Some Characterizations of Parabolas

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Abstract. We study some properties of tangent lines of parabolas. As a result, we establish some characterizations of parabolas.

1. Introduction and Preliminaries

Next to straight lines and circles, one of the most simple and interesting curves in a plane is a parabola. A characterization of ellipse was studied by the present authors in terms of the curvature and the support function ([5]). As was described in [2], a circle is characterized by the fact that the chord joining any two points on it meets the circle at the same angle.

Hammer and Smith ([4]) gave a characterization for a circle in the Euclidean plane and it was generalized to the isoperimetrix of the Minkowski plane ([1]). For some geometric characterizations of ellipses and hyperbolas (respectively, of parabolas), see [5] (respectively, [9]). In this regard, it is interesting to consider what simple geometric properties characterize a parabola.

In this paper, we examine the parabola concerning the chord connecting two points on a parabola and discuss the converse problems of well known properties about the parabola.

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Consider a parabola \( P \), which is given by, say, \( y = f(x) \), where \( f(x) \) is a quadratic polynomial. Then the following are well-known.

**Proposition 1** ([8]). A pair of tangent lines to \( P \) at \( x = x_1 \) and at \( x = x_2 \) meet at \( x = (x_1 + x_2)/2 \).

**Proposition 2** ([7], pp.132-134). For any chord \( AB \) on \( P \) with \( A = (x_1, y_1), B = (x_2, y_2) \), the tangent line to \( P \) at \( x = (x_1 + x_2)/2 \) is parallel to the chord.

**Proposition 3** ([6], p.535). A pair of tangent lines to \( P \) through a point on the directrix of \( P \) intersect at right angle and the chord through the points of tangency always contains the focus of \( P \).

As a matter of fact, it is natural to ask if the converses of such properties hold.

We mainly focus on such in this paper.

### 2. Main Results

In this section, we prove the following:

**Theorem 4** ([8]). A curve \( C \) of class \( C^3 \) given by \( y = f(x) \) is a parabola if it satisfies the following condition.

\( (C_1) \) For any two numbers \( x_1 \) and \( x_2 \), the pair of tangent lines to \( C \) at \( x = x_1 \) and at \( x = x_2 \) meet at \( x = (x_1 + x_2)/2 \).

In [8], it was shown that a curve \( C \) given by \( y = f(x) \) is a parabola if it satisfies \((C_1)\) and \( f(x) \) is analytic.

**Theorem 5.** A curve \( C \) of class \( C^2 \) given by \( y = f(x) \) is a parabola if it satisfies the following condition.

\( (C_2) \) For any chord \( AB \) on \( C \) with \( A = (x_1, y_1), B = (x_2, y_2) \), the tangent line to \( C \) at \( x = (x_1 + x_2)/2 \) is parallel to the chord.

**Theorem 6.** A convex curve \( C \) of class \( C^2 \) is a parabola if it satisfies the following condition.

\( (C_3) \) There are a line \( L \) and a point \( F \) such that for any point \( p \) on \( L \) there are two tangent lines of \( C \) through \( p \) which are perpendicular to each other, and the chord connecting the points of tangency passes through \( F \).

First, suppose that \( C \) satisfies \((C_1)\). Then the tangent lines given by

\[
\begin{align*}
y - f(x_1) &= f'(x_1)(x - x_1), \\
y - f(x_2) &= f'(x_2)(x - x_2)
\end{align*}
\]

have the point of intersection at \( x = (x_1 + x_2)/2 \). Hence we get

\[
2\{f(x_1) - f(x_2)\} = (x_1 - x_2)\{f'(x_1) + f'(x_2)\}. \tag{2}
\]

Differentiating (2) with respect to \( x_1 \), we obtain

\[
f'(x_1) - f'(x_2) = (x_1 - x_2)f''(x_1). \tag{3}
\]
Once more, we differentiate (3) with respect to $x_1$. Then we have

$$ (x_1 - x_2)f'''(x_1) = 0, $$

which shows that $f(x)$ is a quadratic polynomial. This completes the proof of Theorem 4.

From the proof of Theorem 4, we see that the point $x = x_2$ might be fixed.

Second, suppose that $C$ satisfies ($C_2$). Then we have

$$ f(x_1) - f(x_2) = (x_1 - x_2)f'\left(\frac{x_1 + x_2}{2}\right). $$

Differentiating (5) with respect to $x_1$ and $x_2$, respectively, we get

$$ f'(x_1) = f'\left(\frac{x_1 + x_2}{2}\right) + \frac{1}{2}(x_1 - x_2)f''\left(\frac{x_1 + x_2}{2}\right) $$

and

$$ -f'(x_2) = -f'\left(\frac{x_1 + x_2}{2}\right) + \frac{1}{2}(x_1 - x_2)f''\left(\frac{x_1 + x_2}{2}\right). $$

It follows from (6) and (7) that

$$ \frac{f'(x_1) + f'(x_2)}{2} = f'\left(\frac{x_1 + x_2}{2}\right). $$

Differentiating (8) with respect to $x_1$ and $x_2$, respectively, we obtain

$$ f''(x_1) = f''\left(\frac{x_1 + x_2}{2}\right) $$

and

$$ f''(x_2) = f''\left(\frac{x_1 + x_2}{2}\right). $$

It follows from (9) and (10) that $f''(x)$ is a constant, which completes the proof of Theorem 5.

Finally, suppose that $C$ satisfies ($C_3$). Then we may introduce a coordinate system $(x, y)$ of $\mathbb{R}^2$ such that $x$-axis is the line $L$, $F = (b, c)$ and $C$ is given by $y = f(x)$ with $f(x) > 0$. We denote by $V = (a, p)$ the point of $C$ where $p$ is the minimum value of $y = f(x)$.

For any point $(t, 0)$ of $L$, we denote by $m(t)$ and $-\frac{1}{m(t)}$ ($m(t) > 0$) the slopes of the tangent lines to $C$ through $(t, 0)$. Then $C$ is nothing but the envelope of the following 1-parameter family of lines:

$$ y = m(t)(x - t)(x \geq t), $$

$$ y = -\frac{1}{m(t)}(x - t)(x \leq t). $$
When $x \geq a$, letting $F(x, y, t) = m(t)x - y - tm(t)$, the curve $C$ is given by ([3], p.59)

$$F(x, y, t) = m(t)x - y - tm(t) = 0,$$

(12)

$$\frac{\partial F(x, y, t)}{\partial t} = m'(t)x - m(t) - tm'(t) = 0.$$ 

From (12), the curve $C = (x_1, y_1)$ is given by

$$x_1 = t + \frac{m(t)}{m'(t)},$$

(13)

$$y_1 = \frac{m(t)^2}{m'(t)}.$$

When $x \leq a$, using a similar argument as the above, we see that the curve $C = (x_2, y_2)$ is given by

$$x_2 = t - \frac{m(t)}{m'(t)},$$

(14)

$$y_2 = \frac{1}{m'(t)}.$$

Since the curve $C$ is convex, $m(t) : (-\infty, \infty) \to (0, \infty)$ is a strictly increasing function which satisfies

$$\lim_{t \to -\infty} (x_1, y_1) = \lim_{t \to \infty} (x_2, y_2) = V = (a, p).$$

Let’s put $A = (x_1, y_1), B = (x_2, y_2)$. Since the chord $AB$ passes through $F = (b, c)$, (13) and (14) show that

$$m(t)\left(\frac{m(t)^2}{m'(t)}(b - x_1(t)) + y_1(t)\right) = c.$$  

(16)

Since $\lim_{t \to -\infty} m(t) = 0$, it follows from (15) and (16) that $a = b$, hence we have $F = (a, c)$.

Substituting $x_1, y_1$ in (13) into (16), we get a differential equation:

$$m'(t)\{(m^2 - 1)(t - a) + 2cm\} = m(m^2 + 1),$$

(17)

which is equivalent to

$$-m(m^2 + 1)dt + \{(m^2 - 1)(t - a) + 2cm\}dm = 0.$$  

(18)

Letting $M = -m(m^2 + 1)$ and $N = (m^2 - 1)(t - a) + 2cm$, we have

$$\frac{1}{M}(Nt - Mn) = \frac{-4m}{m^2 + 1}.$$  

(19)
Hence an integrating factor $\mu$ of the equation (18) is given by
\begin{equation}
\mu = e^{\int \frac{-m}{m^2+1} \, dm} = (m^2 + 1)^{-2}.
\end{equation}

Multiplying both sides of (18) by $\mu$ in (20), we get
\begin{equation}
\frac{-m}{m^2+1} \, dt + \left\{ \frac{m^2 - 1}{(m^2 + 1)^2} (t - a) + \frac{2cm}{(m^2 + 1)^2} \right\} dm = 0,
\end{equation}
which is an exact differential equation. By integrating (21), we find
\begin{equation}
\frac{(t - a)m + c}{m^2 + 1} = d, \quad d \in R,
\end{equation}
or equivalently,
\begin{equation}
dm^2 - (t - a)m - (c - d) = 0.
\end{equation}

Since $m(t) \to 0$ as $t \to -\infty$, (23) implies that $(a - t)m(t) > 0$ converges to $c - d$ as $t \to -\infty$, hence we see that $c - d > 0$. Since $\lim_{t \to \infty} m(t) = \infty$, (22) shows that $d > 0$. Because $m(t) > 0$, it follows from (23) that
\begin{equation}
m(t) = \frac{1}{2d} \left\{ t - a + \sqrt{(t - a)^2 + \alpha^2} \right\}, \quad \alpha^2 = 4d(c - d).
\end{equation}

Together with (24), (13) and (14) yield, respectively, that
\begin{equation}
\begin{aligned}
y_1 &= \frac{1}{4d}(x_1 - a)^2 + \frac{\alpha^2}{4d}, \quad x_1 \geq a, \\
y_2 &= \frac{d}{\alpha^2}(x_2 - a)^2 + d, \quad x_2 \leq a.
\end{aligned}
\end{equation}

Since $p = f(a)$, it follows from (25) that $d = p$ and $\alpha = 2p$. Thus the curve $C$ is the parabola given by $y = \frac{1}{4p}(x - a)^2 + p$ with focus $F = (a, 2p)$. This completes the proof of Theorem 6.

References


