On Nearly Pairwise Compact Spaces

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Abstract. In this paper, we introduce the notion of near pairwise compactness which generalizes the notion of pairwise compactness.

1. Introduction

Singal and Mathur [10] introduced and studied the notion of near compactness by generalizing the concept of compactness of a topological space. Later the notion of near compactness studied and developed considerably by Carnahan [1], Singal and Mathur [8], Herrington [3], Joseph [4] and others. The notion of near compactness became an important meadow to topologists. Following these trends, Nandi [6] introduced the notion of near compactness in bitopological spaces: A bitopological space \((X, P_1, P_2)\) is said to be \(ij\)-nearly compact if for each \((P_i)\) open cover \(\mathcal{U}\) of \(X\), there exists a finite subcollection \(\mathcal{V} \subset \mathcal{U}\) such that \(\{(P_i)\text{int}(P_j)clV) | V \in \mathcal{V}\}\) covers \(X\). \(X\) is said to be pairwise nearly compact if it is 12- and 21-nearly compact. The notion of pairwise near compactness is defined considering only \((P_i)\) open sets. As such, this notion of pairwise near compactness cannot be a generalization of pairwise compactness (Fletcher et al. [2]). In this paper, we introduce a generalized notion of pairwise compactness and we call it nearly pairwise compact (Definition 2.7). It is also a generalization of near compactness.
2. Preliminaries

Unless or otherwise mentioned, $X$ stands for the bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$. We recall the following definitions.

**Definition 2.1.** A collection $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ is said to be pairwise open if for each $\alpha \in A$, $U_\alpha$ is $(\mathcal{P}_i)$ open for some $i \in \{1, 2\}$ and for each $i \in \{1, 2\}$, $\mathcal{U} \cap \mathcal{P}_i \neq \emptyset$. A pairwise open collection covering $X$ is called a pairwise open cover (Fletcher et al. [2]).

A collection $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$ of subsets of $X$ is said to be pairwise closed (Pahk and Choi [7]) if $\{X - F_\alpha \mid \alpha \in A\}$ is pairwise open.

**Definition 2.2([5]).** In a bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$, the topology $\mathcal{P}_i$ is said to be regular with respect to $\mathcal{P}_j$, if for each $x \in X$ and each $(\mathcal{P}_i)$ closed set $A$ with $x \notin A$, there exist $U \in \mathcal{P}_i$ and $V \in \mathcal{P}_j$ such that $x \in U$, $A \subset V$ and $U \cap V = \emptyset$.

$x$ is said to be pairwise regular if $\mathcal{P}_i$ is regular with respect to $\mathcal{P}_j$ for both $i = 1$ and $i = 2$.

**Definition 2.3([11]).** Let $(X, \mathcal{P}_1, \mathcal{P}_2)$ and $(Y, \mathcal{D}_1, \mathcal{D}_2)$ be two bitopological spaces and $\mathcal{P}_1 \times \mathcal{D}_2$ be the product topology on $X \times Y$ of the topologies $\mathcal{P}_1$ and $\mathcal{D}_2$ on $X$ and $Y$ respectively. Then the bitopological space $(X \times Y, \mathcal{P}_1 \times \mathcal{D}_1, \mathcal{P}_2 \times \mathcal{D}_2)$ is called the product bitopological space of the spaces $(X, \mathcal{P}_1, \mathcal{P}_2)$ and $(Y, \mathcal{D}_1, \mathcal{D}_2)$.

**Definition 2.4([9]).** A set $A \subset X$ is said to be $(i, j)$ regularly open if $A = (\mathcal{P}_i)\text{int}(\mathcal{P}_j)\text{cl}(A)$.

A subset of $X$ is said to be $(i, j)$ regularly closed if its complement is $(i, j)$ regularly open. In other words, a set $A \subset X$ is $(i, j)$ regularly closed iff $A = (\mathcal{P}_i)\text{cl}(\mathcal{P}_j)\text{int}(A)$.

**Definition 2.5([9]).** A bitopological space $X$ is said to be pairwise semiregular iff for each $x \in X$ and each $(\mathcal{P}_i)$ open set $U$ with $x \in U$, there exists a $(\mathcal{P}_i)$ open set $V$ such that $x \in V \subset (\mathcal{P}_i)\text{int}(\mathcal{P}_j)\text{cl}(V) \subset U$.

Obviously, a pairwise regular space is pairwise semiregular.

**Definition 2.6([9]).** A bitopological space $X$ is said to be pairwise almost regular if for each $x \in X$ and each $(i, j)$ regularly closed set $F$ with $x \notin F$, there exist a $(\mathcal{P}_i)$ open set $U$ and a $(\mathcal{P}_j)$ open set $V$, $j \neq i$, $i, j \in \{1, 2\}$, such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.

Equivalently, $X$ is pairwise almost regular iff for each $x \in X$ and each $(i, j)$ regularly open set $U$ with $x \in U$, there exists a $(\mathcal{P}_i)$ open set $V$ such that $x \in V \subset (\mathcal{P}_j)\text{cl}(V) \subset U$.

We introduce the following definitions.

**Definition 2.7.** A bitopological space $X$ is said to be nearly pairwise compact if for each pairwise open cover $\mathcal{W}$ of $X$, there exists a finite subcollection $\mathcal{V} \subset \mathcal{W}$ such that $\{(\mathcal{P}_i)\text{int}(\mathcal{P}_j)\text{cl}(V) \mid V \in \mathcal{V} \cap \mathcal{P}_i, i \in \{1, 2\}\}$ covers $X$. 
Obviously, a pairwise compact space is nearly pairwise compact. The following examples shows that, the notion of pairwise near compactness and near pairwise compactness are independent.

**Example 2.1.** Let $b$ be a fixed real number. We define

\[
P_1 = \{\emptyset, R\} \cup \left\{ \left(-\frac{1}{n}, \infty \right) \mid n \in \mathbb{N} \} \cup \{[b, \infty)\},\right.
\]

\[
P_2 = \{\emptyset, R\} \cup \left\{ \left(-\infty, b - \frac{1}{n} \right) \mid n \in \mathbb{N} \} \cup \{(-\infty, b], [b, \infty)\} \right.
\]

\[
\cup \left\{ \left[R - \left(b - \frac{1}{n}, b\right) \right] \mid n \in \mathbb{N} \} \right\}.
\]

$(R, P_1, P_2)$ is pairwise nearly compact but it is not nearly pairwise compact.

**Example 2.2** (cf. [11], p. 142). Let

\[
P_1 = \{\emptyset, R\} \cup \{(-\infty, n) \mid n \in \mathbb{Z}\},
\]

\[
P_2 = \{\emptyset, R\} \cup \{(n, \infty) \mid n \in \mathbb{Z}\}
\]

where $\mathbb{Z}$ is the set of integers. The bitopological space $(R, P_1, P_2)$ is not $ij$-nearly compact for any $i \in \{1, 2\}$. Hence the space is not pairwise nearly compact. The space is pairwise compact and hence it is also nearly pairwise compact.

**Example 2.3.** Let $b$ be a fixed real number. We define

\[
P_1 = \{\emptyset, R\} \cup \left\{ \left(-\infty, b\right] \cup \left[b, \infty \right) \right\},
\]

\[
P_2 = \{\emptyset, R\} \cup \{\left(b, \infty \right) \cup \{\left(b + \frac{1}{n}, \infty \right) \mid n \in \mathbb{N} \} \right\}.
\]

$(R, P_1, P_2)$ is nearly pairwise compact but it is not pairwise compact.

**Definition 2.8.** A bitopological space $X$ is said to be almost pairwise compact if for each pairwise open cover $U$ of $X$, there exists a finite subcollection $V \subset U$ such that $\{(P_i)_{cl}V \mid V \in U \cap P_i, i \in \{1, 2\}\}$ covers $X$.

It readily follows from definitions, a nearly pairwise compact space is an almost pairwise compact space.

**Definition 2.9.** A cover $C$ of $X$ is said to be a pairwise basic cover if there exist two bases $B_1$ and $B_2$ of the topologies $P_1$ and $P_2$ respectively such that $C \subset B_1 \cup B_2$ and for each $i \in \{1, 2\}$, $C \cap B_i \neq \emptyset$.

**Definition 2.10.** A collection $U$ (resp. $F$) of subsets of $X$ is said to be pairwise regularly open (resp. pairwise regularly closed) if each member of $U$ (resp. $F$) is $(i,j)$regularly open (resp. $(i,j)$regularly closed) for some $i \in \{1, 2\}$ and contains at
least one \((i, j)\) regularly open (resp. \((i, j)\) regularly closed) set for each \(i \in \{1, 2\}\). \(\mathcal{U}\) (resp. \(\mathcal{F}\)) is said to be a pairwise regularly open (resp. pairwise regularly closed) cover if it covers \(X\).

**Definition 2.11.** A bifilter is a collection \(\mathcal{F}\) of nonempty subsets of \(X\) with the following properties:

\[
\begin{align*}
(a) & \quad \mathcal{F} \subset \mathcal{P}_1 \cup \mathcal{P}_2 \text{ and } \mathcal{F} \cap \mathcal{P}_i \neq \emptyset \text{ for each } i \in \{1, 2\}. \\
(b) & \quad \text{If } E, F \in \mathcal{F} \text{ with } E, F \in \mathcal{P}_i \text{ for some } i \in \{1, 2\} \text{ then } E \cap F \in \mathcal{F}. \\
(c) & \quad \text{If } G \in \mathcal{F} \text{ and } H \supseteq G \text{ with } G, H \in \mathcal{P}_i \text{ for some } i \in \{1, 2\} \text{ then } H \in \mathcal{F}.
\end{align*}
\]

**Definition 2.12.** A bifilter \(\mathcal{F}\) on a bitopological space \(X\) is said to be maximal provided

\[
\begin{align*}
(a) & \quad \text{for any bifilter } \mathcal{G}\text{ on } X, \mathcal{G} \subset \mathcal{F}, \\
(b) & \quad \text{if } \mathcal{G}\text{ is a bifilter with } \mathcal{F} \subset \mathcal{G}, \text{ then } \mathcal{F} = \mathcal{G}.
\end{align*}
\]

**Definition 2.13.** A point \(p \in X\) is said to be a bicluster point of a bifilter \(\mathcal{F}\) if for each \(F \in \mathcal{F}\), \(p \in (\mathcal{P}_i)\text{cl} F\) whenever \(F\) is \((\mathcal{P}_i)\text{open}\) for some \(i \in \{1, 2\}\).

**Definition 2.14.** A \((\mathcal{P}_i)\text{open}\) set containing a point \(p \in X\) is said to be a \((\mathcal{P}_i)\text{open}\) neighbourhood (abbreviated as \((\mathcal{P}_i)\text{open nbd}) of \(p\).

**Definition 2.15.** A point \(p\) is said to be a biconvergent point of a bifilter \(\mathcal{F}\) if each \((\mathcal{P}_i)\text{open nbd}\) of \(p\) is a member of \(\mathcal{F}\).

Throughout the paper, \(N\) denotes the set of natural numbers and \(R\), the set of real numbers. For a pairwise open (resp. closed) collection \(\mathcal{U}\) (resp. \(\mathcal{F}\)) of subsets of a bitopological space \((X, \mathcal{P}_1, \mathcal{P}_2)\), we write \(\mathcal{U}'\) (resp. \(\mathcal{F}'\)) to denote the collection of all \((\mathcal{P}_i)\text{open}\) (resp. \((\mathcal{P}_i)\text{closed}\) sets in \(\mathcal{U}\) (resp. \(\mathcal{F}\)). \((\mathcal{F})\text{int} A\) (resp. \((\mathcal{F})\text{cl} A\)) denotes the interior (resp. closure) of a set \(A\) in a topological space \((X, \mathcal{F})\). Always \(i, j \in \{1, 2\}\) and whenever \(i, j\) appear together, \(j \neq i\).

**3. Results**

We now establish the following theorems on nearly pairwise compact spaces.

**Theorem 3.1.** In a bitopological space \(X\), the following statements are equivalent:

\[
\begin{align*}
(a) & \quad \text{\(X\) is nearly pairwise compact.} \\
(b) & \quad \text{Each pairwise basic cover } \mathcal{U}\text{ of } X\text{ possesses a finite subcollection } \mathcal{V} \subset \mathcal{U}\text{ such that } \{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl} V) \mid V \in \mathcal{V} \cap \mathcal{P}_i, i \in \{1, 2\}\}\text{ covers } X. \\
(c) & \quad \text{Each pairwise regularly open cover of } X\text{ has a finite subcover.} \\
(d) & \quad \text{Each pairwise regularly closed collection of subsets of } X\text{ with finite intersection property has nonempty intersection.} \\
(e) & \quad \text{Each pairwise closed collection } \mathcal{F} = \{F_\alpha \mid \alpha \in B\} \text{ of subsets of } X\text{ with the property that for any finite subcollection } \mathcal{E} \subset \mathcal{F}, \bigcap\{(\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int} F_\alpha) \mid F_\alpha \in \mathcal{E}, i \in \{1, 2\}\} \neq \emptyset, \text{ has a nonempty intersection.}
\end{align*}
\]
Proof. \((a) \Rightarrow (b):\) Obvious.

\((b) \Rightarrow (c):\) Let \(\mathcal{G} = \{G_\alpha \mid \alpha \in A\}\) be a pairwise regularly open cover of \(X\) and let \(\mathcal{P}_i\) be a base of the topology \(\mathcal{P}_i\). For each \(G_\alpha \in \mathcal{G}\) with \(G_\alpha \in \mathcal{P}_i\), there exist \(\mathcal{H}_\alpha = \{H_\lambda \mid \lambda \in \Lambda, H_\lambda \in \mathcal{P}_i\}\) such that \(G_\alpha = \bigcup\{H_\lambda \mid H_\lambda \in \mathcal{H}_\alpha\}\). Then \(\mathcal{W} = \{H_\lambda \mid \lambda \in \Lambda, \alpha \in A\}\) is a pairwise basic cover of \(X\). So by \((b)\), we obtain a finite subcollection \(\mathcal{V} = \{H_{\lambda_k} \mid k = 1,2,\ldots,m\}\) of \(\mathcal{W}\) such that \(\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}H_{\lambda_k}) \mid H_{\lambda_k} \in \mathcal{V} \cap \mathcal{P}_i, k = 1,2,\ldots,m\}\) covers \(X\). For each \(H_{\lambda_k} \in \mathcal{P}_i\), there exists a \(G_{\alpha_k} \in \mathcal{P}_i, \alpha_k \in A\) such that \(H_{\lambda_k} \subset G_{\alpha_k}\) which implies \((\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}H_{\lambda_k}) \subset (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_{\alpha_k}) = G_{\alpha_k}\). Then \(\{G_{\alpha_k} \mid k = 1,2,\ldots,m\}\) is a finite subcover of \(\mathcal{G}\).

\((c) \Rightarrow (d):\) We suppose that \(\mathcal{F} = \{F_\alpha \mid \alpha \in I\}\) is a pairwise regularly closed collection of subsets of \(X\) with finite intersection property i.e. for each \(n \in N, \bigcap\{F_{\alpha_k} \mid k = 1,2,\ldots,n\} \neq \emptyset\). If possible, let \(\bigcap\{F_{\alpha} \mid \alpha \in I\} = \emptyset\). Then \(X - F_\alpha \mid \alpha \in I\) is a pairwise regularly open cover of \(X\). So by \((c)\), \(X - F_\alpha \mid \alpha \in I\) has a finite subcover \(X - F_{\alpha_k} \mid k = 1,2,\ldots,m\) which in turn implies \(\bigcap\{F_{\alpha_k} \mid k = 1,2,\ldots,m\} = \emptyset\). This is a contradiction to our assumption. Thus we have \(\bigcap\{F_{\alpha} \mid \alpha \in I\} \neq \emptyset\).

\((d) \Rightarrow (c):\) We suppose that \(\mathcal{F} = \{F_\alpha \mid \alpha \in B\}\) is a pairwise closed collection of subsets of \(X\) with \(X - F_\alpha \in \mathcal{P}_i\) such that for any finite subcollection \(\mathcal{E}\) of \(\mathcal{F}\), \(\bigcap\{\mathcal{P}_i\text{cl}((\mathcal{P}_j)\text{int}F_\alpha) \mid F_\alpha \in \mathcal{E}, i \in \{1,2\}\} \neq \emptyset\). Thus \(\mathcal{P}_i\text{cl}((\mathcal{P}_j)\text{int}F_\alpha) \mid \alpha \in B\) is a pairwise regularly closed collection of subsets of \(X\) with finite intersection property. So by \((d)\), we have \(\bigcap\{\mathcal{P}_i\text{cl}((\mathcal{P}_j)\text{int}F_\alpha) \mid \alpha \in B\} \neq \emptyset\). Since \(F_\alpha\) is \((\mathcal{P}_i)\text{cl}closed, (\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}F_\alpha) \subset F_\alpha\). Thus it follows that \(\bigcap\{F_{\alpha} \mid \alpha \in B\} \neq \emptyset\).

\((e) \Rightarrow (a):\) Suppose \(\mathcal{U} = \{U_\alpha \mid \alpha \in A\}\) is a pairwise open cover of \(X\). If possible, suppose \(X\) is not nearly pairwise compact. So for any finite subcollection \(\{U_{\alpha_k} \mid \alpha_k \in A, k = 1,2,\ldots,m\}\) of \(\mathcal{U}\), \(\{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k}) \mid \alpha_k \in A, k = 1,2,\ldots,m\}\) is not a cover of \(X\). Thus \(\bigcap\{X - (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k}) \mid \alpha_k \in A, k = 1,2,\ldots,m\} \neq \emptyset\). Since \(X - (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}U_{\alpha_k}) \subset (\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}(X - U_{\alpha_k}))\), \(\bigcap\{(\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{int}(X - U_{\alpha_k})) \mid \alpha_k \in A, k = 1,2,\ldots,m\} \neq \emptyset\). Thus \(\{X - U_\alpha \mid \alpha \in A\}\) is a pairwise closed collection of subsets of \(X\) satisfying the properties of \((e)\). Hence \(\bigcap\{X - U_\alpha \mid \alpha \in A\} \neq \emptyset\) which in turn implies \(\bigcup\{U_\alpha \mid \alpha \in A\} \neq X\), which is a contradiction.

\(\Box\)

**Theorem 3.2.** A pairwise semiregular space is nearly pairwise compact iff it is pairwise compact.

Proof. Firstly, suppose \(X\) is pairwise semiregular and nearly pairwise compact. Let \(\mathcal{U} = \{U_\alpha \mid \alpha \in A\}\) be a pairwise open cover of \(X\). For each \(x \in X\), there exists a \(U_{\alpha(x)} \in \mathcal{U}\), \(\alpha(x) \in A\) with \(x \in U_{\alpha(x)}\). Suppose \(U_{\alpha(x)} \in \mathcal{P}_i\). So by pairwise semiregularity, there exists a \((\mathcal{P}_i)\text{open set,} G_x \subset \mathcal{P}_i\text{int}((\mathcal{P}_j)\text{cl}G_x) \subset U_{\alpha(x)}\). Here \(\mathcal{F} = \{(\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_x) \mid x \in X\}\) is a pairwise regularly open cover of \(X\). Using \((c)\) of Theorem 3.1, we obtain a finite subcover \((\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}G_{x_k}) \mid k = 1,2,\ldots,n\) of \(\mathcal{F}\) which in turn implies \(\{U_{\alpha(x_k)} \mid k = 1,2,\ldots,n\}\) is a finite subcover of \(\mathcal{U}\). The converse part is obvious.\(\Box\)
Theorem 3.3. A pairwise almost regular space is nearly pairwise compact if it is almost pairwise compact.

Proof. Let \( \mathcal{U} = \{ U_\alpha \mid \alpha \in A \} \) be a pairwise regularly open cover of a pairwise almost regular and almost pairwise compact space \( X \). For each \( x \in X \), we have a \( U_{\alpha(x)} \in \mathcal{U}, \alpha(x) \in A \) such that \( x \in U_{\alpha(x)} \). Suppose \( U_{\alpha(x)} \in \mathcal{P}_i \). Hence using the notion of pairwise almost regularity, we obtain a \( (\mathcal{P}_i) \) open set \( G_x \) such that \( x \in G_x \subset (\mathcal{P}_j)\text{cl}G_x \subset U_{\alpha(x)} \). Obviously, \( \mathcal{G} = \{ G_x \mid x \in X \} \) is a pairwise open cover of \( X \). So there exists a finite subcollection \( \{ G_{x_k} \mid k = 1, 2, \ldots, n \} \) of \( \mathcal{G} \) such that \( \{ (\mathcal{P}_j)\text{cl}G_{x_k} \mid k = 1, 2, \ldots, n \} \) covers \( X \). Thus \( \{ U_{\alpha(x_k)} \mid k = 1, 2, \ldots, n \} \) is a finite subcover of \( \mathcal{U} \) for \( X \). Hence \( X \) is nearly pairwise compact by (c) of Theorem 3.1.

Lemma 3.1. Each \( (\mathcal{P}_i) \) open cover \( \mathcal{U} \) of a \((j,i)\) regularly closed subset \( F \) of a nearly pairwise compact space \( X \) has a finite subfamily \( \mathcal{V} \) of \( \mathcal{U} \) such that \( \{ (\mathcal{P}_i)\text{cl}((\mathcal{P}_j)\text{cl}A) \mid A \in \mathcal{V} \} \) covers \( F \).

Proof. The proof is straightforward and hence omitted. \( \square \)

Theorem 3.4. Every pairwise Hausdorff, nearly pairwise compact space is pairwise almost regular.

Proof. Suppose \( X \) is a pairwise Hausdorff and nearly pairwise compact space. Let \( G \) be a \((i,j)\) regularly open set and \( x \) be a point of \( X \) with \( x \in G \). For each \( y \in X - G \), we obtain a \((\mathcal{P}_i) \) open set \( U_y \) and a \((\mathcal{P}_j) \) open set \( V_y \) such that \( x \in U_y, y \in V_y \) and \( U_y \cap V_y = \emptyset \). Then \( \mathcal{G} = \{ V_y \mid y \in X - G \} \) is a \((\mathcal{P}_j) \) open cover of the \((i,j)\) regularly closed set \( X - G \). So by Lemma 3.1, \( \mathcal{G} \) has a finite subcollection \( \mathcal{H} = \{ V_{y_k} \mid k = 1, 2, \ldots, n \} \) with \( X - G \subset \bigcup \{ (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}V_{y_k}) \mid k = 1, 2, \ldots, n \} \). We write \( U = \bigcap_{k=1}^{n} U_{y_k} \) and \( V = \bigcup_{k=1}^{n} (\mathcal{P}_i)\text{int}((\mathcal{P}_j)\text{cl}V_{y_k}) \). Here \( U \) is \((\mathcal{P}_i) \) open with \( x \in U \) and \( V \) is \((\mathcal{P}_j) \) open with \( X - G \subset V \) and \( U \cap V = \emptyset \). Thus \( (\mathcal{P}_j)\text{cl}U \subset X - V \). Therefore it follows that \( x \in U \subset (\mathcal{P}_j)\text{cl}U \subset G \). \( \square \)

Theorem 3.5. If the topological space \( (X, \mathcal{F}) \) is nearly compact and the bitopological space \( (Y, \mathcal{D}_1, \mathcal{D}_2) \) is nearly pairwise compact, then the product space \( (X \times Y, \mathcal{I} \times \mathcal{D}_1, \mathcal{I} \times \mathcal{D}_2) \) is nearly pairwise compact.

Proof. Let \( \mathcal{U} \) be a pairwise basic cover of \( X \times Y \). For each \( U \in \mathcal{U} \), we have \( U = G \times H, G \in \mathcal{F} \text{ and } H \in \mathcal{D}_i, i \in \{1, 2\} \). For each \( x \in X \), the space \( \{ x \} \times Y \) is nearly pairwise compact. Hence we get a finite number of elements \( G_x^k \times H_x^k, k = 1, 2, \ldots, n \) of \( \mathcal{U} \) such that \( \{ x \} \times Y \subset \bigcup_{k=1}^{n} (\mathcal{F} \times \mathcal{D}_i)\text{int}((\mathcal{F} \times \mathcal{D}_j)\text{cl}(G_x^k \times H_x^k)) \) where we assume \( H_x^k \in \mathcal{D}_i \). We suppose that all the sets \( G_x^k \times H_x^k \) intersects \( \{ x \} \times Y \). Then \( x \in G_x \), where \( G_x = \bigcap_{k=1}^{n} G_x^k \). The \((\mathcal{F}) \) open cover \( \{ G_x \mid x \in X \} \) of \( X \) has a finite subfamily \( \{ G_{x_1}, G_{x_2}, \ldots, G_{x_m} \} \) such that \( X = \bigcup_{k=1}^{m} (\mathcal{F} \times \mathcal{D}_i)\text{int}((\mathcal{F} \times \mathcal{D}_j)\text{cl}(G_{x_i}^k \times H_{x_i}^k)) \mid k = 1, 2, \ldots, n; l = 1, 2, \ldots, m \} \) covers \( X \times Y \) and \( \{ G_{x_i}^k \times H_{x_i}^k \mid k = 1, 2, \ldots, n; l = 1, 2, \ldots, m \} \) is a finite subcollection of \( \mathcal{U} \). \( \square \)

But the product of two nearly pairwise compact space need not be nearly pairwise compact. For, we consider Example 2.2. The space \((R, \mathcal{P}_1, \mathcal{P}_2)\) is nearly...
pairwise compact, but the product space \((R \times R, \mathcal{P}_1 \times \mathcal{P}_1, \mathcal{P}_2 \times \mathcal{P}_2)\) is not nearly pairwise compact.

**Lemma 3.2.** A bifilter \(\mathcal{F}\) is maximal iff for some \(i \in \{1, 2\}\), each \((\mathcal{P}_i)\)open set \(A\) intersecting every member of \(\mathcal{F}^i\) belongs to \(\mathcal{F}\).

**Proof.** Firstly, suppose \(\mathcal{F}\) is maximal. We write \(\mathcal{G} = \{G \mid G \supset A \cap B\) for some \(B \in \mathcal{F}^i\) and \(G = (\mathcal{P}_i)\)open \(\cup \mathcal{F}^j\). Obviously, \(\mathcal{G}\) is a bifilter with \(\mathcal{G} \supset \mathcal{F}\) and \(A \in \mathcal{G}\). Since \(\mathcal{F}\) is a maximal bifilter, we have \(\mathcal{G} = \mathcal{F}\).

Conversely, suppose the condition holds. If \(\mathcal{F}\) is not maximal, there exists a bifilter \(\mathcal{H}\) such that \(\mathcal{H} \supset \mathcal{F}\). Let \(H \in \mathcal{H}\) and \(H\) be \((\mathcal{P}_i)\)open. Then by definition of a bifilter, \(H\) intersects every member of \(\mathcal{H}^i\) and hence every member of \(\mathcal{F}^i\). Thus \(H \in \mathcal{F}\) and hence we have \(\mathcal{H} = \mathcal{F}\). \(\Box\)

**Lemma 3.3.** A bicluster point of a bifilter is a biconvergent point if it is a maximal bifilter.

**Proof.** Suppose the maximal bifilter \(\mathcal{F}\) has a bicluster point \(p\). Then for each \(F \in \mathcal{F}\), \(p \in (\mathcal{P}_i)\)cl\(\{\alpha\}\) whenever \(F = (\mathcal{P}_i)\)open for some \(i \in \{1, 2\}\). So each \((\mathcal{P}_i)\)open nbd \(V\) of \(p\) intersects every \(F \in \mathcal{F}^i\). Thus by Lemma 3.2, \(V \in \mathcal{F}\) which implies \(p\) is a biconvergent point of \(\mathcal{F}\). \(\Box\)

**Lemma 3.4.** Each pairwise open collection of subsets of \(X\) with finite intersection property is contained in a maximal bifilter.

**Proof.** The proof is straightforward and hence omitted. \(\Box\)

**Theorem 3.6.** Let \(X\) be pairwise almost regular and each bifilter \(\mathcal{A}\) in \(X\) has the following property: For \(A, B \in \mathcal{A}\) with \(A \in \mathcal{P}_1\) and \(B \in \mathcal{P}_2\), \(A \cap B\) is nonempty \((\mathcal{P}_i)\)open for each \(i \in \{1, 2\}\). Then the following statements are equivalent:

1. \(X\) is nearly pairwise compact.
2. Each bifilter in \(X\) has a bicluster point.
3. Each maximal bifilter in \(X\) has a biconvergent point.

**Proof.** 
(a) \(\Rightarrow\) (b): Let \(\mathcal{I} = \{G_\alpha \mid \alpha \in A\}\) be a bifilter. For each \(\alpha \in A\), we write \(F_\alpha = (\mathcal{P}_i)\)cl\(G_\alpha\) if \(G_\alpha \in \mathcal{P}_i\). Then \(\mathcal{F} = \{F_\alpha \mid \alpha \in A\}\) is a pairwise closed collection of subsets of \(X\) with following property: For any finite subcollection \(\mathcal{E} \subset \mathcal{F}\), \(\bigcap\{(\mathcal{P}_i)\)cl\((\mathcal{P}_j)\)int\(F) \mid F \in \mathcal{E}\} \neq \emptyset\). Hence by Theorem 3.1(e), \(\bigcap\{F_\alpha \mid \alpha \in A\} \neq \emptyset\). Thus there exists a \(p \in X\) with \(p \in F_\alpha\) for each \(\alpha \in A\). So \(p\) is a bicluster point of \(\mathcal{I}\).

(b) \(\Rightarrow\) (c): A maximal bifilter is of course a bifilter. So by (b), each maximal bifilter has a bicluster point \(p\). It then follows by Lemma 3.3, \(p\) is a biconvergent point of the maximal bifilter.

(c) \(\Rightarrow\) (a): Let \(\mathcal{W}\) be a pairwise regularly open cover of \(X\). Suppose \(\mathcal{W}\) has no finite subcollection covering \(X\). Again for each \(x \in X\), there exists a \(U_x \in \mathcal{W}\) such that \(x \in U_x\). Suppose \(U_x\) is \((i,j)\)regularly open. Since \(X\) is pairwise almost regular, we obtain a \((\mathcal{P}_i)\)open set \(G_x\) such that \(x \in G_x \subset (\mathcal{P}_j)\)cl\(G_x\) \(\subset U_x\). We
note here that $\mathcal{H} = \{G_x \mid x \in X\}$ is a pairwise open cover of $X$. Also $\mathcal{H}' = \{X - (\mathcal{P}_j)\text{cl}G_x \mid G_x \in \mathcal{I}\}$ is a pairwise open collection of subsets of $X$ with finite intersection property. Now by Lemma 3.4, we obtain a maximal bifilter $\mathcal{E}$ which contains $\mathcal{H}$. So by (c), $\mathcal{E}$ has a biconvergent point $p$. A biconvergent point of a maximal bifilter is also a bicluster point. So if $H$ contains the intersection property. Now by Lemma 3.4, we obtain a maximal bifilter $\mathcal{E}$ which contains $\mathcal{H}$. So by (c), $\mathcal{E}$ has a biconvergent point $p$. A biconvergent point of a maximal bifilter is also a bicluster point. So if $E \in \mathcal{E}$ then $p \in (\mathcal{P}_i)\text{cl}E$ for each $E \in \mathcal{E}$. Hence $p \in (\mathcal{P}_j)\text{cl}(X - (\mathcal{P}_j)\text{cl}G_x)$ for each $G_x \in \mathcal{I}$. Now we show $p \notin G_x$ for any $G_x \in \mathcal{I}$. We need only to prove the case when $p \notin X - (\mathcal{P}_j)\text{cl}G_x$ but $p$ is a $(\mathcal{P}_j)$ limit point of $X - (\mathcal{P}_j)\text{cl}G_x$. If possible, let $p \in G_z$ for some $G_z \in \mathcal{I}$. For definiteness suppose, $G_z$ is $(\mathcal{P}_i)$ open. Now each $(\mathcal{P}_j)$ open set $A$ with $p \in A$ intersects each $E \in \mathcal{E}$ whenever $E$ is $(\mathcal{P}_i)$ open. Again $G_z$ intersects each $E \in \mathcal{E}$ whenever $E$ is $(\mathcal{P}_i)$ open. Therefore by Lemma 3.2, $A, G_z \in \mathcal{E}$. So $A \cap G_z$ is $(\mathcal{P}_i)$ open for each $i \in \{1, 2\}$ and $p \in A \cap G_z \subseteq G_z$. Since $p$ is a $(\mathcal{P}_j)$ limit point of $X - (\mathcal{P}_j)\text{cl}G_x$ we have $(A \cap G_z) \cap (X - (\mathcal{P}_j)\text{cl}G_x) \neq \emptyset$ which is not possible since $G_z \cap (X - (\mathcal{P}_j)\text{cl}G_x) = \emptyset$. Thus our anticipation $p \notin G_x$ for any $G_x \in \mathcal{I}$ is true.

This contradicts the fact that $\mathcal{I}$ is a pairwise open cover of $X$. So $\mathcal{I}$ must have a finite subcover. Hence $X$ is nearly pairwise compact. □

Remark 3.1. Theorem 3.6 also holds good if the expression ‘$X$ be pairwise almost regular’ of the theorem is replaced by ‘$X$ be a bitopological space with each $(X, \mathcal{P}_i)$ being regular’.

We now give an example of a bitopological space which satisfies the conditions of Theorem 3.6.

Example 3.1. For any $a \in R$, we define
\[
\begin{align*}
\mathcal{P}_1 &= \{\emptyset, R, (-\infty, a), (-\infty, a], (a, \infty), R - \{a\}\}, \\
\mathcal{P}_2 &= \{\emptyset, R, (-\infty, a), (-\infty, a]\}.
\end{align*}
\]

The bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise almost regular. The possible bifilters of this space are \{\{(-\infty, a], R\}, \{(-\infty, a), (-\infty, a], R - \{a\}, R\}. Clearly, they satisfy the conditions of Theorem 3.6.

It also follows, the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$ is not pairwise regular.

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References


