SCHUR POWER CONVEXITY OF GINI MEANS

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ABSTRACT. In this paper, the Schur convexity is generalized to Schur $f$-convexity, which contains the Schur geometrical convexity, harmonic convexity and so on. When $f : \mathbb{R}_+ \to \mathbb{R}$ is defined as $f(x) = (x^m - 1)/m$ if $m \neq 0$ and $f(x) = \ln x$ if $m = 0$, the necessary and sufficient conditions for $f$-convexity (is called Schur $m$-power convexity) of Gini means are given, which generalize and unify certain known results.

1. Introduction

Let $p, q \in \mathbb{R}$ and $a, b \in \mathbb{R}_+ := (0, \infty)$. The Gini means [13] are defined as

\[ G_{p,q}(a, b) = \begin{cases} 
\left( \frac{a^p + b^p}{a^q + b^q} \right)^{1/(p-q)}, & p \neq q, \\
\exp \left( \frac{a^p \ln a + b^p \ln b}{a^p + b^p} \right), & p = q.
\end{cases} \]

It is easy to see that the Gini means $G_{p,q}(a, b)$ are continuous on the domain \{$(a, b; p, q)$ : $a, b \in \mathbb{R}_+; p, q \in \mathbb{R}$\} and differentiable with respect to $(a, b)$ for fixed $p, q \in \mathbb{R}$. Also, Gini means are symmetric with respect to $a, b$ and $p, q$.

Gini means $G_{p,q}(a, b)$ contain many classical two variable means, for example, $G_{1,0} = A$ is the arithmetic mean, $G_{0,0} = G$ is the geometric mean, $G_{-1,0} = H$ is the harmonic mean, and more generally, the $p$-th power mean is equal to $G_{p,0}$, $G_{p,p-1}$ is the Lehmer mean. The basic properties of Gini means, as well as their comparison theorems, log-convexities, and inequalities are studied in papers [8, 9, 10, 11, 20, 21, 25, 26, 27, 30, 36, 43, 44, 45, 48].

Schur convexity was introduced by Schur in 1923 [22], and it has many important applications in analytic inequalities [2, 15, 49], linear regression [35], graphs and matrices [7], combinatorial optimization [16], information-theoretic topics [12], Gamma functions [23], stochastic orderings [32], reliability [17], and other related fields.

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In recent years, the Schur convexity and Schur geometrical convexity of $G_{p,q}(a,b)$ have attracted the attention of a considerable number of mathematicians [4, 5, 19, 29, 28, 31, 33]. Sándor [31] proved that the Gini means $G_{p,q}(a,b)$ are Schur convex on $(-\infty,0] \times (-\infty,0]$ and Schur concave on $[0,\infty) \times [0,\infty)$ with respect to $(p,q)$ for fixed $a,b > 0$ with $a \neq b$. Yang [47] improved Sándor’s result and proved that Gini means $G_{p,q}(a,b)$ are Schur convex with respect to $(p,q)$ for fixed $a,b > 0$ with $a \neq b$ if and only if $p+q < 0$ and Schur concave if and only if $p + q > 0$. Wang and Zhang [38, 39] showed that Gini means $G_{p,q}(a,b)$ are Schur convex with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p+q \geq 1$, $p,q \geq 0$ and Schur concave if and only if $p+q \leq 1$, $p \leq 0$ or $p+q \leq 1$, $q \leq 0$. Gu and Shi [14, 34] also discussed the Schur convexity. Recently, Chu and Xia [6] also proved the same result as Wang and Zhang’s.

The Schur geometrical convexity was introduced by Zhang [50]. Wang and Zhang [39] proved Gini means $G_{p,q}(a,b)$ are Schur geometrically convex with respect to $(a,b) \in \mathbb{R}_+^2$ if $p+q \geq 0$ and Schur geometrically concave if $p+q \leq 0$. Gu and Shi [14, 34] also investigated the Schur geometrical convexities of Lehmer mean $G_{p,1-p}(a,b)$ and Gini means $G_{p,q}(a,b)$, respectively.

Recently, Anderson et al. [1] discussed an attractive class of inequalities, which arise from the notion of harmonic convexity. And then it was started to research for Schur harmonic convexity. Chu et al. [3] showed that the Hany symmetric function is Schur harmonic convex and obtained some analytic inequalities including the well-known Weierstrass inequalities. Xia [40] proved that the Lehmer mean $G_{p,p-1}(a,b)$ is Schur harmonic convex (Schur harmonic concave) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p \geq (\leq)0$.

The purpose of this paper is to generalize the notion of Schur convexity and to investigate the so-called Schur power convexity of Gini means $G_{p,q}(a,b)$.

Our main results are as follows.

**Theorem 1.1.** For $m > 0$ and fixed $(p,q) \in \mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur $m$-power convex with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p+q \geq m$ and $\min(p,q) \geq 0$.

**Theorem 1.2.** For $m > 0$ and fixed $(p,q) \in \mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur $m$-power concave with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p+q \leq m$ and $\min(p,q) \leq 0$.

**Theorem 1.3.** For $m < 0$ and fixed $(p,q) \in \mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur $m$-power convex with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p+q \geq m$ and $\max(p,q) \geq 0$.

**Theorem 1.4.** For $m < 0$ and fixed $(p,q) \in \mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur $m$-power concave with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p+q \leq m$ and $\max(p,q) \leq 0$.

**Theorem 1.5.** For $m = 0$ and fixed $(p,q) \in \mathbb{R}^2$, Gini mean $G_{p,q}(a,b)$ is Schur $m$-power convex (Schur $m$-power concave) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $p+q \geq (\leq)0$. 
The organization of the paper is as follows. In Section 2, based on the notions and lemmas of Schur convexity, we introduce the definition of Schur $f$-convex and Schur $f$-concave function, and prove the decision theorem for Schur $f$-convexity. As special case, the definition and decision theorem of Schur power convexity are deduced. In Section 3, some lemmas are given. In Section 4, our main results are proved.

2. Schur $f$-convexity and Schur power convexity

For convenience of readers, we recall some definitions as follows.

**Definition 2.1 ([22, 37]).** Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n(n \geq 2)$.

(i) $x$ is said to be majorized by $y$ (in symbol $x \prec y$) if

$$
\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{for} \quad 1 \leq k \leq n-1, \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},
$$

where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $x$ and $y$ in a decreasing order.

(ii) $x \succeq y$ means $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$. Let $\Omega \subseteq \mathbb{R}^n(n \geq 2)$. The function $\phi : \Omega \to \mathbb{R}$ is said to be increasing if $x \succeq y$ implies $\phi(x) \geq \phi(y)$. $\phi$ is said to be decreasing if and only if $-\phi$ is increasing.

(iii) $\Omega \subseteq \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$ for all $x, y$ and all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iv) Let $\Omega \subseteq \mathbb{R}^n(n \geq 2)$ be a set with nonempty interior. Then $\phi : \Omega \to \mathbb{R}$ is said to be Schur convex if $x \prec y$ on $\Omega$ implies $\phi(x) \leq \phi(y)$. $\phi$ is said to be Schur concave if $-\phi$ is Schur convex.

**Definition 2.2 ([22]).** (i) $\Omega \subseteq \mathbb{R}^n(n \geq 2)$ is called a symmetric set, if $x \in \Omega$ implies $xP \in \Omega$ for every $n \times n$ permutation matrix $P$.

(ii) The function $\phi : \Omega \to \mathbb{R}^n$ is called symmetric if for every permutation matrix $P$, $\phi(xP) = \phi(x)$ for all $x \in \Omega$.

For the Schur convexity, there is the following well-known result.

**Lemma 2.1 ([22, 37]).** Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric set with nonempty interior $\Omega^0$ and $\phi : \Omega \to \mathbb{R}$ be continuous on $\Omega$ and differentiable in $\Omega^0$. Then $\phi$ is Schur convex (Schur concave) on $\Omega$ if and only if $\phi$ is symmetric on $\Omega$ and

$$
(x_1 - x_2) \left( \frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0.
$$

Next, let us define the Schur $f$-convexity as follows.

**Definition 2.3.** Let $\Omega = \mathbb{U}^n(U \subseteq \mathbb{R})$ and $f$ be a strictly monotone function defined on $\mathbb{U}$. Assume that

$$
f(x) = (f(x_1), f(x_2), \ldots, f(x_n)) \quad \text{and} \quad f(y) = (f(y_1), f(y_2), \ldots, f(y_n)).
$$


Remark 2.2. Let \( \Omega = \mathbb{U}^n(\mathbb{U} \subseteq \mathbb{R}) \) and \( f \) be a strictly monotone function defined on \( \mathbb{U} \) and \( f(\Omega) = \{ f(x) : x \in \Omega \} \). Then function \( \phi : \Omega \to \mathbb{R} \) is Schur \( f \)-concave if and only if \( \phi \circ f^{-1} \) is Schur convex (Schur concave) on \( f(\Omega) \).

Indeed, if function \( \phi : \Omega \to \mathbb{R} \) is Schur \( f \)-concave, then \( \forall x', y' \in f(\Omega) \), there are \( x, y \in \Omega \) such that \( x' = f(x), y' = f(y) \). If \( f(x) \prec f(y) \), that is, \( x' \prec y' \), then \( \phi(x) \leq \phi(y) \), that is, \( \phi((f^{-1}(x'))) \leq \phi((f^{-1}(y'))) \). This shows that \( \phi \circ f^{-1} \) is Schur convex on \( f(\Omega) \). Conversely, if \( \phi \circ f^{-1} \) is Schur convex on \( f(\Omega) \), then \( \forall x, y \in \Omega \) such that \( f(x) \prec f(y) \), we have \( \phi((f^{-1}(f(x)))) \leq \phi((f^{-1}(f(y)))) \), that is, \( \phi(x) \leq \phi(y) \). This indicates \( \phi \) is Schur \( f \)-convex on \( \Omega \).

In the same way, we can show that \( \phi \) is Schur \( f \)-concave on \( \Omega \) if and only if \( \phi \circ f^{-1} \) is Schur concave on \( f(\Omega) \).

Remark 2.2. Let \( \Omega \subseteq \mathbb{R}^n(n \geq 2) \) be a symmetric set and the function \( \phi : \Omega \to \mathbb{R} \) be Schur \( f \)-concave (Schur \( f \)-convex). Then \( \phi \) is symmetric on \( \Omega \).

In fact, for any \( x \in \Omega \) and every permutation matrix \( P \), we have \( xP \in \Omega \). Note \( xP \) is another permutation of \( x \), hence \( f(x) \prec f(xP) \prec f(x) \). Since \( \phi \) is Schur \( f \)-convex (Schur \( f \)-concave), we have \( \phi(x) \leq (\geq) \phi(xP) \leq (\geq) \phi(x) \), that is, \( \phi(xP) = \phi(x) \) for all \( x \in \Omega \). This shows that \( \phi \) is symmetric on \( \Omega \).

By Lemma 2.1 and Remarks 2.1, 2.2, we have the following:

**Theorem 2.1.** Assume that \( \Omega = \mathbb{U}^n(\mathbb{U} \subseteq \mathbb{R}) \) is a symmetric set with nonempty interior \( \Omega^0 \), \( f \) is a strictly monotone and derivable function defined on \( \mathbb{U} \), and \( \phi : \Omega \to \mathbb{R} \) is continuous on \( \Omega \) and differentiable in \( \Omega^0 \). Then \( \phi \) is Schur \( f \)-convex (Schur \( f \)-concave) on \( \Omega \) if and only if \( \phi \) is symmetric on \( \Omega \) and

\[
(f(x_1) - f(x_2)) \left( \frac{1}{f'(x_1)} \frac{\partial \phi}{\partial x_1} - \frac{1}{f'(x_2)} \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0
\]

holds for any \( x = (x_1, x_2, \ldots, x_n) \in \Omega^0 \) with \( x_1 \neq x_2 \).

**Proof.** We easily check that \( \phi \circ f^{-1} \) is symmetric on \( f(\Omega) \) if and only if \( \phi \) is symmetric on \( \Omega \).

By Remark 2.1 and Lemma 2.1, \( \phi \circ f^{-1} \) is Schur convex (Schur concave) if and only if \( \phi \circ f^{-1} \) is symmetric on \( f(\Omega) \) and

\[
(y_1 - y_2) \left( \frac{\partial (\phi \circ f^{-1})}{\partial y_1} - \frac{\partial (\phi \circ f^{-1})}{\partial y_2} \right) \geq (\leq) 0
\]

holds for any \( y \in f(\Omega)^0 \) with \( y_1 \neq y_2 \). Substituting \( f^{-1}(y) = x \) yields (2.3), where \( x = (x_1, x_2, \ldots, x_n) \in \Omega^0 \) with \( x_1 \neq x_2 \).

This proof is finished. \( \square \)
Putting \( f(x) = 1, \ln x, x^{-1} \) in Definition 2.3 yield the Schur convexity, Schur geometrical convexity and Schur harmonic convexity. It is clear that the Schur \( f \)-convexity is a generalization of the Schur convexity mentioned above. In general, we have:

**Definition 2.4.** Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = (x^m - 1)/m \) if \( m \neq 0 \) and \( f(x) = \ln x \) if \( m = 0 \). Then function \( \phi : \Omega(\subseteq \mathbb{R}^n) \to \mathbb{R} \) is said to be Schur \( m \)-power convex on \( \Omega \) if \( f(x) \prec f(y) \) on \( \Omega \) implies \( \phi(x) \leq \phi(y) \).

\( \phi \) is said to be Schur \( m \)-power concave if \( -\phi \) is Schur \( m \)-power convex.

For the Schur power convexity, by Theorem 2.1 we have:

**Corollary 2.1.** Let \( \Omega \subseteq \mathbb{R}^n_+ \) be a symmetric set with nonempty interior \( \Omega^0 \) and \( \phi : \Omega \to \mathbb{R} \) be continuous on \( \Omega \) and differentiable in \( \Omega^0 \). Then \( \phi \) is Schur \( m \)-power convex (Schur \( m \)-power concave) on \( \Omega \) if and only if \( \phi \) is symmetric on \( \Omega \) and

\[
\begin{align*}
(2.4) & \quad \frac{x_1^m - x_2^m}{m} \left( x_1^{1-m} \frac{\partial \phi}{\partial x_1} - x_2^{1-m} \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0 & \quad \text{if } m \neq 0, \\
(2.5) & \quad \left( \ln x_1 - \ln x_2 \right) \left( x_1^{1-m} \frac{\partial \phi}{\partial x_1} - x_2^{1-m} \frac{\partial \phi}{\partial x_2} \right) \geq (\leq) 0 & \quad \text{if } m = 0
\end{align*}
\]

holds for any \( x = (x_1, x_2, \ldots, x_n) \in \Omega^0 \) with \( x_1 \neq x_2 \).

3. Lemmas

To prove the main results, we need the following useful lemmas.

**Lemma 3.1.** For fixed \((p, q) \in \mathbb{R}^2\), Gini means \( G_{p,q}(a, b) \) is Schur \( m \)-power convex (Schur \( m \)-power concave) with respect to \((a, b) \in \mathbb{R}_+^2\) if and only if \( g(t) \geq (\leq) 0 \) for all \( t > 0 \), where

\[
(3.1) \quad g(t) := g_{p,q}(t) = \left\{ \begin{array}{ll}
\frac{(p-q) \sinh At + p \sinh Bt + q \sinh Ct}{p-q} & \text{if } p \neq q, \\
\sinh(2p-m)t - \sinh mt + 2pt \cosh mt & \text{if } p = q,
\end{array} \right.
\]

and

\[
(3.2) \quad A = p + q - m, \quad B = p - q - m, \quad C = p - q + m.
\]

**Proof.** Let \( m \neq 0 \) and \( G = G_{p,q} := G_{p,q}(a, b) \) defined by (1.1).

For \( p \neq q \), some simple partial derivative calculations yield

\[
\begin{align*}
\frac{\partial \ln G}{\partial a} & = \frac{1}{G} \frac{\partial G}{\partial a} = \frac{1}{p-q} \left( \frac{pa^{p-1} - qa^{q-1}}{a^p + b^p} \right), \\
\frac{\partial \ln G}{\partial b} & = \frac{1}{G} \frac{\partial G}{\partial b} = \frac{1}{p-q} \left( \frac{pb^{p-1} - qb^{q-1}}{a^q + b^q} \right)
\end{align*}
\]

Therefore, we have

\[
a^{1-m} \frac{\partial \phi}{\partial a} - b^{1-m} \frac{\partial \phi}{\partial b} = G \left( \frac{a^{p-m} - b^{p-m}}{a^p + b^p} \right),
\]

where

\[
(3.2) \quad G = \left( \frac{a^q - b^q}{a^{q-m} + b^{q-m}} \right).
\]
Substituting \( \ln \sqrt{a/b} = t \) and using \( \sinh x = \frac{1}{2}(e^x - e^{-x}), \cosh x = \frac{1}{2}(e^x + e^{-x}) \), the right hand side above can be written as

\[
a^{1-m} \frac{\partial \phi}{\partial a} - b^{1-m} \frac{\partial \phi}{\partial b} = \frac{G(ab)^{-m/2}}{p-q} \left( \frac{p \sinh(p-m)t}{\cosh pt} - q \frac{\sinh(q-m)t}{\cosh qt} \right)
\]

Using the “product into sum” formula for hyperbolic functions and (3.1), we have

\[
\Delta := \frac{a^m - b^m}{m} \left( a^{1-m} \frac{\partial G_{p,q}}{\partial a} - b^{1-m} \frac{\partial G_{p,q}}{\partial b} \right) = d_{p,q}(t) \cdot g_{p,q}(t),
\]

where

\[
d_{p,q}(t) = \frac{a^m - b^m}{m(a-b) 2(ab)^{m/2}} \frac{(a-b)G_{p,q}}{\cosh pt \cosh qt} (p \neq q)
\]

and \( g_{p,q}(t) \) is defined by (3.1).

In the case of \( p = q \), since \( G_{p,q}(a, b) \in C^1 \) we have

\[
\frac{\partial G_{p,p}}{\partial a} = \lim_{q \to p} \frac{\partial G_{p,q}}{\partial a}, \quad \frac{\partial G_{p,p}}{\partial b} = \lim_{q \to p} \frac{\partial G_{p,q}}{\partial b}.
\]

It follows that

\[
\Delta = \frac{a^m - b^m}{m} \left( a^{1-m} \frac{\partial G_{p,p}}{\partial a} - b^{1-m} \frac{\partial G_{p,p}}{\partial b} \right)
\]

Summarizing two cases above yield

\[
\Delta = \left\{ \begin{array}{ll}
g_{p,q}(t) \cdot d_{p,q}(t) & \text{if } p \neq q, \\
g_{p,p}(t) \lim_{q \to p} d_{p,q}(t) & \text{if } p = q.
\end{array} \right.
\]

Since \( \Delta \) is symmetric with respect to \( a \) and \( b \), without loss of generality we assume \( a > b \). It is easy to verify that \( \frac{a^m - b^m}{m(a-b)} > 0, \frac{a-b}{2(ab)^{m/2}} > 0, \) and \( \frac{1}{\cosh pt \cosh qt} > 0 \) for \( t = \ln \sqrt{a/b} > 0 \), which implies that \( d_{p,q}(t) \) and its limit at
Let \( g(t) = g_{p,q}(t) \) be defined by (3.1). Then

\[
\lim_{t \to 0^+, t > 0} \frac{g_{p,q}(t)}{2t} = p + q - m.
\]

**Proof.** It is easy to check that \( g(0) = 0 \).

In the case of \( p \neq q \), applying L'Hospital's rule yields

\[
\lim_{t \to 0^+, t > 0} \frac{g_{p,q}(t)}{2t} = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{(p - q)A + pB + qC}{2(p - q)} = p + q - m.
\]

In the case of \( p = q \), we have

\[
\lim_{t \to 0^+, t > 0} \frac{g_{p,p}(t)}{2t} = 2p - m.
\]

This completes the proof. \( \square \)

**Lemma 3.3.** Let \( m > 0 \) and \( \beta = \max(\{|A|, |B|, |C|\}) \) where \( A, B, C \) are defined by (3.1). Then

(i) If \( p > q \), then

\[
\lim_{t \to \infty} \frac{2\beta g_{p,q}(t)}{e^{\beta t}} = \begin{cases} 
    p + q - m & \text{if } p > q > m \text{ or } 0 > p > q, \\
    \frac{p - m}{p - q} & \text{if } p > q = m, \\
    \frac{2(q - m)}{p - q} & \text{if } p = 0 > q, \\
    \frac{p - q + m}{p - q} & \text{if } p > 0, q < m, p > q;
\end{cases}
\]

(ii) If \( p = q \), then

\[
\lim_{t \to \infty} \frac{2\beta g_{p,p}(t)}{e^{\beta t}} = \begin{cases} 
    2p - m & \text{if } p > m \text{ or } p < 0, \\
    -2m & \text{if } p = 0, \\
    \infty & \text{if } 0 < p \leq m.
\end{cases}
\]

**Proof.** (3.4)-(3.5) easily follows from the following limit relations:

\[
\lim_{t \to \infty} \frac{2 \cosh \alpha t}{e^{\beta t}} = \begin{cases} 
    1 & \text{if } \beta = |\alpha|, \\
    0 & \text{if } \beta > |\alpha|,
\end{cases}
\]

\[
\lim_{t \to \infty} \frac{2 \alpha t \sinh \alpha t}{e^{\beta t}} = \begin{cases} 
    \infty & \text{if } \beta = |\alpha|, \\
    0 & \text{if } \beta > |\alpha|.
\end{cases}
\]

(i) If \( p > q \), then \( \beta = \max(|A|, |B|, |C|) = \max(|A|, |C|) \) because \(|C|^2 - |B|^2 = 4m(p - q) > 0 \). We have

\[
(p - q) \lim_{t \to \infty} \frac{2\beta g_{p,q}(t)}{e^{\beta t}} = (p - q) \lim_{t \to \infty} \frac{2}{e^{\beta t}} \frac{\partial g_{p,q}(t)}{\partial t}.
\]
\[
= \lim_{t \to \infty} 2 \frac{(p - q) A \cosh At + pB \cosh Bt + qC \cosh Ct}{e^{\beta t}}
\]

\[
= \begin{cases} 
(p - q) A & \text{if } |A| > |C|, \text{ i.e., } p(q - m) > 0, \\
(p - q) A + qC & \text{if } |A| = |C|, \text{ i.e., } p(q - m) = 0, \\
qC & \text{if } |A| < |C|, \text{ i.e., } p(q - m) < 0.
\end{cases}
\]

\[
= \begin{cases} 
(p - q)(p + q - m) & \text{if } p > q > m \text{ or } 0 > p > q, \\
p^2 & \text{if } p > q = m, \\
-2q(q - m) & \text{if } p = 0 > q, \\
q(p - q + m) & \text{if } p > 0, q < m, p > q.
\end{cases}
\]

Dividing by \((p - q)\) in the above limit relation yields (3.4).

(ii) If \(p = q\), then \(\beta = \max(|A|, |B|, |C|) = \max(2p - m, m)\). We have

\[
\lim_{t \to \infty} 2 \frac{\beta g_{p,p}(t)}{e^{\beta t}} = \lim_{t \to \infty} 2 \frac{\partial g_{p,p}(t)}{e^{\beta t}}
\]

\[
= \begin{cases} 
2p - m & \text{if } |2p - m| > m, \text{ i.e., } p > m \text{ or } p < 0, \\
\infty & \text{if } |2p - m| = m, p \neq 0, \text{ i.e., } p = m, \\
-2m & \text{if } |2p - m| = m, p = 0, \text{ i.e., } p = 0, \\
\infty & \text{if } |2p - m| < m, \text{ i.e., } 0 < p < m,
\end{cases}
\]

which implies (3.5).

This completes the proof. \(\square\)

4. Proof of main results

Proof of Theorem 1.1. Assume that

\[ E_1 = \{(p, q) : p + q - m \geq 0, \min(p, q) \geq 0\} \quad (m > 0). \]

By Lemma 3.1, to prove Theorem 1.1, it suffices to prove that \(g_{p,q}(t) \geq 0\) for all \(t > 0\) if and only if \((p, q) \in E_1\).

**Necessity.** We prove that \((p, q) \in E_1\) is the necessary conditions for \(g(t) = g_{p,q}(t) \geq 0\) for all \(t > 0\). It is obvious that

\[(4.1) \quad \lim_{t \to 0+, t > 0} \frac{g_{p,q}(t)}{2t} \geq 0 \text{ and } \lim_{t \to \infty} \frac{2 \beta g_{p,q}(t)}{e^{\beta t}} \geq 0.\]

Now, we get the necessary conditions from (4.1) together with (3.4) and (3.5).

To this aim, we distinguish three cases.

(i) **Case 1:** \(p > q\). By (4.1) together with (3.3) and (3.4), we have

**Subcase 1:**

\[
\begin{align*}
& p + q - m \geq 0, \\
& p + q - m \geq 0, \\
& p > q > m \text{ or } 0 > p > q.
\end{align*}
\]

which implies \((p, q) \in \{(p, q) : p > q > m\} := E_{11}\).
Subcase 2: 
\[
\begin{align*}
& \quad \quad \left\{ \begin{array}{c}
p + q - m \geq 0, \\
p^2 - m \geq 0, \\
p > q = m
\end{array} \right. \\
& \quad \implies p > q = m,
\end{align*}
\]
which implies \((p, q) \in \{(p, q) : p > q = m\} := E_{12}.

Subcase 3: 
\[
\begin{align*}
& \quad \quad \left\{ \begin{array}{c}
p + q - m \geq 0, \\
2(q - m) \geq 0, \\
p = 0 > q
\end{array} \right. \\
& \quad \implies \text{which is impossible.}
\end{align*}
\]

Subcase 4: 
\[
\begin{align*}
& \quad \quad \left\{ \begin{array}{c}
p + q - m \geq 0, \\
p > 0, \\
q \leq m,
\end{array} \right. \\
& \quad \implies \left\{ \begin{array}{c}
p + q - m \geq 0, \\
p > 0, \\
0 < q < m, \\
p > q
\end{array} \right.
\end{align*}
\]
which implies \((p, q) \in \{(p, q) : p > q \geq m\} := E'_{14}.

(i) Case 1: \(p < q\). Since \(g_{p,q}(t)\) is symmetric with respect to \(p\) and \(q\), we get \((p, q) \in E'_{11} \cup E'_{12} \cup E'_{14}\), where
\[
\begin{align*}
E'_{11} &= \{(p, q) : q > p \geq m\}, \\
E'_{12} &= \{(p, q) : q > p = m\}, \\
E'_{14} &= \{(p, q) : p + q - m \geq 0, q > 0, 0 < p < m, q > p\}.
\end{align*}
\]

(ii) Case 2: \(p = q\). By (4.1) together with (3.3) and (3.5), we have

Subcase 1: 
\[
\begin{align*}
& \quad \quad \left\{ \begin{array}{c}
p + q - m \geq 0, \\
2p - m \geq 0, \\
p > q \geq m \\
\end{array} \right. \\
& \quad \implies \text{which is impossible.}
\end{align*}
\]

Subcase 2: 
\[
\begin{align*}
& \quad \quad \{p + q - m \geq 0, \\
& \quad \quad -2m \geq 0, \\
& \quad \quad p = 0
\end{align*}
\]
which is impossible.

Subcase 3: 
\[
\begin{align*}
& \quad \quad \left\{ \begin{array}{c}
p + q - m \geq 0, \\
\infty \geq 0, \\
0 < p \leq m
\end{array} \right. \\
& \quad \implies \frac{m}{2} \leq p = q < m.
\end{align*}
\]
The above three subcases imply \((p, q) \in \{(p, q) : p = q \geq \frac{m}{2}\} := E_{10}.

Summarizing all the cases (i), (i') and (ii) yields
\[
(p, q) \in (E_1 \cup E_{12} \cup E_{14}) \cup (E'_{11} \cup E'_{12} \cup E'_{14}) \cup E_{10} = E_1.
\]

Sufficiency. We prove the condition \((p, q) \in E_1\) is sufficient for \(g(t) = g_{p,q}(t) \geq 0\) for all \(t > 0\). Since \(g(0) = 0\), it is enough to prove \(g'(t) \geq 0\) if \((p, q) \in E_1\). For symmetry, we may assume again that \(p \geq q\).
Noting
\((p - q)A = pB + qC\) or \(pB = (p - q)A - qC\),
we have
\[
(p - q)g'(t) = (p - q)A \cosh At + pB \cosh Bt + qC \cosh Ct
\]
\[
= (p - q)A(\cosh At + \cosh Bt) + qC(\cosh Ct - \cosh Bt)
\]
\[
= (p - q)A(\cosh At + \cosh Bt) + 2qC \sinh(p - q)t \sinh mt.
\]
(4.2)

If \(p > q\) and \((p, q) \in E_1\), then \(A = p + q - m \leq 0\), \(q = \min(p, q) \geq 0\), \(C = p - q + m > 0\). It follows that \((p - q)g'(t) \geq 0\) for \((p, q) \in E_1\).

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Assume that
\[
E_2 = \{(p, q) : p + q - m \leq 0, p \geq q, q \leq 0\} \quad (m > 0),
\]
\[
E'_2 = \{(p, q) : p + q - m \leq 0, q \geq p, p \leq 0\} \quad (m > 0),
\]
then
\[
E_2 \cup E'_2 = \{(p, q) : p + q - m \leq 0 \text{ and } \min(p, q) \leq 0\} \quad (m > 0).
\]

By Lemma 3.1, to prove Theorem 1.2, it suffices to show that \(g_{p,q}(t) \leq 0\) for all \(t > 0\) if and only if \((p, q) \in E_2 \cup E'_2\).

Necessity. If \(g_{p,q}(t) \leq 0\) for all \(t > 0\), then
\[
\lim_{t \to 0^+} g_{p,q}(t) \leq 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{2\beta g_{p,q}(t)}{e^{\beta t}} \leq 0.
\]
(4.4)

Similarly, we divide the proof of necessity into three cases.

(i) Case 1: \(p > q\). By (4.4) together with (3.3) and (3.4), we have

Subcase 1:
\[
\begin{align*}
p + q - m & \leq 0, \\
p + q - m & \leq 0, \\
p > q & > m \quad \text{or} \quad 0 > p > q
\end{align*}
\]
which implies \((p, q) \in \{(p, q) : 0 > p > q\} := E_{21}\).

Subcase 2:
\[
\begin{align*}
p + q - m & \leq 0, \\
p^2 - m & \leq 0, \\
p > q & = m
\end{align*}
\]

which is impossible.

Subcase 3:
\[
\begin{align*}
p + q - m & \leq 0, \\
2(q - m) & \leq 0, \\
p & = 0 > q
\end{align*}
\]
which implies \((p, q) \in \{(p, q) : p = 0 > q\} := E_{23}\).
Subcase 4:
\[
\begin{cases}
  p + q - m \leq 0, \\
  \frac{g(p-q+m)}{p-q} \leq 0, \\
  p > 0, \\
  q < m, \\
  p > q
\end{cases}
\implies \begin{cases}
  p + q - m \leq 0, \\
  p > 0 \geq q
\end{cases}
\]
which implies \((p, q) \in \{(p, q) : p + q - m \leq 0, p > 0 \geq q\} := E_{24}.

(i') Case 1': \(p < q\). Since \(g_{p,q}(t)\) is symmetric with respect to \(p\) and \(q\), so \((p, q) \in E_{21}' \cup E_{23}' \cup E_{24}'\), where
\[
E_{21}' = \{(p, q) : 0 > q > p\},
E_{23}' = \{(p, q) : q = 0 > p\},
E_{24}' = \{(p, q) : p + q - m \leq 0, q > 0 \geq p\}.
\]

(ii) Case 2: \(p = q\). By (4.4) together with (3.3) and (3.5), we have
Subcase 1:
\[
\begin{cases}
  p + q - m \leq 0, \\
  2p - m \leq 0 \\
\end{cases}
\implies p = q < 0.
\]
Subcase 2:
\[
\begin{cases}
  p + q - m \leq 0, \\
  -2m \leq 0 \\
  p = 0
\end{cases}
\implies p = q = 0.
\]
Subcase 3:
\[
\begin{cases}
  p + q - m \leq 0, \\
  \infty \leq 0 \\
  0 < p \leq m
\end{cases}
\implies \text{which is impossible.}
\]
The above three subcases imply \((p, q) \in \{(p, q) : p = q \leq 0\} := E_{20}.

Summarizing all the cases (i), (i') and (ii) yields
\((p, q) \in (E_{21} \cup E_{23} \cup E_{24}) \cup (E_{21}' \cup E_{23}' \cup E_{24}') \cup E_{20} = E_2 \cup E_2'.

Sufficiency. Similarly to proof of sufficiency of Theorem 1.1, by (4.2) and (4.3) we easily prove \(g'(t) \leq 0\) if \((p, q) \in E_2 \cup E_2'. Hence g_{p,q}(t) = g(t) \leq g(0) = 0\) for all \(t > 0\).

The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. Let \(g_{p,q,m}(t) := g_{p,q}(t)\) be defined by (3.1) and
\[
p' = -p, \quad q' = -q, \quad m' = -m.
\]
We easily verify that, for \(p, q, p', q', m, m' \in \mathbb{R},
\]
\[
g_{p,q,m}(t) = -g_{p',q',m'}(t).
\]
From this and Lemma 3.1, for \( m < 0 \), Gini mean \( G_{p,q}(a,b) \) is Schur \( m \)-power convex if and only if \( G_{p',q'}(a,b) \) is Schur \( m' \)-power concave with respect to \( (a, b) \in \mathbb{R}^2_+ \), which, by Theorem 1.2, if and only if

\[
p' + q' \leq m' \quad \text{and} \quad \min(p', q') \leq 0,
\]

that is,

\[
p + q \geq m \quad \text{and} \quad \max(p, q) \geq 0.
\]

Theorem 1.3 follows. \( \square \)

**Proof of Theorem 1.4.** Similarly as in the proof of Theorem 1.3, for \( m < 0 \), Gini mean \( G_{p,q}(a,b) \) is Schur \( m \)-power concave if and only if \( G_{p',q'}(a,b) \) is Schur \( m' \)-power convex with respect to \( (a, b) \in \mathbb{R}^2_+ \), which, by Theorem 1.1, if and only if

\[
p' + q' \geq m' \quad \text{and} \quad \min(p', q') \geq 0,
\]

that is,

\[
p + q \leq m \quad \text{and} \quad \max(p, q) \leq 0.
\]

The proof of Theorem 1.4 ends. \( \square \)

**Proof of Theorem 1.5.** By Lemma 3.1, to prove Theorem 1.5, it is enough to prove that \( g_{p,q}(t) \geq (\leq)0 \) for all \( t > 0 \) if and only if \( p + q \geq (\leq)0 \) for \( m = 0 \). To this end, we divide the proof into two cases.

(i) **Case 1:** \( p \neq q \). By (3.1), we have

\[
g_{p,q}(t) = \frac{(p-q) \sinh(p+q)t + (p+q) \sinh(p-q)t}{p-q} = \left\{ \begin{array}{ll}
\frac{t(p+q)}{(p+q)t} \left( \frac{\sinh(p+q)t}{(p+q)t} + \frac{\sinh(p-q)t}{(p-q)t} \right) & \text{if } p+q \neq 0, \\
0 & \text{if } p+q = 0.
\end{array} \right.
\]

Since \( \sinh u > 0 \) for all \( u \neq 0 \) and \( t > 0 \), we obtain \( \text{sgn} (g_{p,q}(t)) = \text{sgn}(p+q) \).

(ii) **Case 2:** \( p = q \). By (3.1), we have

\[
g_{p,p}(t) = \left\{ \begin{array}{ll}
2pt \left( \frac{\sinh(2pt)}{2pt} + 1 \right) & \text{if } p \neq 0, \\
0 & \text{if } p = 0.
\end{array} \right.
\]

It is obvious that \( \text{sgn} (g_{p,p}(t)) = \text{sgn}(p) \).

In brief, \( g_{p,q}(t) \geq (\leq)0 \) for all \( t > 0 \) if and only if \( p + q \geq (\leq)0 \).

The proof of Theorem 1.5 is finished. \( \square \)

**References**


Y. M. Chu and X. M. Zhang, Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave, J. Math. Kyoto Univ. 48 (2008), no. 1, 229–238.


M. Merkle, Convexity, Schur-convexity and bounds for the gamma function involving the digamma function, Rocky Mountain J. Math. 28 (1998), no. 3, 1053–1066.


