RIGHT AND LEFT FREDHOLM OPERATOR MATRICES

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Abstract. We consider right and left Fredholm operator matrices of the form $\begin{bmatrix} A & C \\ T & S \end{bmatrix}$, which are linear and bounded on the Banach space $Z = X \oplus Y$.

1. Introduction

Let $Z$ be an infinite dimensional Banach space, such that $Z = X \oplus Y$ for some closed subspaces $X$ and $Y$. This sum will be also denoted by $X \oplus Y$. If $W$ is a finite dimensional subspace of $X$, then $\dim W$ denotes its dimension. If $W$ is infinite dimensional, then we simply write $\dim W = \infty$. However, if $U$ is a closed subspace of a Hilbert space, then $\dim_H(U)$ denotes the orthogonal dimension of $U$.

Let $\mathcal{L}(X, Y)$ denote the set of all linear bounded operators from $X$ to $Y$. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X, X)$. The set of all finite rank operators from $X$ to $Y$ is denoted by $\mathcal{F}(X, Y)$. For $A \in \mathcal{L}(X, Y)$ we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$ to denote the range and the null-space of $A$, respectively.

If $Z = X \oplus Y$, then any $M \in \mathcal{L}(Z)$ can be decomposed as the following operator matrix

$$M = \begin{bmatrix} A & C \\ T & S \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \rightarrow \begin{bmatrix} X \\ Y \end{bmatrix}$$

for some $A \in \mathcal{L}(X)$, $C \in \mathcal{L}(Y, X)$, $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$. On the other hand, any choice of $A, C, T, S$ (linear and bounded operators on the corresponding subspaces), produces a linear and bounded operator $M$ on the space $Z$. Moreover, $M$ is finite rank if and only if all $A, C, T, S$ are finite rank operators.

If $A$ and $C$ are fixed, then we use the notation $M_{(T, S)}$ to show that $M$ depends on $T$ and $S$. For given $A$ and $C$, we are interested to find $T$ and $S$, such that $M_{(T, S)}$ is right or left Fredholm operator.

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For this purpose we need to review some properties of right and left Fredholm operators [9]. An operator $A \in \mathcal{L}(X,Y)$ is right Fredholm, if $\text{def}(A) = \dim Y/\mathcal{R}(A) < \infty$, and $\mathcal{N}(A)$ is complemented in $X$. Notice that if $A$ is right Fredholm, then it follows that $\mathcal{R}(A)$ has to be a closed and complemented subspace of $Y$. The set of all right Fredholm operators from $X$ to $Y$ is denoted by $\Phi_{r}(X,Y)$. It is well-known that $A \in \Phi_{r}(X,Y)$ if and only if there exist $B \in \mathcal{L}(Y,X)$ and $F \in \mathcal{F}(X)$ such that $AB = I_{Y} + F$ holds.

An operator $A \in \mathcal{L}(X,Y)$ is left Fredholm, if $\text{null}(A) = \dim \mathcal{N}(A) < \infty$, and $\mathcal{R}(A)$ is closed and complemented in $Y$. The set of all left Fredholm operators from $X$ to $Y$ is denoted by $\Phi_{l}(X,Y)$. It is well-known that $A \in \Phi_{l}(X,Y)$ if and only if there exist $B \in \mathcal{L}(Y,X)$ and $F \in \mathcal{F}(X)$ such that $BA = I_{X} + F$ holds.

If $A \in \Phi_{r}(X,Y)$ and $B \in \Phi_{r}(Y,Z)$, then $BA \in \Phi_{r}(X,Z)$. The similar result holds for the class $\Phi_{l}$. The set of Fredholm operators is defined as $\Phi(X,Y) = \Phi_{r}(X,Y) \cap \Phi_{l}(X,Y)$.

We formulate the following well-known results.

**Lemma 1.1.** Let $X, Y, Z$ be Banach spaces and let $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Y,Z)$. If $BA \in \Phi(X,Z)$, then the following holds: $A \in \Phi(X,Y)$ if and only if $B \in \Phi(Y,Z)$.

**Lemma 1.2.** Let $X, Y$ be Banach spaces, and let $A \in \Phi_{r}(X,Y)$, $P \in \mathcal{F}(X,Y)$. Then $A + P \in \Phi_{r}(X,Y)$. The analogous result holds for classes $\Phi_{l}$ and $\Phi$.

**Lemma 1.3.** Let $M_{1}, M_{2}$ and $N$ be the vector subspaces of the vector space $X$. If $M_{1} \subseteq M_{2}$, then $\dim M_{1}/(M_{1} \cap N) \leq \dim M_{2}/(M_{2} \cap N)$.

Properties of right (left) Fredholm and related operators can be found in [6] and [9]. For the importance and applications of operator matrices we refer to [1], [2], [3], [4], [5], [7], [8] and [10]. Particularly, this paper is related to the research in [4] and [7], where the left and right invertibility of $M_{(T,S)}$ is considered.

### 2. Right Fredholm operators

We consider right Fredholm properties of $M_{(T,S)}$.

**Theorem 2.1.** Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y,X)$ be given. The following statements are equivalent:

(a) $[A \ C] \in \Phi_{r}(X \oplus Y,Y,X) \setminus \Phi(X \oplus Y,X)$, and there exists an operator $J \in \Phi_{l}(Y,\tilde{\mathcal{N}}([A \ C])) \setminus \Phi(Y,\tilde{\mathcal{N}}([A \ C]))$.

(b) $M_{(T,S)} \in \Phi_{r}(X \oplus Y,X) \setminus \Phi(X \oplus Y,X)$ for some $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y)$.

**Proof.** (a) $\implies$ (b): Suppose that $[A \ C] \in \Phi_{r}(X \oplus Y,Y,X) \setminus \Phi(X \oplus Y,X)$. It follows that $\mathcal{N}([A \ C])$ is infinite dimensional. By the assumption, there exists an operator $J \in \Phi_{l}(Y,\tilde{\mathcal{N}}([A \ C])) \setminus \Phi(Y,\tilde{\mathcal{N}}([A \ C]))$, so $\mathcal{N}(J)$ is finite.
dimensional and $N([A \ C])/R(J)$ is infinite dimensional. The operator $J$ has the form

$$J = \begin{bmatrix} E \\ G \end{bmatrix} : Y \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$ 

Since $R(J)$ is closed and complemented in $N([A \ C])$, and $N([A \ C])$ is closed and complemented in $X \oplus Y$, we obtain that there exist closed subspaces $V$ and $W$ such that $N([A \ C]) = R(J) \oplus V$ and $X \oplus Y = N([A \ C]) \oplus W = R(J) \oplus V \oplus W$. Notice that $V$ is infinite dimensional.

There exists a closed subspace $Y_1$ such that $Y = N(J) \oplus Y_1$. Now, the reduction operator $J : Y_1 \to R(J)$ is invertible, so let $K_1 : R(J) \to Y_1$ denote its inverse. Define the operator $K \in L(X \oplus Y, Y)$ in the following way:

$$Kx = \begin{cases} K_1x, & x \in R(J), \\ 0, & x \in V \oplus W. \end{cases}$$

Then $K \in L(X \oplus Y, Y)$ is a right Fredholm operator, such that $N(K) = V \oplus W$.

The operator $K$ has the matrix form

$$K = \begin{bmatrix} T & S \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to Y.$$ 

We also have

(1) \[ KJ = \begin{bmatrix} T & S \end{bmatrix} \begin{bmatrix} E \\ G \end{bmatrix} = I_Y - P_1, \]

where $P_1$ is the projection from $Y$ onto the finite dimensional subspace $N(J)$, parallel to $Y_1$.

From $R(J) \subset N([A \ C])$ we get that

(2) \[ [A \ C] \begin{bmatrix} E \\ G \end{bmatrix} = 0. \]

Since $[A \ C] \in \Phi_r(X \oplus Y, X)$, we have the following decompositions of spaces: $X \oplus Y = N([A \ C]) \oplus W$ and $X = R([A \ C]) \oplus U$, where $U$ is finite dimensional. Since the reduction $[A \ C] : W \to R([A \ C])$ is invertible, define $L_1 : R([A \ C]) \to W$ to be its inverse. Then consider the operator $L \in L(X, X \oplus Y)$, which is defined as follows:

$$Lx = \begin{cases} L_1x, & x \in R([A \ C]), \\ 0, & x \in U. \end{cases}$$

The operator $L$ has the matrix form

$$L = \begin{bmatrix} D \\ F \end{bmatrix} : X \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$ 

Then $L \in \Phi_l(X, X \oplus Y)$, $R(L) = W$, and

(3) \[ [A \ C]L = [A \ C] \begin{bmatrix} D \\ F \end{bmatrix} = I_X - P_2, \]
where $P_2$ is the projection from $X$ onto the finite dimensional subspace $U$, parallel to $R([A\ C])$. Since $N([T\ S]) = V \oplus W$, we conclude that

\[(4)\quad [T\ S]\begin{bmatrix} D \\ F \end{bmatrix} = 0.\]

Finally, from (1), (2), (3) and (4), we get that for $M = [A\ C\ T\ S],\ N = [D\ E\ F\ G]$ the following holds:

\[(5)\quad MN = [A\ C\ T\ S]\begin{bmatrix} D \\ E \\ F \\ G \end{bmatrix} = [I_X\ 0 \\ 0\ I_Y] + \begin{bmatrix} -P_2 \\ 0 \\ 0 \\ -P_1 \end{bmatrix}.
\]

Since $\begin{bmatrix} -P_2 \\ 0 \\ 0 \\ -P_1 \end{bmatrix}$ is finite rank, we conclude that $M$ is right Fredholm. Moreover, we notice that $N(M) = N([A\ C]) \cap N([T\ S]) = V$, $R(N) = R([D\ F]) \oplus R([E\ G]) = W \oplus R(J)$, $X \oplus Y = R(J) \oplus V \oplus W$.

Since $V$ is infinite dimensional, we obtain that both $M$ and $N$ are not Fredholm operators.

$(b) \implies (a)$: Suppose that there exist some $T \in L(X, Y)$ and $S \in L(Y)$ such that $M(T, S) \in \Phi_r(X \oplus Y) \setminus \Phi(X, Y)$. Then there exist operators $N \in L(X \oplus Y)$ and $P \in L(Y \oplus X)$ such that $MN = I + P$. The last equality holds in the matrix form as follows:

\[
\begin{bmatrix} A \\ T \\ C \\ S \end{bmatrix}\begin{bmatrix} D \\ E \\ F \\ G \end{bmatrix} = \begin{bmatrix} I_X \\ 0 \\ 0 \\ I_Y \end{bmatrix} + \begin{bmatrix} P_{11} \\ P_{12} \\ P_{21} \\ P_{22} \end{bmatrix},
\]

where all $P_{ij}$ are finite rank operators. It also follows that $N = [D\ E\ F\ G] \in \Phi_l(X \oplus Y)$.

In particular, we obtain

\[
[A\ C]\begin{bmatrix} D \\ F \end{bmatrix} = I_X + P_{11},
\]

so $[A\ C]$ is right Fredholm. The operator $I_X + P_{11}$ is Fredholm. If we suppose that $[A\ C]$ is Fredholm, by Lemma 1.1 it follows that $[D\ F]$ is also Fredholm. Since

\[
R\left(\begin{bmatrix} D \\ E \\ F \\ G \end{bmatrix}\right) = R\left(\begin{bmatrix} D \\ F \end{bmatrix}\right) \cup R\left(\begin{bmatrix} E \\ G \end{bmatrix}\right),
\]

it follows that $[D\ E\ F\ G]$ belongs to $\Phi_r(X \oplus Y)$, so $[D\ E\ F\ G]$ is Fredholm. By Lemma 1.1 again, we obtain that $[A\ C]$ is Fredholm (since $I + P$ is Fredholm from Lemma 1.2). The last statement is not possible, so we obtain that $[A\ C] \in \Phi_r(X \oplus Y, X) \setminus \Phi(X \oplus Y, Y)$.

Denote with $L = [\frac{E}{F}] \in L(Y, X \oplus Y)$. We have $[T\ S]L = I_Y + P_{22}$, so $L \in \Phi_l(Y, X \oplus Y) \setminus \Phi(Y, X \oplus Y)$. Otherwise, if $L$ is Fredholm, then also $[D\ E\ F\ G]$ is Fredholm, so $[D\ E\ F\ G]$ is Fredholm.
Since we have the following decomposition of space $X \oplus Y = \mathcal{N}([A \ C]) \oplus W$, the operator $L$ has the matrix form
\[
L = \begin{bmatrix} J & 0 \\ K & 0 \end{bmatrix} : Y \to \begin{bmatrix} \mathcal{N}([A & C]) \\ W \end{bmatrix}.
\]

From the fact that
\[
\mathcal{R}(P_{12}) = \mathcal{R}([A & C]\mathcal{L}) = \mathcal{R} \left( \begin{bmatrix} J \\ K \end{bmatrix} \right) = [A & C](\mathcal{R}(K))
\]
is a finite space and the reduction $[A & C] : W \to \mathcal{R}([A & C])$ is a bijection, we obtain that $\mathcal{R}(K)$ is a finite dimensional subspace of $W$.

Since $L \in \Phi(Y, X \oplus Y) \setminus \Phi(Y, X \oplus Y)$, we have the following decompositions of spaces $Y = \mathcal{N}(\mathcal{L}) \oplus U$ and $X \oplus Y = \mathcal{R}(\mathcal{L}) \oplus U_1$, where $\dim \mathcal{N}(\mathcal{L}) < \infty$ and $\dim U_1 = \infty$. The reduction operator $L : U \to \mathcal{R}(\mathcal{L})$ is invertible, so let $L_1 : \mathcal{R}(\mathcal{L}) \to U$ be its inverse.

As it was shown, $\mathcal{R}(K)$ is a finite dimensional subspace, so $Y_1 = L_1(\mathcal{R}(K))$ have to be a finite dimensional subspace of $U$ and there exists a closed subspace $Y_2$ such that $U = Y_1 \oplus Y_2$.

Now, the operator $L$ has the following matrix form
\[
L = \begin{bmatrix} J & 0 & 0 \\ 0 & K & 0 \end{bmatrix} : \begin{bmatrix} Y_2 \\ Y_1 \\ \mathcal{N}(\mathcal{L}) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(\mathcal{N}([A & C]) \\ N \end{bmatrix},
\]
where $Y_1$ is finite dimensional. We obtain that $\mathcal{N}(J) = Y_1 \oplus \mathcal{N}(\mathcal{L})$, so $\dim \mathcal{N}(J) < \infty$.

From the fact that $[T & S]L = I_Y + P_{22}$ follows that
\[
L_1(\mathcal{N}([T & S]) \cap \mathcal{R}(\mathcal{L})) \subseteq \mathcal{N}(I_Y + P_{22}).
\]
Since $I_Y + P_{22}$ is a Fredholm operator, we have that $L_1(\mathcal{N}([T & S]) \cap \mathcal{R}(\mathcal{L}))$ is finite dimensional, so $\mathcal{N}([T & S]) \cap \mathcal{R}(\mathcal{L})$ is also a finite dimensional subspace.

Denote with $V = \mathcal{N}([A & C]) \cap \mathcal{N}([T & S]) \cap \mathcal{R}(J)$. Further,
\[
V \subseteq \mathcal{N}([T & S]) \cap \mathcal{R}(J) \subseteq \mathcal{N}([T & S]) \cap \mathcal{R}(L),
\]
so it follows that $\dim V < \infty$. Then, there exists a closed subspace $V_1$ such that $\mathcal{N}(M(T,S)) = \mathcal{N}([A & C]) \cap \mathcal{N}([T & S]) = V \oplus V_1$. Since $\mathcal{N}(M(T,S))$ is infinite dimensional, then $V_1$ is also an infinite dimensional subspace.

Now, applying Lemma 1.3 on the spaces $\mathcal{N}([A & C]) \cap \mathcal{N}([T & S]), \mathcal{N}([A & C])$ and $\mathcal{R}(J)$, we obtain
\[
\dim V_1 = \dim(\mathcal{N}([A & C]) \cap \mathcal{N}([T & S]))/V \leq \dim \mathcal{N}([A & C])/\mathcal{R}(J).
\]
We conclude that $\dim \mathcal{N}([A & C])/\mathcal{R}(J) = \infty$.

Lastly, we proved for the operator $J : Y \to \mathcal{N}([A & C])$ that $\dim \mathcal{N}(J) < \infty$ and $\dim \mathcal{N}([A & C])/\mathcal{R}(J) = \infty$.

So, there exists the operator $J \in \Phi(Y, \mathcal{N}([A & C]) \setminus \Phi(Y, \mathcal{N}([A & C]))$.
3. Left Fredholm operators

Now we investigate the left Fredholm properties of \( M_{(T,S)} \). We consider two separate cases according to the dimension of \( Y \).

**Theorem 3.1.** Let \( X \) be infinite dimensional, and let \( Y \) be finite dimensional. For given \( A \in \mathcal{L}(X) \) and \( C \in \mathcal{L}(Y,X) \), the following statements are equivalent:

(a) \( M_{(T,S)} \in \Phi_l(X \oplus Y) \backslash \Phi(X \oplus Y) \) for every \( T \in \mathcal{L}(X,Y) \) and every operator \( S \in \mathcal{L}(Y) \);

(b) \( A \in \Phi_l(X) \backslash \Phi(X) \).

**Proof.** Before the proof of the equivalence, note that

\[
\mathcal{N} \left( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right) = \mathcal{N}(A) \oplus Y, \quad \mathcal{R} \left( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right) = \mathcal{R}(A) \oplus \{0\}.
\]

Since \( Y \) is finite dimensional, we have that \( A \in \Phi_l(X) \backslash \Phi(X) \) if and only if \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \notin \Phi_l(X \oplus Y) \backslash \Phi(X \oplus Y) \).

(a) \( \implies \) (b): Suppose that \( M_{(T,S)} \) is left Fredholm but not Fredholm, for every \( T \in \mathcal{L}(X,Y) \) and every \( S \in \mathcal{L}(Y) \). We have that \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & C \\ T & S \end{bmatrix} \) where \( \begin{bmatrix} 0 & -C \\ -T & 0 \end{bmatrix} \) is a finite rank operator. Applying Lemma 1.2, we obtain that \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) is a left Fredholm operator.

Suppose that \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \) is Fredholm. Applying Lemma 1.2 to \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \) we conclude that \( M_{(T,S)} \) has to be Fredholm, which does not hold. Hence, \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \) is left Fredholm but not Fredholm, so we have that \( A \in \Phi_l(X) \backslash \Phi(X) \).

(b) \( \implies \) (a): Suppose that \( A \) is left Fredholm but not Fredholm, so the operator \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \) is also left Fredholm but not Fredholm.

Let \( T \in \mathcal{L}(X,Y) \) and \( S \in \mathcal{L}(Y) \) be arbitrary operators. Then the operator \( M_{(T,S)} \) is a finite-rank perturbation of \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \). Indeed, \( \begin{bmatrix} A & C \\ T & S \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & C \\ T & S \end{bmatrix} \), where \( \begin{bmatrix} 0 & C \\ T & S \end{bmatrix} \) is a finite rank operator because \( Y \) is a finite dimensional space.

Applying Lemma 1.2 to \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \) we get that \( M_{(T,S)} \) is a left Fredholm operator. If we suppose that \( M_{(T,S)} \) is Fredholm, from Lemma 1.2, we conclude that \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \) have to be Fredholm, which does not hold. We obtain that \( M_{(T,S)} \) is left Fredholm but not Fredholm operator. \( \square \)

**Theorem 3.2.** Let \( X \) and \( Y \) be infinite dimensional, such that \( Y \) is isomorphic to \( Z = X \oplus Y \). Let \( A \in \mathcal{L}(X) \) and \( C \in \mathcal{L}(Y,X) \) be arbitrary. Then \( M_{(T,S)} \in \Phi_l(X \oplus Y) \backslash \Phi(X \oplus Y) \) for some \( T \in \mathcal{L}(X,Y) \) and \( S \in \mathcal{L}(Y) \).

**Proof.** Since \( Y \) is isomorphic with \( Z \), then \( Y = Y_1 \oplus Y_2 \), where \( X \) is isomorphic to \( Y_1 \), and \( Y \) is isomorphic to \( Y_2 \). Let \( T \in \mathcal{L}(X,Y_1) \) and \( S \in \mathcal{L}(Y,Y_2) \) be those isomorphisms. Then \( T \in \mathcal{L}(X,Y) \) is left invertible with a left inverse \( K \in \mathcal{L}(Y,X) \) and \( N(K) = Y_2 \). Also, \( S \in \mathcal{L}(Y,Y_2) \) is left invertible with a left inverse \( L \) and \( N(L) = Y_1 \). Then

\[
\begin{bmatrix} 0 & K \\ 0 & L \end{bmatrix} \begin{bmatrix} A & C \\ T & S \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix}.
\]
so \( M_{(T,S)} \) is left invertible. It follows that \( M_{(T,S)} \) is left Fredholm for chosen operators \( T \) and \( S \). Suppose that \( M_{(T,S)} \) is Fredholm. Since \(
abla = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix} \) is Fredholm, from Lemma 1.1 it follows that \( \nabla \) is also Fredholm. However, we notice \( \nabla(N) = X \), which is infinite dimensional. Hence, \( N \) is not Fredholm. Then \( M_{(T,S)} \) is not Fredholm also, i.e., \( M_{(T,S)} \in \Phi_1(X \oplus Y) \setminus \Phi(X \oplus Y) \). □

We formulate a corollary for Hilbert space operators.

**Corollary 3.1.** Let \( X \) and \( Y \) be infinite dimensional and mutually orthogonal subspaces of a Hilbert space \( Z = X \oplus Y \). Suppose that \( \dim_H Y = \dim_H Z \). Let \( A \in \mathcal{L}(X) \) and \( C \in \mathcal{L}(Y,X) \) be arbitrary. Then \( M_{(T,S)} \in \Phi_1(X \oplus Y) \setminus \Phi(X \oplus Y) \) for some \( T \in \mathcal{L}(X,Y) \) and \( S \in \mathcal{L}(Y) \).

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