nabla_x f(x) &= \int_{\partial D}f(y)\nabla_x P(x,y)\,d\sigma(y)\\ &= \int_{\partial D}(f(y)-\mathcal{P}f(x))\,\nabla_x P(x,y)\,d\sigma(y).\end{aligned}$$

By (3), we have

$$|\nabla_x P(x,y)| \lesssim \quad \delta(x)^{-1} \left| P(x,y) \right|.$$

Combining the above two inequalities, we get that

$$\delta(x)^{1-\alpha} |\nabla_x f(x)| \lesssim \delta(x)^{-\alpha} \int_{\partial D} |f(y) - f(x)| \, |P(x,y)| \; d\sigma(y). \tag{8}$$

□

By the following examples below, we know that the converse of each step of the inequalities in Theorem 1.3 are false if f is not harmonic except the first step.

Example 4.1. Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^N$. Define f by $f(x) = |x|^\alpha$. Then

$$\|f\|_{\Lambda_\alpha(\mathbb{B})} \lesssim 1.$$

However

$$\begin{aligned} HL_\alpha(\nabla f) &= \sup_{|x|<1} (1 - |x|)^{1-\alpha} |\nabla f(x)| \\ &\sim \sup_{|x|<1} (1 - |x|)^{1-\alpha} |x|^{\alpha - 1} = \infty. \end{aligned}$$

Example 4.2. Let $\chi(x)$ be a smooth function such that $0 \le \chi \le 1$ and

$$\chi(x) = \begin{cases} 1 & \text{if} \quad |x| \le \frac{1}{3} \\ 0 & \text{if} \quad |x| \ge \frac{2}{3}. \end{cases}$$

We define f_n by $f_n(x) = \chi(x) n |x|^\alpha$. Then

$$\begin{aligned} \Delta_{f_n,\alpha}(\mathbb{B}) &= \sup_{\substack{x,y \in \mathbb{B} \\ x \neq y}} \frac{|f_n(x) - f_n(y)|}{|x-y|^\alpha} \\ &\ge \sup_{|x| \le 1/3} \frac{n|x|^\alpha}{|x|^\alpha} = n. \end{aligned}$$

Thus $\|f_n\|_{\Lambda_\alpha(\mathbb{B})} \to \infty \quad \text{as} \quad n \to \infty$. However, $\|f_n\|_{\Lambda_\alpha(\partial\mathbb{B})} = 0$.

The only remaining relation is between $\mathcal{G}_{f,\alpha}$ and $\Delta_{f,\alpha}(\partial D)$.

Remark 1. Note that $\mathcal{P}f$ is harmonic and $\mathcal{P}f \equiv f$ on the boundary if f is continuous. Then $\Delta_{f,\alpha}(\partial D) \sim \Delta_{\mathcal{P}f,\alpha}(D) \sim \mathcal{G}_{\mathcal{P}f,\alpha}$ by Theorem 1.4. By the definition of $\mathcal{G}_{f,\alpha}$,

$$\mathcal{G}_{f,\alpha} = \mathcal{G}_{\mathcal{P}f,\alpha}.$$

Therefore, $\mathcal{G}_{f,\alpha} \sim \Delta_{f,\alpha}(\partial D)$ if f is continuous on the boundary.

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