PURE INJECTIVE REPRESENTATIONS OF QUIVERS

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Abstract. Let $R$ be a ring and $Q$ be a quiver. In this paper we give another definition of purity in the category of quiver representations. Under such definition we prove that the class of all pure injective representations of $Q$ by $R$-modules is preenveloping. In case $Q$ is a left rooted semi-co-barren quiver and $R$ is left Noetherian, we show that every cotorsion flat representation of $Q$ is pure injective. If, furthermore, $R$ is $n$-perfect and $F$ is a flat representation $Q$, then the pure injective dimension of $F$ is at most $n$.

1. Introduction

Throughout the paper rings are associative with identity and modules are unital (unless otherwise specified). If $Q$ is a quiver (a directed graph), then an arrow from a vertex $v_1$ to a vertex $v_2$ is denoted by $a : v_1 \rightarrow v_2$. The set of vertices (resp. arrows) of a quiver $Q$ is denoted by $V_Q$ (resp. $E_Q$). For a given arrow $a$ of $Q$, $i(a)$ denotes the initial vertex of $a$ and $t(a)$ denotes the terminal vertex of $a$. A quiver may be thought as a category in which the objects are vertices and the morphisms are paths, a path is a sequence of arrows. A representation $X$ by modules of a quiver $Q$ is then a covariant functor $X : Q \rightarrow \text{R-Mod}$. Thus a representation $X$ is determined by giving a module $X(v)$ to each vertex $v \in V_Q$ and a homomorphism $X(a) : X(v_1) \rightarrow X(v_2)$ to each arrow $a \in E_Q$. A morphism $f$ between representations $X$ and $Y$ is a natural transformation. The category of representations of a quiver $Q$ by left $R$-modules over a ring $R$ is denoted by $\text{Rep}(Q, R)$. This is a Grothendieck category with projective generators and injective cogenerators (see [1] and [2]).

A quiver $Q$ is called left(right) rooted if there is no path of the form $\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots$ in $Q$. The study of special objects in the category of representations of quivers has a long history in the literature. Especially flat representations of a left rooted quiver, and existence of flat covers in the category $\text{Rep}(Q, R)$, when $Q$ is left rooted, have been studied in [3], [8].

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Let $\text{Flat}(Q)$ be the class of all flat representations of a left rooted quiver $Q$ and $\text{Flat}(Q)^\perp$, be the class of all object $C$ of $\text{Rep}(Q, R)$ such that $\text{Ext}^1_Q(F, C) = 0$ for every $F \in \text{Flat}(Q)$. A representation $C$ of $Q$ is called cotorsion if $C \in \text{Flat}(Q)^\perp$. Therefore $\text{Flat}(Q)^\perp$ is the class of all cotorsion representations of $Q$. Cotorsion representations of a left rooted quiver are characterized in [11].

Let us first recall some notations and results of [2, Corollary 6.7] that we need throughout. Let $Q$ be a quiver. We can define the opposite quiver $Q^{\text{op}} = (V_Q, E_Q^{\text{op}})$ to $Q$ such that its set of vertices is $V_Q$ and its set of arrows is $E_Q^{\text{op}}$, in which $v \rightarrow w \in E_Q^{\text{op}}$ if and only if $w \rightarrow v \in E_Q$. Note that $\text{Rep}(Q^{\text{op}}, R^{\text{op}})$ is the category of representations of $Q^{\text{op}}$ by right $R$-modules.

Let $\mathcal{X}$ be a representation of $Q$, the representation $\mathcal{X}^+ \in \text{Rep}(Q^{\text{op}}, R^{\text{op}})$ is given by the following:

i) For any $v \in V_Q$, $\mathcal{X}^+ = \text{Hom}_Z(\mathcal{X}(v), Q/\mathbb{Z})$,

ii) For any $a \in E_Q$ such that $a: v \rightarrow w$, $\mathcal{X}^+(a): \mathcal{X}^+(w) \rightarrow \mathcal{X}^+(v)$.

Let $\text{Inj}(Q^{\text{op}})$ be the class of all injective representations of $Q^{\text{op}}$. By the following proposition, there is a fully faithful functor $\text{Flat}(Q) \longrightarrow \text{Inj}(Q^{\text{op}})$.

**Proposition 1.1.** Let $Q$ be a left rooted quiver. A representation $F$ of $Q$ is flat in $\text{Rep}(Q, R)$ if and only if $F^+$ is an injective object of $\text{Rep}(Q^{\text{op}}, R^{\text{op}})$.

**Proof.** See [2, Corollary 6.7]. □

The category $\text{Rep}(Q, R)$ is a locally finitely presented additive category, so there is a categorical notion of purity in terms of the finitely presented representations. By [9], every representation of $Q$ has a pure injective envelope. Actually if $F$ is a flat representation, it can be easily shown that the pure injective envelope (in the sense of [9]) is flat if, and only if, it coincides with its cotorsion envelope. Therefore the main result of this work, Theorem 3.6, can not be followed from [9]. So, for this end, we had to make a new definition of purity. We give many propositions to show that our notion of purity is well-behaved. For example in Theorem 2.13 we prove that the classical relation between flatness and purity is true in $\text{Rep}(Q, R)$.

It is possible that the categorical notion of purity and our notion of purity are the same, but our notion of purity has some advantage. For instance, let $0 \rightarrow \mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z} \rightarrow 0$ be an exact sequence of representations, then there exists the following commutative diagram with exact rows and pure exact columns

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{X}^+ \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{Y}^+ \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{Z}^+ \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

in $\text{Rep}(Q, R)$. But from the categorical notion of purity it is not clear if we can deduce this important diagram.
In Section 2 we give the definition of purity in Rep(\(Q, R\)) and study its properties, and show that under such definition each representation \(X\) of \(Q\) is a pure subrepresentation of a pure injective representation. In Section 3, over a left Noetherian ring we give necessary and sufficient conditions for \(X \in \text{Rep}(Q, R)\) to be flat and pure injective when \(Q\) is a left rooted semi-co-barren quiver. If \(R\) is \(n\)-perfect, then the finiteness of the length of pure injective resolution of \(X\) will be discussed.

**Setup:** Throughout this paper \(Q\) is a left rooted quiver.

## 2. Purity and pure injectivity

This section is devoted to the study of purity in Rep(\(Q, R\)). As a rich reference to the concepts of purity and pure injectivity and its properties in the category of \(R\)-modules, see [10], [12] and [13].

**Definition 2.1.** Let \(A\) be an abelian category and \(C\) be a class of objects of \(A\). For an object \(A\) of \(A\), an object \(C \in C\) is called a \(C\)-envelope of \(A\) if there is a morphism \(\varphi : A \rightarrow C\) such that the following hold.

(i) For any morphism \(\varphi' : A \rightarrow C'\) with \(C' \in C\), there is a morphism \(f : C \rightarrow C'\) with \(\varphi' = f \varphi\).

(ii) If an endomorphism \(f : C \rightarrow C\) is such that \(\varphi = f \varphi\), then \(f\) must be an automorphism.

If (i) holds, \(\varphi : A \rightarrow C\) is called a \(C\)-preenvelope. Sometimes we call \(C\) or the map \(\varphi\) a \(C\)-envelope (preenvelope) of \(A\). For more details on the concept of (pre)enveloping classes and their properties, see [6] and [12].

A morphism \(f : \mathcal{X} \rightarrow \mathcal{Y}\) of representations is called a monomorphism if \(f\) has a cancelation property from the left, that is if for each representation \(Z\) of \(Q\) and each morphisms \(g, h : Z \rightarrow \mathcal{X}\) which \(fh = fg\), then \(g = h\). On the other hand a morphism \(f : \mathcal{X} \rightarrow \mathcal{Y}\) is a monomorphism if and only if for every \(v \in V_Q\), the morphism \(f(v) : \mathcal{X}(v) \rightarrow \mathcal{Y}(v)\) is a monomorphism of \(R\)-modules. Now we make our definition of purity in Rep(\(Q, R\)).

**Definition 2.2.** A monomorphism \(f : \mathcal{X} \rightarrow \mathcal{Y}\) in Rep(\(Q, R\)) is a pure monomorphism if \(f^+ : \mathcal{Y}^+ \rightarrow \mathcal{X}^+\) is a split epimorphism in the category Rep(\(Q^{op}, R^{op}\)).

**Remark 2.3.** If \(\mathcal{X} \rightarrow \mathcal{Y}\) is a pure monomorphism in Rep(\(Q, R\)), then \(\mathcal{X}(v) \rightarrow \mathcal{Y}(v)\) is a pure monomorphism of \(R\)-modules for any vertex \(v\) of \(Q\). But the converse is not true. To see this let \(M\) be an arbitrary \(R\)-module. Consider the quiver \(Q : v_1 \xrightarrow{a} v_2\), and the representations \(\mathcal{M}_1 : M \xrightarrow{\mathcal{M}_1(a)} \hat{M} \oplus M\) (where \(\mathcal{M}_1(a)\) is injection) and \(\mathcal{M}_2 : M \oplus M \xrightarrow{\mathcal{M}_2(a)} M \oplus M\) (where \(\mathcal{M}_2(a)\) is identity). We see that \(\mathcal{M}_1(v_1) \rightarrow \mathcal{M}_2(v_1)\) and \(\mathcal{M}_1(v_2) \rightarrow \mathcal{M}_2(v_2)\) are pure monomorphisms of \(R\)-modules but \(\mathcal{M}_1 \xrightarrow{\theta} \mathcal{M}_2\), where \(\theta(a) = (\mathcal{M}_1(a), \mathcal{M}_2(a))\), is not pure in Rep(\(Q, R\)).
Example 2.4. Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a monomorphism in $\text{Rep}(Q,R)$ such that for each vertex $v$ of $Q$, $\mathcal{X}(v) \rightarrow \mathcal{Y}(v)$ is a pure monomorphism of $R$-modules, and for each arrow $a : v \rightarrow w$, $\mathcal{X}(a) : \mathcal{X}(v) \rightarrow \mathcal{X}(w)$ is a split epimorphism. Then $\mathcal{X} \rightarrow \mathcal{Y}$ is a pure monomorphism in $\text{Rep}(Q,R)$.

Proposition 2.5. (i) Any split exact sequence

$$0 \rightarrow \mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z} \rightarrow 0$$

in $\text{Rep}(Q,R)$ is pure exact.

(ii) Any direct limit of pure exact sequences is pure.

(iii) Let $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{Z}$ be a sequence of subrepresentations of $\mathcal{Z}$. If $\mathcal{X}$ is a pure subrepresentation of $\mathcal{Z}$, then it is also pure as a subrepresentation of $\mathcal{Y}$.

Proof. (i) Since the exact sequence

$$0 \rightarrow \mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z} \rightarrow 0$$

is split, $\mathcal{Y} = \mathcal{X} \oplus \mathcal{Z}$. Therefore $\mathcal{Y}^+ = \mathcal{X}^+ \oplus \mathcal{Z}^+$ and so

$$0 \rightarrow \mathcal{Z}^+ \rightarrow \mathcal{Y}^+ \rightarrow \mathcal{X}^+ \rightarrow 0$$

is split exact in $\text{Rep}(Q^{\text{op}}, R^{\text{op}})$. Thus

$$0 \rightarrow \mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z} \rightarrow 0$$

is pure exact sequence in $\text{Rep}(Q,R)$.

(ii) Let

$$(0 \rightarrow \mathcal{X}_i \rightarrow \mathcal{Y}_i \rightarrow \mathcal{Z}_i \rightarrow 0)_{i \in I}$$

be a direct system of pure exact sequences in $\text{Rep}(Q,R)$. Then

$$(0 \rightarrow \mathcal{Z}_i^+ \rightarrow \mathcal{Y}_i^+ \rightarrow \mathcal{X}_i^+ \rightarrow 0)_{i \in I}$$

is an inverse system of split exact sequences in $\text{Rep}(Q^{\text{op}}, R^{\text{op}})$. Therefore, for each $i \in I$ we have the following split short exact sequence

$$0 \rightarrow \lim_{\leftarrow i \in I} \mathcal{Z}_i^+ \rightarrow \lim_{\leftarrow i \in I} \mathcal{Y}_i^+ \rightarrow \lim_{\leftarrow i \in I} \mathcal{X}_i^+ \rightarrow 0,$$

and then we have the split short exact sequence

$$0 \rightarrow (\lim_{\leftarrow i \in I} \mathcal{Z}_i^+)^+ \rightarrow (\lim_{\leftarrow i \in I} \mathcal{Y}_i^+)^+ \rightarrow (\lim_{\leftarrow i \in I} \mathcal{X}_i^+)^+ \rightarrow 0.$$ 

This implies that the short exact sequence

$$0 \rightarrow \lim_{\leftarrow i \in I} \mathcal{X}_i \rightarrow \lim_{\leftarrow i \in I} \mathcal{Y}_i \rightarrow \lim_{\leftarrow i \in I} \mathcal{Z}_i \rightarrow 0,$$

is pure exact sequence in $\text{Rep}(Q,R)$.

(iii) Let

$$\mathcal{X} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$$
be a sequence of subrepresentations in \( \text{Rep}(Q, R) \). We have the following commutative diagram in \( \text{Rep}(Q, R) \):

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow i_2 \\
\mathcal{Y} \\
\downarrow i_3 \\
\mathcal{Z}
\end{array}
\]

Therefore we have the commutative diagram:

\[
\begin{array}{c}
\mathcal{Z}^+ \\
\downarrow i_2^+ \\
\mathcal{X}^+ \\
\downarrow i_3^+ \\
\mathcal{Y}^+
\end{array}
\]

Rep\(\text{Rep}(Q^{op}, R^{op})\), and there exists \( k_1 : \mathcal{X}^+ \rightarrow \mathcal{Z}^+ \) such that \( i_1^+ k_1 = 1_{\mathcal{X}^+} \). So

\[
i_2^+ (i_3^+ k_1) = (i_2^+ i_3^+ k_1) = i_1^+ k_1 = 1_{\mathcal{X}^+}.
\]

Thus the composition

\[
\mathcal{Y}^+ \xrightarrow{i_1^+} \mathcal{X}^+ \xrightarrow{i_2^+ k_1} \mathcal{Y}^+
\]

is \( 1_{\mathcal{X}^+} \). Therefore

\[
\mathcal{Y}^+ \xrightarrow{i_1^+} \mathcal{X}^+
\]

admits a section, and hence \( \mathcal{X} \hookrightarrow \mathcal{Y} \) is a pure monomorphism in \( \text{Rep}(Q, R) \).

Proposition 2.6. Let

\[
\mathcal{E} : 0 \rightarrow \mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z} \rightarrow 0
\]

be an exact sequence in \( \text{Rep}(Q, R) \).

(i) Let \( \mathcal{E} \) be a pure exact sequence in \( \text{Rep}(Q, R) \). Then \( \mathcal{Y} \) is flat in \( \text{Rep}(Q, R) \) if and only if \( \mathcal{X} \) and \( \mathcal{Z} \) are flat.

(ii) Let \( \mathcal{Z} \) be a flat object of \( \text{Rep}(Q, R) \). Then \( \mathcal{X} \) is a flat object of \( \text{Rep}(Q, R) \) if and only if \( \mathcal{Y} \) is flat.

Proof. (i) Let \( \mathcal{E} \) be a pure exact sequence of \( \text{Rep}(Q, R) \). Then the exact sequence

\[
0 \rightarrow \mathcal{Z}^+ \rightarrow \mathcal{Y}^+ \rightarrow \mathcal{X}^+ \rightarrow 0
\]

is split. So \( \mathcal{Y}^+ \) is injective if and only if \( \mathcal{X}^+ \) and \( \mathcal{Z}^+ \) are injective. Therefore \( \mathcal{Y} \) is a flat representation if and only if \( \mathcal{X} \) and \( \mathcal{Z} \) are flat.

(ii) The exact sequence

\[
0 \rightarrow \mathcal{Z}^+ \rightarrow \mathcal{Y}^+ \rightarrow \mathcal{X}^+ \rightarrow 0
\]

is split. Thus

\[
\mathcal{Y}^+ = \mathcal{X}^+ \oplus \mathcal{Z}^+.
\]

So \( \mathcal{X}^+ \) is injective if and only if \( \mathcal{Y}^+ \) is injective. Therefore \( \mathcal{X} \) is flat if and only if \( \mathcal{Y} \) is flat.
Recall that a representation $Z$ of $Q$ is called pure injective if for any pure exact sequence

$$0 \rightarrow X \rightarrow Y$$

in $\text{Rep}(Q, R)$, the sequence

$$\text{Hom}_Q(Y, Z) \rightarrow \text{Hom}_Q(X, Z) \rightarrow 0$$

is exact in the category of abelian groups.

**Proposition 2.7.** Let $Q$ be any quiver and $X$ be a representation of $Q$. Then

(i) The canonical monomorphism $X \rightarrow X^{++}$ is pure and $X^{++}$ is a pure injective representation.

(ii) A representation $X$ is pure injective if and only if it is a direct summand of $X^{++}$. Moreover, $Y^+$ is pure injective, for any representation $Y$ of $\text{Rep}(Q^{\text{op}}, R^{\text{op}})$.

**Proof.** In the first place we show that $X \rightarrow X^{++}$ is a pure injection. By definition, it suffices to give a section $X^{+++} \rightarrow X^{++}$. But the canonical map $X^{++} \rightarrow X^{+++}$ does the job. Suppose then that $0 \rightarrow N \rightarrow M$ is a pure monomorphism and $f : N \rightarrow X^{++}$ is a map. Let $s : N^{++} \rightarrow M^{++}$ be such that $i^+s = 1_{N^{++}}$ and set $g = sf^+ : X^{+++} \rightarrow M^{++}$. Clearly $g^+i^{++} = f^{++}$.

Note that since $X \rightarrow X^{++}$ is a pure monomorphism, there exists a map $t : X^{+++} \rightarrow X^{++}$ which is a retraction for $\varphi_X^{++}$, i.e., $t \varphi_X^{++} = 1_X^{+++}$. On the other hand, since the canonical map $\varphi_N$ is natural, we infer that $\varphi_{M^+} = i^{++}\varphi_N$ and $f^{++}\varphi_N = \varphi_X^{++}f$. Now define $h = t\varphi_X^{++}$ and observe that $hi = f$. Thus the map $\text{Hom}(M, X^{++}) \rightarrow \text{Hom}(N, X^{++})$ is a surjection and therefore $X^{++}$ is pure injective.

(ii) Suppose $X$ is a pure injective representation. Then from the canonical monomorphism $\varphi_X$, which is pure by (i), one obtains a map $f : X^{++} \rightarrow X$ satisfying $1_X = f \varphi_X$. The converse is obvious from (i). Meanwhile, Since $X^{+}$ is a summand of $X^{+++} = (X^{++})^+$, we deduce that $X^{+}$ is pure injective. □

**Corollary 2.8.** Every injective representation of $Q$ is pure injective.

**Proof.** Let $I$ be an injective representation of $Q$. Then the canonical monomorphism $0 \rightarrow I \rightarrow I^{++}$ is a split monomorphism and hence $I$ is pure injective. □

**Remark 2.9.** If $X$ is a pure injective representation, then it possesses a pure injective $R$-module in each vertex. But the converse need not be true.

**Example 2.10.** If $X$ is a representation of $Q$ such that for any vertex $v$ of $Q$, $X(v)$ is pure injective and for any arrow $a$ of $Q$, $X(a)$ is a split epimorphism, then $X$ is a pure injective representation of $Q$.

**Corollary 2.11.** Every representation of $Q$ has pure injective preenvelope.
Proof. Let $\mathcal{X}$ be a representation of $Q$ and $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of representations such that $\mathcal{Y}$ is pure injective. It is known that $\mathcal{X} \rightarrow \mathcal{X}^{++}$ is a pure monomorphism. So there exists a morphism of representations $g : \mathcal{X}^{++} \rightarrow \mathcal{Y}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi_X} & \mathcal{X}^{++} \\
\downarrow f & & \downarrow g \\
\mathcal{Y} & \xrightarrow{\downarrow g} & \mathcal{Y}
\end{array}
$$

is commutative. This completes the proof. \hfill $\square$

Remark 2.12. By [11, Theorem 2.6], a representation $\mathcal{C}$ of $Q$ is cotorsion if and only if it is cotorsion in each vertex. Therefore, by Remark 2.9 every pure injective object of $\text{Rep}(Q, R)$ is a cotorsion representation of $Q$.

Theorem 2.13. An object $Z$ of $\text{Rep}(Q, R)$ is flat if and only if any exact sequence

$$
0 \rightarrow \mathcal{X} \rightarrow \mathcal{Y} \rightarrow Z \rightarrow 0
$$

is pure in $\text{Rep}(Q, R)$.

Proof. If $Z$ is flat in $\text{Rep}(Q, R)$, then for any exact sequence

(2.13.1) \hspace{1cm} o \rightarrow \mathcal{X} \rightarrow \mathcal{Y} \rightarrow Z \rightarrow 0

in $\text{Rep}(Q, R)$, the sequence $0 \rightarrow Z^+ \rightarrow \mathcal{Y}^+ \rightarrow \mathcal{X}^+ \rightarrow 0$ is split exact in $\text{Rep}(Q^{op}, R^{op})$, because $Z^+$ is injective. Hence (2.13.1) is pure.

Let every exact sequence

$$
0 \rightarrow \mathcal{X} \rightarrow \mathcal{Y} \rightarrow Z \rightarrow 0
$$

be pure in $\text{Rep}(Q, R)$. It suffices to show that $Z^+$ is injective in $\text{Rep}(Q^{op}, R^{op})$. For this end, let

(2.13.2) \hspace{1cm} o \rightarrow Z^+ \xrightarrow{f} \mathcal{X} \rightarrow \mathcal{Y} \rightarrow 0

be an exact sequence in $\text{Rep}(Q^{op}, R^{op})$. Consider the following pullback diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{h} & \mathcal{P} & \xrightarrow{k} & Z & \rightarrow 0 \\
| & | & \downarrow h & | & \downarrow g & | \\
0 & \xrightarrow{f^+} & \mathcal{X}^+ & \xrightarrow{f^+} & Z^{++} & \rightarrow 0
\end{array}
$$

in $\text{Rep}(Q, R)$ with exact rows. By assumption the top row is pure and hence split, because $\mathcal{Y}^+$ is pure injective. So there is a morphism $h' : Z \rightarrow \mathcal{P}$ of representations such that $hh' = 1_Z$. It follows that $g_1 = gh' : Z \rightarrow \mathcal{X}^+$ is a morphism of representations such that $f^+g_1 = f^+gh' = ihh' = i$. Now
consider the following commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Z^+ & \overset{f}{\longrightarrow} & X & \overset{j}{\longrightarrow} & Y & \longrightarrow & 0 \\
0 & \longrightarrow & Z^{++} & \overset{f^{++}}{\longrightarrow} & X^{++} & \longrightarrow & Y^{++} & \longrightarrow & 0 \\
\end{array}
\]

It follows that \(g_1^+kf = g_1^{++}f^{++}j = i^{++}j = 1_{Z^{++}}\). Therefore (2.13.2) is split and hence \(Z^+\) is an injective object of \(\text{Rep}(Q^{op}, R^{op})\). Then by Proposition 1.1, \(Z\) is flat in \(\text{Rep}(Q, R)\). □

3. Pure injective dimension of flat representations

In this section we define the notion of a semi-co-barren quiver and give a characterization of a pure injective flat object in the category of representations of a semi-co-barren quiver.

**Definition 3.1.** A quiver \(Q\) is called semi-co-barren if for every \(v \in V_Q\), \(\{a \in E_Q \mid t(a) = v\}\) is a finite set.

**Example 3.2.** Let \(Q\) be a quiver whose connected components are barren trees. Then \(Q^{op}\) is a semi-co-barren quiver. Recall that a tree \(T\) with a root \(v\) is said to be barren if the number of vertices \(n_i\) of the \(i\)th state of \(T\) is finite for every natural number \(i\) and the sequence of positive natural numbers \(n_1, n_2, \ldots\) stabilizes, for more details see [4] and [5].

**Set up:** In this section we let \(R\) be a left Noetherian ring and \(Q\) be a semi-co-barren quiver or a quiver of type \(A_\infty\) in the sense of [3].

**Lemma 3.3.** Let \(F\) be a representation of \(Q\). Then \(F\) is flat if and only if \(F^{++}\) is a flat representation of \(Q^{op}\).

**Proof.** Let \(F\) be a flat representation of \(Q\). Since \(Q\) is a semi-co-barren quiver and \(R\) is left Noetherian then for every \(v \in V_Q\), \(\bigoplus_{t(a)=v} F^{++}(i(a)) \longrightarrow F^{++}(v)\) is a split monomorphism of flat \(R\)-modules. Therefore by [8] and [3], \(F^{++}\) is a flat representation of \(Q^{op}\).

The converse is a direct consequence of Proposition 2.6(i). □

**Theorem 3.4.** Let \(F\) be a flat representation of \(Q\). Then the followings are equivalent:

(i) \(F\) is pure injective.

(ii) \(F\) is cotorsion.

(iii) \(F\) is isomorphic to a direct summand of \(F^{++}\).

**Proof.** (i)⇒(ii) Let \(F\) be a pure injective representation of \(Q\). \(F\) is cotorsion in each vertex and hence by [11, Theorem 2.6] it is cotorsion object of \(\text{Rep}(Q, R)\).

(ii)⇒(iii) Let \(F\) be a cotorsion representation of \(Q\). By Lemma 3.3, \(F^{++}\) and \(F^{++}/F\) are flat representations of \(Q\). So
\[
\begin{array}{ccccccc}
0 & \longrightarrow & F & \overset{\lambda F}{\longrightarrow} & F^{++} & \longrightarrow & \text{Coker}\lambda F & \longrightarrow & 0,
\end{array}
\]
is split and hence $F$ is pure injective.

(iii)$\Rightarrow$(i) This is trivial. □

**Corollary 3.5.** Let $F$ be a flat object of $\text{Rep}(\mathcal{Q}, R)$. Then $F$ is pure injective if and only if $F(v)$ is pure injective $R$-module for all $v \in V_\mathcal{Q}$.

**Proof.** Assume that for any $v \in V_\mathcal{Q}$, $F(v)$ is pure injective $R$-module. Thus for any $v \in V_\mathcal{Q}$, $F(v)$ is cotorsion. So by [11, Theorem 2.6], $F$ is cotorsion object in $\text{Rep}(\mathcal{Q}, R)$. Then by Theorem 3.4 it is pure injective.

The converse is trivial. □

Let $\text{Pinj}(\mathcal{Q})$ be the class of all pure injective objects in $\text{Rep}(\mathcal{Q}, R)$. In Section 2, we proved that $\text{Pinj}(\mathcal{Q})$ is preenveloping. So every object $X$ in $\text{Rep}(\mathcal{Q}, R)$ has a unique (up to homotopy equivalence) pure injective resolution. Then for a given representation $X$ of $\mathcal{Q}$, the pure injective dimension of $X$ can be defined as follows

$$\text{pid} X = \min \{ n \mid X \text{ has a pure injective resolution of length } n \}.$$ 

Recall that a ring $R$ is called $n$-perfect if for each flat $R$-module $F$, $\text{cd} F$ (the cotorsion dimension of $F$) is at most $n$ (for more details see [7]). In the following theorem we show that, if $R$ is $n$-perfect, then

$$\sup \{ \text{cd} F \mid \text{for each flat representation } F \} = \sup \{ \text{pid} F \mid \text{for each flat representation } F \}.$$ 

**Theorem 3.6.** Let $R$ be an $n$-perfect ring and $F$ be a flat representation of $\mathcal{Q}$. Then $\text{pid} F \leq n$.

**Proof.** Let $F$ be a flat representation of $\mathcal{Q}$ and

$$0 \longrightarrow F \xrightarrow{\sigma} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} \cdots,$$

be a pure injective resolution of $X$ such that

$$C^i = (C^{i-1}/\text{Im} \delta^{i-2})^{++}$$

for each $i \geq 2$. By Proposition 2.6(i) and Lemma 3.3, for each $i \geq 1$, $\text{Coker} \delta^{i-1}$ is a flat representation of $\mathcal{Q}$. Furthermore, for all $v \in V_\mathcal{Q}$, the exact sequence

$$0 \longrightarrow F(v) \xrightarrow{\sigma(v)} C^0(v) \xrightarrow{\delta^0(v)} C^1(v) \xrightarrow{\delta^1(v)} \cdots \xrightarrow{\delta^{n-1}(v)} C^n(v) \xrightarrow{\delta^n(v)} \cdots,$$

is a pure injective resolution of $F(v)$ by pure injective flat $R$-modules. We know that for any $v \in V_\mathcal{Q}$, $\text{cd} F(v) \leq n$. Then for any $v \in V_\mathcal{Q}$, $\text{Coker} \delta^{n-1}(v)$ is a cotorsion flat $R$-module. Therefore by [11, Theorem 2.6] $\text{Coker} \delta^{n-1}$ is cotorsion flat, and by Theorem 3.4, it is pure injective flat in $\text{Rep}(\mathcal{Q}, R)$. Then $\text{pid} F \leq n$. □

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References


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