Industrial Engineering
& Management Systems
Vol 12, No 1, March 2013, pp.16-22
ISSN 1598-7248 │ EISSN 2234-6473 │
http://dx.doi.org/10.7232/iems.2013.12.1.016

Comonotonic Uncertain Vector and Its Properties

Shengguo Li, Bo Zhang
School of Mathematics and Statistics, Huazhong Normal University, Hubei, China

Jin Peng*
Institute of Uncertain Systems, Huanggang Normal University, Hubei, China

(Received: August 17, 2012 / Revised: December 18, 2012 / Accepted: March 6, 2013)

ABSTRACT

This paper proposes a new concept of comonotonicity of uncertain vector based on the uncertainty theory. In order to understand the comonotonicity of uncertain vector, some equivalent definitions are presented. Following the proposed concept, some basic properties of comonotonic uncertain vector are investigated. In addition, the operational law is given for calculating the uncertainty distributions of monotone functions of comonotonic uncertain variables. With the help of operational law, the comonotonic uncertain vector is applied to the premium pricing problems. At last, some numerical examples are given to illustrate the application.

Keywords: Uncertainty Theory, Uncertain Vector, Uncertain Distribution, Comonotonicity

* Corresponding Author, E-mail: pengjin01@tsinghua.org.cn

1. INTRODUCTION

There exist two basic types of structures among variables: independence and correlation. Compared with independence, correlation contains more complex structure forms. How to deal with the problems involving correlative variables mathematically has become a challenging problem.

In many real problems, when one variable fluctuates because of indeterminacy, others also may show similar fluctuations. In order to deal with this phenomenon, the comonotonicity theory is accepted widely. Over the last decade, researchers in economics, financial mathematics and actuarial science have introduced results related to the concept of comonotonicity in their respective fields of interest. In risk analysis and decision making, except for the research on risk measures (Artzner *et al.*, 1999; Choudhry, 2006; Peng, 2009; Szego, 2004), it is necessary to consider the structures of risks on the purpose of minimizing the loss. Thanks to the works such as Roell (1987) and Yaari (1987), comonotonicity has become an important concept in economic theories of decision under risk and indeterminacy. In some models of decision theory, the independence axiom of expected utility has been replaced by a comonotonicity axiom (Kast and Lapied, 2003; Wakker, 1996). In actuarial science, Dhaene *et al*. (2002) used the concept of comonotonicity based on inverse distribution function as a tool to make safe decisions. Besides, comonotonicity can be also found in insurance, risk management and stop-loss premium principles (Cheung, 2008; Dhaene *et al*., 2000, 2006; Ludkovski and Ruschendorf, 2008; Wang and Dhaene, 1998). Recently, Ekelanda *et al*. (2012) extended the concept of comonotonicity to the case of multivariate in order to cope with the multivariate risks problems.

In the area of fuzzy set theory, the idea of comonotonicity can be also found as a tool to deal with fuzziness that was established by Zadeh (1965). For example, the comonotonic function is introduced to portray fuzzy risks in the risk decision theory. The properties and applications of comonotonic function can be found in the literature of fuzzy measure and risk analysis (Benvenuti and Vivona, 1996, 2000; Bernadette *et al*., 2002; Mesiar and Ouyang, 2009; Modave and Grabisch, 1998; Narukawa and Torra, 2009).

As we see above, the theory and application of comonotonicity relate to either probability or fuzziness. In fact, sometimes, many indeterminacy phenomena behave neither like randomness nor like fuzziness. For example, in order to calculate the insurance premium for a new type of risk, an insurer needs to obtain a large number of claim data about the loss derived from this type of risk. However, very often we are lacking in the observed data about it, not only for economic reasons, but also for technical difficulties. It seems that we have to invite some domain experts to evaluate their belief degree that loss will occur. Since human beings usually overweigh unlikely events, the belief degree may have much larger variance than the real frequency. Thus, the law of large numbers is no longer valid and the probability theory is no longer applicable.

In order to deal with such indeterminacy, Liu (2007) founded an uncertainty theory, and it was refined by Liu (2010b). The uncertainty theory is a consistent mathematical system which is suitable to cope with the indeterminacy, and especially suitable for the situation with subjective estimation or lack of historical data. That is, when the sample size is too small (even no-sample) to estimate a probability distribution, we should deal with it with the uncertainty theory. When the sample size becomes large enough, the uncertainty disappears. Mean while, the problem at hand becomes probabilistic, and we should use the probability theory instead of the uncertainty theory. The uncertainty theory provides a new approach to study the risk and insurance problems. Liu (2009) introduced the uncertainty theory in finance and promoted the research on uncertain financial markets. Liu (2010a) proposed the uncertain risk analysis and uncertain reliability analysis to quantify risk and deal with system reliability. Peng (2009) and Peng and Li (2010) presented value at risk (VaR), tail value at risk (TVaR) and spectral measure of uncertain variable in risk analysis. Peng (2013) presented two types of risk metrics of loss function for uncertain system.

In this paper, we aim to introduce a comonotonic uncertain vector within the framework of uncertainty theory and apply it to premium pricing.

The remainder of this paper is organized as follows. Section 2 presents preliminaries in the uncertainty theory. The concept of comonotonic uncertain vector is formally defined in Section 3. Some basic properties of the proposed comonotonic uncertain vector are investigated in Section 4. In Section 5, the applications of comonotonic uncertain vector are discussed. The last section contains some concluding remarks.

2. PRELIMINARIES

Let Γ be a nonempty set. A collection L of subsets of Γ is a σ-algebra. Uncertain measure M introduced by Liu (2007) is a set function if it satisfies the following axioms:

(1) (Normality) $M\{\Gamma\}=1$;
(2) (Self-duality) $M\{\Lambda\}+M\{\Lambda^c\}=1$ for any $\Lambda\in L$;
(3) (Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have $M\{\bigcup\Lambda_i\}\le\sum_i M\{\Lambda_i\}$.
(4) (Product axiom) Let (Γ_k, L_k, M_k) be uncertain space for $k=1,2,\cdots$. Then the product uncertain measure M is an uncertain measure on the product σ-algebra $\prod L_k$ satisfying

$$M\left\{\prod_{k=1}^{\infty}\Lambda_k\right\}=\min_{1\le k\le\infty}M\{\Lambda_k\}.$$

An uncertain variable is defined as a measurable function from an uncertain space (Γ, L, M) to the set of real numbers. A n-dimensional uncertain vector is a measurable function from an uncertainty space to the set of n-dimensional real vectors.

The uncertain variables ξ and η are identically distributed if $M\{\xi\in B\}=M\{\eta\in B\}$ for any Borel set B of real numbers, denoted $\xi\sim\eta$.

The uncertainty distribution $\Phi:\Re\to[0,1]$ of an uncertain variable ξ is defined by Liu (2007) as

$$\Phi(x)=M\{\gamma\in\Gamma\,|\,\xi(\gamma)\le x\}$$

and the inverse function Φ^{-1} is called the inverse uncertainty distribution of ξ.

The uncertainty distribution is an important concept to describe uncertain variable. Next, we will introduce some uncertain variables and uncertainty distributions of them.

An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(x)=\begin{cases}0, & x<a\\ \dfrac{x-a}{b-a}, & a\le x<b\\ 1, & x\ge b,\end{cases}$$

denoted by $L(a, b)$ where a and b are real numbers with $a<b$, and the inverse uncertainty distribution is

$$\Phi^{-1}(\alpha)=(1-\alpha)a+\alpha b,\quad 0<\alpha<1.$$

An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x)=\left(1+\exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1},\quad x\in\Re,$$

denoted by $N(e,\sigma)$, where e and σ are real numbers with $\sigma>0$, and the inverse uncertainty distribution is

$$\Phi^{-1}(\alpha)=e+\frac{\sqrt{3}\sigma}{\pi}\ln\frac{\alpha}{1-\alpha},\quad 0<\alpha<1.$$

An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x)=\begin{cases}0, & x<a\\ \dfrac{x-a}{2(b-a)}, & a\le x<b\\ \dfrac{x+c-2b}{2(c-b)}, & b\le x<c\\ 1, & c\le x,\end{cases}$$

denoted by Z(a, b, c) where a, b and c are real numbers with $a<b<c$, and the inverse uncertainty distribution is

$$\Phi^{-1}(\alpha)=\begin{cases}(1-2\alpha)a+2\alpha b, & \alpha<0.5\\ (2-2\alpha)b+(2\alpha-1)c, & \alpha\ge 0.5.\end{cases}$$

Let $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ be an uncertain vector. Then the joint uncertainty distribution $\Phi: \Re\rightarrow[0, 1]$ is defined by

$$\Phi(x_1, x_2,\cdots, x_n)=M\{\xi_1\le x_1, \xi_2\le x_2, \cdots, \xi_n\le x_n\}$$

for any real numbers $x_1, x_2, \cdots, x_n$.

The expected value of uncertain variable ξ is defined by Liu (2007) as

$$E[\xi]=\int_0^{+\infty}M\{\xi\ge\gamma\}d\gamma-\int_{-\infty}^{0}M\{\xi\le\gamma\}d\gamma$$

provided that at least one of the two integrals is finite.

In order to discuss the problems of multiple uncertain variables, Liu (2010b) introduces two functions as following.

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly increasing if

$$f(x_1, x_2, \cdots, x_n)<f(y_1, y_2, \cdots, y_n)$$

whenever $x_i\le y_i$ for $i=1, 2, \cdots, n$ and $x_j<y_j$ for at least one index j.

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly decreasing if

$$f(x_1, x_2, \cdots, x_n)>f(y_1, y_2,\cdots, y_n)$$

whenever $x_i\le y_i$ for $i=1, 2, \cdots, n$ and $x_j<y_j$ for at least one index j.

3. COMONOTONIC UNCERTAIN VECTOR

We first recall the comonotonicity of a set of n-vectors in R^n. For n-dimensional vectors $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, y_2, \cdots, y_n)$ in $A\subseteq R^n$, the notation $x\prec y$ will be used for the componentwise order which is defined by $x_i\le y_i$ for $i=1, 2, \cdots, n$.

Definition 1 (Dhaene *et al*., 2002). The set $A\subseteq R^n$ is called to be comonotonic if for any x and y in A, either $x\prec y$ or $y\prec x$.

Example 1. Let $A=\{(x, y)|y=x\in R\}$, then $A\subseteq R^2$ is a comonotonic set.

In order to give the comonotonicity of uncertain vector, we will first define the notion of support of n-dimensional uncertain vector.

Definition 2. Let $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ be an uncertain vector. Any set $A\subseteq R^n$ will be called a support of ξ if $M\{\xi\in A\}=1$.

Definition 3. A n-dimensional uncertain vector is said to be comonotonic if it has a comonotonic support.

Definition 4. We say that uncertain variables $\xi_1, \xi_2,\cdots, \xi_n$ are comonotonic if uncertain vector $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ is comonotonic.

The comonotonicity of an uncertain vector shows that the higher the value of one component, the higher the value of any other component.

Next, we give some equivalent definitions by following theorem to know more characterizations of comonotonicity.

Theorem 1. An uncertain vector $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ is comonotonic if and only if one of the following equivalent conditions holds:
(1) $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ has a comonotonic support;
(2) For all $x=(x_1, x_2, \cdots, x_n)$, we have
$\Phi_\xi(x)=\min\{\Phi_{\xi_1}(x_1), \Phi_{\xi_2}(x_2), \cdots, \Phi_{\xi_n}(x_n)\}$;
(3) For $\eta\sim L(0, 1)$, we have

$$\xi\sim(\Phi^{-1}{}_{\xi_1}(\eta), \Phi^{-1}{}_{\xi_2}(\eta), \cdots, \Phi^{-1}{}_{\xi_n}(\eta));$$

(4) There exist an uncertain variable ς and non-decreasing functions f_i, $i=1, 2, \cdots, n$ such that

$$\xi\sim(f_1(\varsigma), f_2(\varsigma), \cdots, f_n(\varsigma)).$$

Proof. (1) ⇒ (2). Assume that uncertain vector ξ has a comonotonic support B. We define

$$A_j=\{y\in B\,|\,y_j\le x_j\},\quad i=1, 2, \cdots, n$$

for $x=(x_1, x_2, \cdots, x_n)$. Then there must be an i such that $A_i=\bigcap_{j=1}^{n}A_j$. We only need to prove that $A_k\subseteq A_l$ or $A_l\subseteq A_k$ holds for $k\ne 1$ and $k, l\in\{1, 2, \cdots, n\}$. In fact, if $A_k\subseteq A_l$, then there exists $y\in A_k$ and $y\in A^c{}_l$, i.e., $y_k\le x_k$ and $y_l>x_l$. Hence, $\forall z\in A_l$, we have $z_l\le x_l<y_l$. Because the comonotonicity of B, the inequality $z_k\le y_k\le x_k$ holds which implies $\forall z\in A_k$. And $A_l\subseteq A_k$ can be proved. Therefore, we find

$$\begin{aligned}\Phi_\xi(x)&=M\{\xi\in x\}=M\{\xi\in\bigcap_{j=1}^{n}A_j\}\\&=M\{\xi\in A_i\}=\Phi_{\xi_i}(x_i)\\&=\min\{\Phi_{\xi_1}(x_1),\Phi_{\xi_2}(x_2), \cdots, \Phi_{\xi_n}(x_n)\}.\end{aligned}$$

The last equality holds since $\Phi_{\xi_i}(x_i)\le\Phi_{\xi_j}(x_j)$ can be obtained from $A_i\subseteq A_j$ for all $j=1, 2, \cdots, n$.

(2) ⇒ (3). For all $x=(x_1, x_2, \cdots, x_n)$, assume that

$$\Phi_{\xi}(x)=\min\{\Phi_{\xi_1}(x_1), \Phi_{\xi_2}(x_2), \cdots, \Phi_{\xi_n}(x_n)\}.$$

Then, we have

$$\begin{aligned} &M\{\Phi_{\xi_1}^{-1}(\eta)\le x_1, \Phi_{\xi_2}^{-1}(\eta), \cdots, \Phi_{\xi_n}^{-1}(\eta)\le x_n\} \\ &=M\{\eta\le \min\{\Phi_{\xi_1}(x_1), \Phi_{\xi_2}(x_2), \cdots, \Phi_{\xi_n}(x_n)\}\} \\ &=\min\{\Phi_{\xi_1}(x_1), \Phi_{\xi_2}(x_2), \cdots, \Phi_{\xi_n}(x_n)\} \end{aligned}$$

which just is the distribution function of uncertain vector $(\Phi^{-1}{}_{\xi_1}(\eta), \Phi^{-1}{}_{\xi_2}(\eta), \cdots, \Phi^{-1}{}_{\xi_n}(\eta))$.

(3) ⇒ (4). It is obvious.

(4) ⇒ (1). Let B be the support of uncertain variable ς. Then the set $C=\{(f_1(z), f_2(z), \cdots, f_n(z))\,|\, z\in B\}$ is the support of uncertain vector $(f_1(\varsigma), f_2(\varsigma), \cdots, f_n(\varsigma))$. Meanwhile, C is also the support of ξ because of $\xi \sim (f_1(\varsigma), f_2(\varsigma), \cdots, f_n(\varsigma))$. Besides, the functions f_i, $i=1, 2, \cdots, n$ show C is comonotonic. Hence, ξ has a comonotonic support.

The theorem states that we can depict the support of comonotonic uncertain vector by the inverse distribution function of component of uncertain vector.

Example 2. For uncertain variables $\xi=L(0, 1)$ and $\eta=N(0, 1)$, the inverse distribution functions are

$$\Phi_{\xi}^{-1}(\alpha)=\alpha, \quad 0<\alpha<1$$

and

$$\Phi_{\eta}^{-1}(\alpha)=e+\frac{\sqrt{3}\sigma}{\pi}\ln\frac{\alpha}{1-\alpha}, \quad 0<\alpha<1.$$

We can prove easily the uncertain measure of (ξ, η) is 1 in the set $\{(\Phi_{\xi}^{-1}(\alpha), \Phi_{\eta}^{-1}(\alpha))\,|\,0<\alpha<1\}$ which just is the support of comonotonic uncertain vector (ξ, η).

4. SOME PROPERTIES OF COMONOTONIC UNCERTAIN VECTOR

Theorem 2. An uncertain vector $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ is comonotonic if and only if (ξ_i, ξ_j) is comonotonic for all $i, j\in\{1, 2, \cdots, n\}$.

Proof. Assume that the comonotonic support of ξ is $B\subseteq R^n$. For any i, j, it is obvious that (i, j)-projection B_{ij} is the comonotonic support of (ξ_i, ξ_j). The necessity is proved. Conversely, suppose that B_{ij} is the comonotonic support of (ξ_i, ξ_j). Then there is a comonotonic set $B\subseteq R^n$ so that its (i, j)-projection is B_{ij}. Moreover, we have that $(\xi\in B)=\{(\xi_i, \xi_j)\in B_{ij}\}$ which shows $M\{\xi\in B\}=1$. Hence, $B\subseteq R^n$ is the comonotonic support of ξ and ξ is comonotonic.

Theorem 3. Let $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ be a comonotonic uncertain vector, and $f_1(x), \cdots, f_n(x)$ strictly increasing functions. Then $\eta=(f_1(\xi_1), \cdots, f_n(\xi_n))$ is comonotonic.

Proof. Let Φ_{ξ_i} and $\Phi_{f_i(\xi_i)}$ be uncertainty distributions of ξ_i and $f_i(\xi_i)$, $i=1, \cdots, n$, respectively. It follows from $\Phi_{f_i(\xi_i)}(x_i)=\Phi_{\xi_i}(f_i^{-1}(x_i))$, $i=1, \cdots, n$ that

$$\begin{aligned} \Phi_{\eta}(x)&=M\{f_1(\xi_1)\le x_1, \cdots, f_n(\xi_n)\le x_n\} \\ &=M\{\xi_1\le f_1^{-1}(x_1), \cdots, \xi_n\le f_n^{-1}(x_n)\} \\ &=\min\{\Phi_{f_1(\xi_1)}(x_1), \cdots, \Phi_{f_n(\xi_n)}(x_n)\} \end{aligned}$$

for any $x=(x_1, x_2, \cdots, x_n)$. Thus uncertain vector η is comonotonic.

Theorem 4. Let ξ and η be uncertain variables. Then ξ and η are comonotonic if and only if

$$(\xi(\gamma_1)-\xi(\gamma_2))(\eta(\gamma_1)-\eta(\gamma_2))\ge 0$$

for almost all $\gamma_1, \gamma_2\in T$.

Proof. Necessity: ξ and η are comonotonic implies that uncertain vector (ξ, η) is comonotonic. Then there is a comonotonic support B. Let $\Lambda=\{\gamma\in\,|\,(\xi, \eta)(\gamma)\in B\}$, we have $M\{\Lambda\}=1$. It follows from B is comonotonic support that $(\xi(\gamma_1)-\xi(\gamma_2))(\eta(\gamma_1)-\eta(\gamma_2))\ge 0$ for almost all $\gamma_1, \gamma_2\in\Lambda$.
Sufficiency: Let Λ be the set so that for almost all $\gamma_1, \gamma_2\in\Lambda$, $(\xi(\gamma_1)-\xi(\gamma_2))(\eta(\gamma_1)-\eta(\gamma_2))\ge 0$. Then, $M\{\Lambda\}=1$. Clearly, the set $B=\{(\xi(\gamma), \eta(\gamma))\in\Lambda\}$ is comonotonic and $M\{(\xi, \eta)\in B\}=1$ which indicates (ξ, η) is a comonotonic uncertain vector, i.e., ξ and η are comonotonic.

Theorem 5. Let ξ and η be comonotonic uncertain variables and $a, b\in R$. Then $\{\xi\le a\}\le\{\eta\le b\}$ a.s. or alternatively $\{\eta\le b\}\le\{\xi\le a\}$ a.s.

Proof. For $a, b\in R$, let $\Lambda_1=\{\xi\le a\}\cap\{\eta>b\}$ and $\Lambda_2=\{\eta\le b\}\cap\{\xi>a\}$. Then for $\theta_1\in\Lambda_1$ and $\theta_2\in\Lambda_2$, we have $(\xi(\theta_1)-\xi(\theta_2))(\eta(\theta_1)-\eta(\theta_2))<0$ since ξ and η are comonotonic. This shows $M\{\Lambda_1\}=0$ or $M\{\Lambda_2\}=0$. The theorem is proved.

For any finite comonotonic uncertain variables, we can prove the general conclusion by Theorem 5.

Theorem 6. Let $\xi_1, \xi_2, \cdots, \xi_n$ be comonotonic uncertain variables and $a_1, a_2, \cdots, a_n\in R$. Then we have

$$\{\xi_{i_1}\le a_{i_1}\}\subset\{\xi_{i_2}\le a_{i_2}\}\subset\cdots\subset\{\xi_{i_n}\le a_{i_n}\}$$

where $i_1, i_2, \cdots, i_n$ in is a permutation of $1, 2, \cdots, n$.

Similarly, the following two theorems can be obtained easily.

Theorem 7. Let ξ and η be comonotonic uncertain variables and $a, b \in R$. Then $\{\xi \geq a\} \leq \{\eta \geq b\}$ a.s. or alternatively $\{\eta \geq b\} \leq \{\xi \geq a\}$ a.s.

Theorem 8. Let $\xi_1, \xi_2, \cdots, \xi_n$ be comonotonic uncertain variables and $a_1, a_2, \cdots, a_n \in R$. Then we have

$$\{\xi_{i_1} \geq a_{i_1}\} \subset \{\xi_{i_2} \geq a_{i_2}\} \subset \cdots \subset \{\xi_{i_n} \geq a_{i_n}\}$$

where $i_1, i_2, \cdots, i_n$ in is a permutation of $1, 2, \cdots, n$.

Theorem 9. Let $\xi_1, \xi_2, \cdots, \xi_n$ be comonotonic uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2 \cdots, x_n)$ is a strictly increasing function, then $\xi = f(\xi_1, \xi_2 \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)).$$

Proof. We first prove $M\{\xi \leq \Psi^{-1}(\alpha)\} \geq \alpha\}$. It follows from $f(x_1, x_2, \cdots, x_n)$ is a strictly increasing function and Theorem 6 that

$$\begin{aligned}
&M\{\xi \leq \Psi^{-1}(\alpha)\} \\
&\quad = M\{f(\xi_1, \cdots, \xi_n) \leq f(\Phi_1^{-1}(\alpha), \cdots, \leq \Phi_n^{-1}(\alpha))\} \\
&\quad \geq M\{(\xi_1 \leq \Phi_1^{-1}(\alpha) \cap \cdots \cap \xi_n \leq \Phi_n^{-1}(\alpha)\} \\
&\quad = M\{\xi_1 \leq \Phi_1^{-1}(\alpha) \wedge \cdots \wedge M\{\xi_n \leq \Phi_n^{-1}(\alpha)\} \\
&\quad = \alpha.
\end{aligned}$$

On the other hand, there exists an index i such that

$$\begin{aligned}
&\{f(\xi_1, \cdots, \xi_n) \leq f(\Phi_1^{-1}(\alpha), \cdots, \leq \Phi_n^{-1}(\alpha))\} \\
&\subset \{\xi_i \leq \Phi_i^{-1}(\alpha)\}
\end{aligned}$$

which shows $M\{\xi \leq \Psi^{-1}(\alpha)\} \leq M\{\xi_1 \leq \Phi_1^{-1}(\alpha) = \alpha.$
Then $M\{\xi \leq \Psi^{-1}(\alpha)\} = \alpha$, that is, $\Psi^{-1}(\alpha)$ is the inverse uncertainty distribution of ξ.

Theorem 10. Let $\xi_1, \xi_2, \cdots, \xi_n$ be comonotonic uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is a strictly decreasing function, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Theorem 11. Let $\xi_1, \xi_2, \cdots, \xi_n$ be comonotonic uncertain variables with finite expected values. Then for any non-negative real numbers $a_1, a_2, \cdots, a_n \in R$, we have

$$E[\sum_{i=1}^{n} a_i \xi_i] = \sum_{i=1}^{n} a_i E[\xi_i].$$

Proof. Let $\Phi_1, \Phi_2, \cdots, \Phi_n$ be the distribution function of $\xi_1, \xi_2, \cdots, \xi_n$, respectively. Taking $f = \sum_{i=1}^{n} a_i x_i$ and applying the Theorem 9, we have the inverse uncertainty distribution of $\sum_{i=1}^{n} a_i x_i$ is $\Psi^{-1}(\alpha) = \sum_{i=1}^{n} a_i \Phi_i^{-1}(\alpha)$.

Notice that Liu (2007) has proved $E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha$ whenever the expected value of uncertain variable ξ exists. Then, we have $E[\sum_{i=1}^{n} a_i \xi_i] = \int_0^1 \Psi^{-1}(\alpha) d\alpha = \sum_{i=1}^{n} a_i E[\xi_i].$

5. APPLICATIONS

Premium pricing is one of the most important problems in actuarial science which are inseparable from risk analysis. There exist different premium principles due to diverse perception of risk. Generally, individual risks are usually assumed to be mutually independent in risk analysis. But sometimes, in insurance portfolio of risks, one risk fluctuation originated from uncertainty may influence others. For this reason, it is not proper to estimate the risk under the independence assumption. Therefore, the comonotonicity is more accurate than independence for the dependent structure of risks. Next, we discuss the application of comonotonic uncertain variables in premium pricing. In this section, the risk involved is regarded as an uncertain variable. And, the insurance pricing discussed is based on the uncertainty theory (Peng, 2010; Peng and Li, 2011). The premium pricing of uncertain variable ξ is defined by Peng and Li (2011) as

$$M_g[\xi] = \int_0^{\infty} g(\Psi(x)) dx = \int_0^1 \Phi^{-1}(\alpha) g'(1-\alpha) d\alpha$$

where $\Psi = 1 - \Phi$ and g is a distortion function.

Let $\xi_1, \xi_2, \cdots, \xi_n$ be n risks with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Without loss of generality, we assume that the exist of uncertainty causes risks $\xi_1, \xi_2, \cdots, \xi_n$ to become high simultaneously. It is obvious that $\xi_1, \xi_2, \cdots, \xi_n$ are comonotonic risks with inverse distributions $\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)$. It follows from Theorem 9 that the inverse distribution of $\xi_1 + \cdots + \xi_n$ is $\Phi_1^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha)$.

For any non-negative real $a_1, a_2, \cdots, a_n$, it is obtained that the premium pricing of portfolio $\xi = a_1\xi_1 + \cdots + a_n \xi_n$ is $M_g[\sum_{i=1}^{n} a_i \xi_i] = \sum_{i=1}^{n} a_i M_g[\xi_i]$.

This means that the uncertain premium satisfies the linearity for comonotonic uncertain losses. Especially, taking distortion function $g(x) = x$, we can get the uncertain net premium pricing.

Example 3. Let $\xi = Z(1, 2, 3)$ and $\eta = Z(2, 3, 4)$ be two comonotonic zigzag uncertain variables with distributions

$$\Phi(x)=\begin{cases}0, & x<1\\ (x-1)/2, & 1\le x<3\\ 1, & x\ge 3\end{cases}$$

and

$$\Psi(x)=\begin{cases}0, & x<2\\ (x-2)/2, & 2\le x<4\\ 1, & x\ge 4.\end{cases}$$

For non-negative real $a_1=0.4$ and $a_2=0.6$, we can calculate the uncertain net premium pricing of portfolio $0.4\xi+0.6\eta$ is $E[0.4\xi+0.6\eta]=2.6$.

Example 4. Taking (Γ, L, M) to be $\{\gamma_1,\gamma_2,\gamma_3\}$ with $M\{\gamma_1\}=0.7$, $M\{\gamma_2\}=0.3$ and $M\{\gamma_3\}=0.2$. Then $M\{\gamma_1, \gamma_2\}=0.8$, $M\{\gamma_1,\gamma_3\}=0.7$, $M\{\gamma_2,\gamma_3\}=0.3$. Define two uncertain losses as follows,

$$\xi(\gamma)=\begin{cases}0, & \gamma=\gamma_1\\ 1, & \gamma=\gamma_2\\ 2, & \gamma=\gamma_3,\end{cases}\qquad \eta(\gamma)=\begin{cases}0, & \gamma=\gamma_1\\ 2, & \gamma=\gamma_2\\ 3, & \gamma=\gamma_3.\end{cases}$$

Note that ξ and η are comonotonic, and their sum is

$$(\xi+\eta)(\gamma)=\begin{cases}0, & \gamma=\gamma_1\\ 3, & \gamma=\gamma_2\\ 5, & \gamma=\gamma_3.\end{cases}$$

Thus $E[\xi]=0.5, E[\eta]=0.8,$ and $E[\xi+\eta]=1.3.$ This fact implies $E[\xi+\eta]=E[\xi]+E[\eta]$.

The above examples imply that when the risks are comonotonic, the uncertain net premium of portfolio is the linear sum of every risk premium. However, generally speaking, the uncertain net premium is not necessarily linear if the comonotonicity is not assumed in uncertainty theory.

Example 5. Taking (Γ, L, M) to be $\{\gamma_1,\gamma_2,\gamma_3\}$ with $M\{\gamma_1\}=0.7, M\{\gamma_2\}=0.3$ and $M\{\gamma_3\}=0.2.$ Then $M\{\gamma_1,\gamma_2\}$ $M\{\gamma_1,\gamma_2\}=0.8, M\{\gamma_1,\gamma_3\}=0.7, M\{\gamma_2,\gamma_3\}=0.3.$ Define two uncertain losses as follows,

$$\xi(\gamma)=\begin{cases}1, & \gamma=\gamma_1\\ 0, & \gamma=\gamma_2\\ 2, & \gamma=\gamma_3,\end{cases}\qquad \eta(\gamma)=\begin{cases}0, & \gamma=\gamma_1\\ 2, & \gamma=\gamma_2\\ 3, & \gamma=\gamma_3.\end{cases}$$

Note that ξ and η are not comonotonic, and their sum is

$$(\xi+\eta)(\gamma)=\begin{cases}1, & \gamma=\gamma_1\\ 2, & \gamma=\gamma_2\\ 5, & \gamma=\gamma_3.\end{cases}$$

Thus $E[\xi]=0.9, E[\eta]=0.8,$ and $E[\xi+\eta]=1.9.$ This fact implies $E[\xi+\eta]>E[\xi]+E[\eta]$.
If the uncertain losses defined by

$$\xi(\gamma)=\begin{cases}0, & \gamma=\gamma_1\\ 1, & \gamma=\gamma_2\\ 2, & \gamma=\gamma_3,\end{cases}\qquad \eta(\gamma)=\begin{cases}0, & \gamma=\gamma_1\\ 3, & \gamma=\gamma_2\\ 1, & \gamma=\gamma_3.\end{cases}$$

Then ξ and η are not comonotonic, and

$$(\xi+\eta)(\gamma)=\begin{cases}0, & \gamma=\gamma_1\\ 4, & \gamma=\gamma_2\\ 3, & \gamma=\gamma_3.\end{cases}$$

Thus $E[\xi]=0.5, E[\eta]=0.9,$ and $E[\xi+\eta]=1.2.$ This fact implies $E[\xi+\eta]<E[\xi]+E[\eta].$

6. CONCLUSION

The comonotonic idea often appears in actuarial science and finance. In the past, the theory and application of comonotonicity are mainly related to the probability theory. The uncertainty theory provides a new approach to deal with the comonotonic uncertain phenomena. In this paper, the concept of comonotonic uncertain vector was introduced and some properties were investigated under the framework of the uncertainty theory. It was mainly proved that the expected value operator satisfies the linearity for comonotonic uncertain variables. In application, the comonotonicity of uncertain variables was applied to premium pricing problems and some interesting results were obtained. It was found that the obtained results are different from those in stochastic insurance, and for the purpose of comparison, some numerical examples were given.

ACKNOWLEDGMENT

This work is supported by the National Natural Science Foundation (No. 61273044), the Key Project of Hubei Provincial Natural Science Foundation (No. 2012 FFA065), and the Scientific and Technological Innovation Team Project (No. T201110) of Hubei Provincial Department of Education, China.

REFERENCES

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999), Coherent measures of risk, *Mathematical Finance*, **9**(3), 203-228.
Benvenuti, P. and Vivona, D. (1996), General theory of the fuzzy integral, *Mathware and Soft Computing*,

3(1/2), 199-209.

Benvenuti, P. and Vivona, D. (2000), Comonotone aggregation operators, *Rendiconti di mathematica e delle sue applicazioni*, **20**, 323-336.

Bernadette, B., Julio, G., and Luis, M. (2002), *Technologies for Constructing Intelligent Systems 2: Tools*, Springer-Verlag, Heidelberg.

Cheung, K. C. (2008), Characterization of comonotonicity using convex order, *Insurance: Mathematics and Economics*, **43**(3), 403-406.

Choudhry, M. (2006), *An Introduction to Value-at-Risk* (4th ed.), John Wiley and Sons, Hoboken, NJ.

Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R., and Vyncke, D. (2002), The concept of comonotonicity in actuarial science and finance: theory, *Insurance: Mathematics and Economics*, **31**(1), 3-33.

Dhaene, J., Vanduffel, S., Goovaerts, M. J., Kaas, R., Tang, Q., and Vyncke, D. (2006), Risk measure and comonotonicity: a review, *Stochastic Models*, **22**, 573-606.

Dhaene, J., Wang, S., Young, V., and Goovaerts, M. J. (2000), Comonotonicity and maximal stop-loss premiums, *Bulletin of the Swiss Association of Actuaries*, **2**, 99-113.

Ekelanda, I., Galichon, A., and Henry, M. (2012), Comonotonic measures of multivariate risks, *Mathematical Finance*, **22**, 109-132.

Kast, R. and Lapied, A. (2003), Comonotonic book making and attitudes to uncertainty, *Mathematical Social Sciences*, **46**(1), 1-7.

Liu, B. (2007), *Uncertainty Theory* (2nd ed.), Springer-Verlag, Berlin.

Liu, B. (2009), Some research problems in uncertainty theory, *Journal of Uncertain Systems*, **3**(1), 3-10.

Liu, B. (2010a), Uncertain risk analysis and uncertain reliability analysis, *Journal of Uncertain Systems*, **4**(3), 163-170.

Liu, B. (2010b), *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin.

Ludkovski, M. and Ruschendorf, L. (2008), On comonotonicity of pareto optimal risk sharing, *Statistics & Probability Letters*, **78**, 1181-1188.

Mesiar, R. and Ouyang, Y. (2009), General chebyshev type inequalities for sugeno integrals, *Fuzzy Sets and Systems*, **160**(1), 58-64.

Modave, F. and Grabisch, M. (1998), Preference representation by the choquet integral: the commensurability hypothesis, *Proceedings of the 7th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, Paris, France, 164-171.

Narukawa, Y. and Torra, V. (2009), Multidimensional generalized fuzzy integral, *Fuzzy Sets and Systems*, **160**(6), 802-815.

Peng, J. (2009), Value at risk and tail value at risk in uncertain environment, *Proceedings of the 8th International Conference on Information and Management Sciences*, Kunming, China, 787-793.

Peng, J. (2013), Risk metrics of loss function for uncertain system, *Fuzzy Optimization and Decision Making*, **12**(1), 53-64.

Peng, J. and Li, S. (2010), Spectral measures of uncertain risk, *Proceedings of the 1st International Conference on Uncertainty Theory*, Urumchi, China, 115-121.

Peng, J. and Li, S. (2011), Distortion risk measures of uncertain systems, *Proceedings of the 9th International Conference on Reliability, Maintainability and Safety*, Guiyang, China, 460-467.

Peng, Y. (2010), A new type of risk measuring and insurance pricing based on uncertainty theory, *Journal of Huanggang Normal University*, **30**(6), 27-31.

Roell, A. (1987), Risk aversion in Quiggin and Yaari's rank-order model of choice under uncertainty, *The Economic Journal*, **97**, 143-159.

Szego, G. P. (2004), *Risk Measures for the 21st Century*, John Wiley and Sons, Hoboken, NJ.

Wakker, P. P. (1996), The sure-thing principle and the comonotonic sure-thing principle: an axiomatic analysis, *Journal of Mathematics and Economics*, **25**(2), 213-227.

Wang, S. and Dhaene, J. (1998), Comonotonicity, correlation order and premium principles, *Insurance: Mathematics and Economics*, **22**(3), 235-242.

Yaari, M. E. (1987), The dual theory of choice under risk, *Econometrica*, **55**(1), 95-115.

Zadeh, L. A. (1965), Fuzzy sets, *Information and Control*, **8**, 338-353.