ROBUST DUALITY FOR GENERALIZED INVEX PROGRAMMING PROBLEMS

Moon Hee Kim

ABSTRACT. In this paper we present a robust duality theory for generalized convex programming problems under data uncertainty. Recently, Jeyakumar, Li and Lee [Nonlinear Analysis 75 (2012), no. 3, 1362–1373] established a robust duality theory for generalized convex programming problems in the face of data uncertainty. Furthermore, we extend results of Jeyakumar, Li and Lee for an uncertain multiobjective robust optimization problem.

1. Introduction

Consider the standard nonlinear programming problem with inequality constraints

$$\text{(P)} \quad \inf_{x \in \mathbb{R}^n} \{f(x) : g_i(x) \leqq 0, \ i = 1, \ldots, m\},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions. The problem in the face of data uncertainty in the constraints can be captured by the following nonlinear programming problem:

$$\text{(UP)} \quad \inf_{x \in \mathbb{R}^n} \{f(x) : g_i(x, v_i) \leqq 0, \ i = 1, \ldots, m\},$$

where v_i is an uncertain parameter and $v_i \in \mathcal{V}_i$ for some convex compact set $\mathcal{V}_i$ in $\mathbb{R}^q$ and $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is continuously differentiable. Robust optimization, which has emerged as a powerful deterministic approach for studying mathematical programming under uncertainty ([4]-[5], [6]), associates with the uncertain program (UP) its robust counterpart [1],

$$\text{(RP)} \quad \inf_{x \in \mathbb{R}^n} \{f(x) : g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ \ i = 1, \ldots, m\},$$

where the uncertain constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets $\mathcal{V}_i$, $i = 1, \ldots, m$. Recently,

Jeyakumar, Li and Lee [7] established a robust duality theory for generalized convex programming problems in the face of data uncertainty. Furthermore, we extend results of Jeyakumar, Li and Lee [7] for an uncertain multiobjective robust optimization problem.

2. Optimality results

Consider an uncertain multiobjective robust optimization problem:

(MRP)　　　　Minimize　$(f_1(x), \ldots, f_l(x))$

　　　　　　　subject to　$g_j(x, v_j) \leqq 0, \ \forall v_j \in \mathcal{V}_j, \ j = 1, \ldots, m,$

where v_i is an uncertain parameter and $v_i \in \mathcal{V}_i$ for some convex compact set $\mathcal{V}_i$ in $\mathbb{R}^q$, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, l$ and $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $j = 1, \ldots, m$, are continuously differentiable.

Let F be the set of all the robust feasible solutions of (MRP) and $J(\bar{x}) = \{j \mid \exists v_j \in \mathcal{V}_j \text{ s.t. } g_j(\bar{x}, v_j) = 0, j = 1, \ldots, m\}$. A robust feasible solution $\bar{x}$ of (MRP) is a robust weakly efficient solution of (MRP) if there does not exist a robust feasible solution x of (MRP) such that

$$f_i(x) < f_i(\bar{x}), \quad i = 1, \ldots, l.$$

Now we define an Extended Mangasarian-Fromovitz constraint qualification for (MRP) as follows:

There exists $d \in \mathbb{R}^n$ such that for any $j \in J(\bar{x})$ and any $v_j \in \mathcal{V}_j$,

$$\nabla_1 g_j(\bar{x}, v_j)^T d < 0.$$

We use $\nabla_1 g$ to denote the derivative of g with respect to the first variable.

Now we present necessary optimality theorems for robust weakly efficient solutions for (MRP).

Theorem 2.1 ([8])**.** *Let $\bar{x} \in F$ be a robust weakly efficient solution of (MRP). Suppose that $g_j(\bar{x}, \cdot)$ are concave on $\mathcal{V}_j$, $j = 1, \ldots, m$. Then there exist $\lambda_i \geqq 0$, $i = 1, \ldots, l$, $\mu_j \geqq 0$, $j = 1, \ldots, m$, not all zero, and $\bar{v}_j \in \mathcal{V}_j$, $j = 1, \ldots, m$ such that*

(1)　　　　　　　$\displaystyle\sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0,$

(2)　　　　　　　$\mu_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \ldots, m.$

Moreover, if we further assume that the Extended Mangasarian-Fromovitz constraint qualification holds, then there exist $\lambda_i \geqq 0$, $i = 1, \ldots, l$, not all zero, $\mu_j \geqq 0$, $j = 1, \ldots, m$, and $\bar{v}_j \in \mathcal{V}_j$, $j = 1, \ldots, m$ such that (1) and (2) hold.

We provide a robust sufficient optimality condition under the following generalized η-convexity conditions at $(x^*, v_j) \in F \times \mathcal{V}_j$ for each $x \in \mathbb{R}^n$ there exist

$\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty), \ i = 1, \ldots, l, \ \beta_j : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty), \ j = 1, \ldots, m,$
$\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$,

$$(3) \qquad f_i(x) - f_i(x^*) \geqq \alpha_i(x, x^*) \nabla f_i(x^*)^T \eta(x, x^*),$$

$$(4) \qquad g_j(x, v_j) - g_j(x^*, v_j) \geqq \beta_j(x, x^*) \nabla_1 g_j(x^*, v_j)^T \eta(x, x^*).$$

We now present a robust sufficient optimality condition for the uncertain programming problem.

Theorem 2.2. *Let $\lambda_i \geqq 0, \ i = 1, \ldots, l$, not all zero, $\mu_j \geqq 0, \ j = 1, \ldots, m,$
$\bar{v}_j \in \mathcal{V}_j, \ j = 1, \ldots, m$ and $\bar{x} \in F$ satisfy the following condition:*

$$(5) \qquad \sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0,$$

$$(6) \qquad \mu_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \ldots, m.$$

*Suppose that for each $x \in F$ there exist $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty), \ i = 1, \ldots, l,$
$\beta_j : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty), \ j = 1, \ldots, m$ such that (3) and (4) hold. Then $\bar{x}$ is
a robust weakly efficient solution of* (MRP).

Proof. Suppose that $\bar{x}$ is not a robust weakly efficient solution for (MRP). Then there exists $x^* \in F$ such that

$$f_i(x^*) < f_i(\bar{x}), \ i = 1, \ldots, l, \quad g_j(x^*, \bar{v}_j) \leqq 0.$$

By the generalized η-convexity of f_i at $\bar{x} \in F$, there exists $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty), \ i = 1, \ldots, l$ such that

$$f_i(x^*) - f_i(\bar{x}) \geqq \alpha_i(x^*, \bar{x}) \nabla f_i(\bar{x})^T \eta(x^*, \bar{x}), \ i = 1, \ldots, l.$$

Thus $\lambda_i \geqq 0, \ i = 1, \ldots, l$, not all zero, implies that

$$\alpha_i(x^*, \bar{x}) \sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x})^T \eta(x^*, \bar{x}) < 0, \ i = 1, \ldots, l.$$

Therefore from (5), we have

$$-\alpha_i(x^*, \bar{x}) \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j)^T \eta(x^*, \bar{x}) < 0, \ i = 1, \ldots, l.$$

By the generalized η-convexity of g_j at $(\bar{x}, \bar{v}_j) \in F \times \mathcal{V}_j$, there exists $\beta_j : \mathbb{R}^n \times \mathbb{R}^n \to (0, +\infty), \ j = 1, \ldots, m$ such that

$$-\alpha_i(x^*, \bar{x}) \sum_{j=1}^{m} \frac{\mu_j}{\beta_j(x^*, \bar{x})} (g_j(x^*, \bar{v}_j) - g_j(\bar{x}, \bar{v}_j)) < 0, \ i = 1, \ldots, l.$$

Hence $\mu_j g_j(x^*, \bar{v}_j) > \mu_j g_j(\bar{x}, \bar{v}_j), \ j = 1, \ldots, m$. Since $\mu_j g_j(\bar{x}, \bar{v}_j) = 0,$

$$\mu_j g_j(x^*, \bar{v}_j) > 0, \ j = 1, \ldots, m,$$

which contradicts the fact that $\mu_j g_j(x^*, \bar{v}_j) \leqq 0, \ j = 1, \ldots, m.$ $\qquad \square$

3. Duality results

In this section, we establish Wolfe type robust duality between (MRP) and (WD).

(WD)   Maximize $\left(f_1(u) + \sum_{j=1}^{m} \mu_j g_j(u, v_j), \ldots, f_l(u) + \sum_{j=1}^{m} \mu_j g_j(u, v_j) \right)$

subject to $\sum_{i=1}^{l} \lambda_i \nabla_1 f_i(u) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(u, v_j) = 0,$

$$\lambda_i \geqq 0, \ i = 1, \ldots, l, \ \sum_{i=1}^{l} \lambda_i = 1,$$

$$\mu_j \geqq 0, \ v_j \in \mathcal{V}_j, \ j = 1, \ldots, m.$$

Theorem 3.1 (Weak Duality). *Let x be feasible for (MRP) and $(\bar{x}, \bar{v}, \lambda, \mu)$ be feasible for (WD). Suppose that $f_i(\cdot), \ i = 1, \ldots, l$ and $g_j(\cdot, \bar{v}_j), \ j = 1, \ldots, m$ are convex and $g_j(\bar{x}, \cdot)$ are concave on $\mathcal{V}_j$. Then*

$$\left(f_1(x), \ldots, f_l(x) \right) \not< \left(f_1(\bar{x}) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}, \bar{v}_j), \ldots, f_l(\bar{x}) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}, \bar{v}_j) \right).$$

Proof. Let x be feasible for (MRP) and $(\bar{x}, \bar{v}, \lambda, \mu)$ be feasible for (WD). Suppose that $f_i(x) < f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}, \bar{v}_j)$. Since $g_j(x, \bar{v}_j) \leqq 0, \ \mu_j \geqq 0, \ j = 1, \ldots, m, \ \sum_{j=1}^{m} \mu_j g_j(x, \bar{v}_j) \leqq 0,$

$$f_i(x) + \sum_{j=1}^{m} \mu_j g_j(x, \bar{v}_j) < f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}, \bar{v}_j).$$

Since $f_i(\cdot), \ i = 1, \ldots, l$ and $g_j(\cdot, \bar{v}_j), \ j = 1, \ldots, m$ are convex,

$$f_i(x) - f_i(\bar{x}) \geqq \nabla f_i(\bar{x})^T (x - \bar{x}),$$
$$g_i(x, \bar{v}_j) - g_j(\bar{x}, \bar{v}_j) \geqq \nabla_1 g_j(\bar{x}, \bar{v}_j)^T (x - \bar{x}).$$

Hence $\lambda_i \geqq 0, \ i = 1, \ldots, l, \ \sum_{i=1}^{l} \lambda_i = 1, \ \mu_j \geqq 0, \ j = 1, \ldots, m,$

$$0 > \left[\sum_{i=1}^{l} \lambda_i f_i(x) + \sum_{j=1}^{m} \mu_j g_j(x, \bar{v}_j) \right] - \left[\sum_{i=1}^{l} \lambda_i f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j g_j(\bar{x}, \bar{v}_j) \right]$$

$$\geqq \left[\sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) \right]^T (x - \bar{x}).$$

Therefore, $\left[\sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) \right]^T (x - \bar{x}) < 0$. This is a contradiction, since $\sum_{i=1}^{l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0$. $\qquad\square$

Theorem 3.2 (Strong Duality). *Let $\bar{x}$ be a robust weakly efficient solution of (MRP). Assume that the Extended Mangasarian-Fromovitz constraint qualification holds. Then, there exists $(\bar{v}, \bar{\lambda}, \bar{\mu})$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is feasible for (WD) and the objective values of (MRP) and (WD) are equal. If $f_i(\cdot)$, $i = 1, \ldots, l$, $g_j(\cdot, \bar{v}_j)$, $j = 1, \ldots, m$ are convex at $\bar{x}$ and $g_j(\bar{x}, \cdot)$ are concave on $\mathcal{V}_j$, then $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a robust weakly efficient solution of (WD).*

Proof. Since $\bar{x}$ is a robust weakly efficient solution of (MRP) at which the Extended Mangasarian-Fromovitz constraint qualification is satisfied, then by Theorem 2.1, there exist $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, l$, not all zero, $\bar{\mu}_j \geqq 0$, $j = 1, \ldots, m$, and $\bar{v}_j \in \mathcal{V}_j$, $j = 1, \ldots, m$, such that

$$\sum_{i=1}^{l} \bar{\lambda}_i \nabla_1 f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0,$$

$$\bar{\mu}_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \ldots, m.$$

Thus $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is feasible for (WD) and the objective values of (MRP) and (WD) are equal. Moreover, $f_i(\bar{x}) = f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j)$, $i = 1, \ldots, l$. If $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a weak duality, then there exists feasible $(\widetilde{x}, \widetilde{v}, \widetilde{\lambda}, \widetilde{\mu})$ for (WD) such that

$$f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{x}, \bar{v}_j) \not< f_i(\widetilde{x}) + \sum_{j=1}^{m} \widetilde{\mu}_j g_j(\widetilde{x}, \widetilde{v}_j), \ i = 1, \ldots, l.$$

Hence $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a (WD)-feasible, $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a robust weakly efficient solution of (WD). □

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School of Free Major
Tongmyong University
Busan 608-711, Korea
E-mail address: mooni@tu.ac.kr