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HYPONORMALITY OF TOEPLITZ OPERATORS ON THE WEIGHTED BERGMAN SPACES

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Abstract. In this note we consider the hyponormality of Toeplitz operators T_φ on the Weighted Bergman space $A^2_\alpha(\mathbb{D})$ with symbol in the class of functions $f+\overline{g}$ with polynomials f and g of degree 2.

1. Introduction

A bounded linear operator A on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^*A - AA^*$ is positive (semidefinite). Let $\mathbb{D}$ be the open unit disk in the complex plane. For $-1 < \alpha < \infty$, the weighted Bergman space $A^2_\alpha(\mathbb{D})$ of the unit disk $\mathbb{D}$ is the space of analytic functions in $L^2(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z).$$

The space $L^2(\mathbb{D}, dA_\alpha)$ is a Hilbert space with the inner product

$$\langle f,\ g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z) \qquad (f,\ g \in L^2(\mathbb{D}, dA_\alpha)).$$

If $\alpha = 0$ then $A^2_0(\mathbb{D})$ is the Bergman space $A^2(\mathbb{D})$. For any nonnegative integer n, let

$$e_n(z) = \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}}\ z^n \quad (z \in \mathbb{D}),$$

where $\Gamma(s)$ stand for the usual Gamma function. It is easy to check that $\{e_n\}$ is an orthonormal basis for $A^2_\alpha(\mathbb{D})$ ([10]). For $\varphi \in L^\infty(\mathbb{D})$, the

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Toeplitz operator T_φ and the Hankel operator H_φ on $A^2_\alpha(\mathbb{D})$ are defined by

$$T_\varphi f := P_\alpha(\varphi \cdot f) \quad \text{and} \quad H_\varphi f = (I - P_\alpha)(\varphi \cdot f) \qquad (f \in A^2_\alpha(\mathbb{D}),$$

where P_α denotes the orthogonal projection that map from $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha(\mathbb{D})$. The reproducing kernel in $A^2_\alpha(\mathbb{D})$ is given by

$$K_z^{(\alpha)}(\omega) = \frac{1}{(1 - \overline{z}\omega)^{2+\alpha}},$$

for $z, \omega \in \mathbb{D}$. We thus have

$$(T_\varphi f)(z) = \int_{\mathbb{D}} \frac{\varphi(\omega) f(\omega)}{(1 - z\overline{\omega})^{2+\alpha}} dA_\alpha(\omega),$$

for $f \in A^2_\alpha(\mathbb{D})$ and $\omega \in \mathbb{D}$.

The hyponormality of Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$ has been studied by C. Cowen [1], R.E. Curto, I.S. Hwang and W.Y. Lee [2], [3], [5] and others [4]. Recently, in [7], the hyponormality of T_φ on the weighted Bergman space $A^2_\alpha(\mathbb{D})$ was studied. In [1], Cowen characterized the hyponormality of Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. Here we shall employ an equivalent variant of cowen's theorem that was first proposed by T. Nakazi and K. Takahashi [8].

Cowen's Theorem ([1], [8]). *For $\varphi \in L^\infty(\mathbb{T})$, write*

$$\mathcal{E}(\varphi) := \left\{k \in H^\infty : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\right\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

The solution is based on a dilation theorem of Sarason [9]. For the weighted Bergman space, no dilation theorem (similar to Sarason's theorem) is available. In [6], the first named author characterized the hyponormality of T_φ on $A^2_\alpha(\mathbb{D})$ in terms of the coefficients of the trigonometric polynomial φ under certain assumptions about the coefficients of φ on the *weighted Bergman space* when $\alpha \ge 0$.

Theorem A ([6]). *Let $\varphi(z) = \overline{g(z)} + f(z)$, where $f(z) = a_1 z + a_2 z^2$ and $g(z) = a_{-1} z + a_{-2} z^2$. If $a_1\overline{a_2} = a_{-1}\overline{a_{-2}}$ and $\alpha \ge 0$, then*

T_φ on $A^2_\alpha(\mathbb{D})$ is hyponormal

$$\Longleftrightarrow \begin{cases} \frac{1}{\alpha+3}(|a_2|^2 - |a_{-2}|^2) \ge \frac{1}{2}(|a_{-1}|^2 - |a_1|^2) & \text{if } |a_{-2}| \le |a_2| \\ \\ 4(|a_{-2}|^2 - |a_2|^2) \le |a_1|^2 - |a_{-1}|^2 & \text{if } |a_2| \le |a_{-2}|. \end{cases}$$

In this note we consider the hyponormality of Toeplitz operators T_φ on $A^2_\alpha(\mathbb{D})$ with symbol in the class of functions $f+\overline{g}$ with polynomials f and g of degree 2. Since the hyponormality of operators is translation invariant we may assume that $f(0)=g(0)=0$. The following relations can be easily proved:

(1.1) $T_{\varphi+\psi} = T_\varphi + T_\psi \quad (\varphi,\, \psi \in L^\infty)$;
(1.2) $T^*_\varphi = T_{\overline{\varphi}} \quad (\varphi \in L^\infty)$;
(1.3) $T_{\overline{\varphi}} T_\psi = T_{\overline{\varphi}\psi}$ if φ or ψ is analytic.

The purpose of this paper is to prove the Theorem A for the Toeplitz operators on $A^2_\alpha(\mathbb{D})$ when $-1<\alpha<0$.

2. Main result.

We need several auxiliary lemmas to prove Theorem 1. We begin with:

Lemma 1. ([6]). *For any s,t nonnegative integers,*

$$P_\alpha(\overline{z}^t z^s) = \begin{cases} \dfrac{\Gamma(s+1)\Gamma(s-t+\alpha+2)}{\Gamma(s+\alpha+2)\Gamma(s-t+1)} z^{s-t} & \text{if } s \ge t \\ 0 & \text{if } s < t. \end{cases}$$

Write

$$k_i(z) := \sum_{n=0}^{\infty} c_{2n+i} z^{2n+i} \quad (i=0,1).$$

We then have:

Lemma 2. ([6]). *For $0 \le m \le 2$ and $i=0,1$, we have*

(i) $$||\overline{z}^m k_i(z)||^2_\alpha = \sum_{n=0}^{\infty} \frac{(2n+m+i)!\Gamma(\alpha+2)}{\Gamma(2n+m+i+\alpha+2)} |c_{2n+i}|^2$$

(ii) $||P_\alpha(\overline{z}^m k_i(z))||^2_\alpha$

$$= \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{(2n+i)!^2\Gamma(2n+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(2n+i+\alpha+2)^2\Gamma(2n+i-m+1)} |c_{2n+i}|^2 & \text{if } m \le i \\ \displaystyle\sum_{n=1}^{\infty} \frac{(2n+i)!^2\Gamma(2n+i-m+\alpha+2)\Gamma(\alpha+2)}{\Gamma(2n+i+\alpha+2)^2\Gamma(2n+i-m+1)} |c_{2n+i}|^2 & \text{if } m > i. \end{cases}$$

We are ready for:

Theorem 1. *Let* $\varphi(z) = \overline{g(z)} + f(z)$*, where* $f(z) = a_1 z + a_2 z^2$ *and* $g(z) = a_{-1}z + a_{-2}z^2$*. If* $a_1\overline{a_2} = a_{-1}\overline{a_{-2}}$ *and* $-1 < \alpha < \infty$*, then*

$$T_\varphi \text{ on } A^2_\alpha(\mathbb{D}) \text{ is hyponormal}$$
$$\iff \begin{cases} \frac{1}{\alpha+3}(|a_2|^2 - |a_{-2}|^2) \geq \frac{1}{2}(|a_{-1}|^2 - |a_1|^2) & \text{if } |a_{-2}| \leq |a_2| \\ \\ 4(|a_{-2}|^2 - |a_2|^2) \leq |a_1|^2 - |a_{-1}|^2 & \text{if } |a_2| \leq |a_{-2}|. \end{cases}$$

Proof. By Theorem A, we may assume that $-1 < \alpha < 0$. For $i = 0, 1$, put

$$K_i := \Big\{ k_i(z) \in A^2_\alpha(\mathbb{D}) : k_i(z) = \sum_{n=0}^{\infty} c_{2n+i} z^{2n+i} \Big\}.$$

Then a straightforward calculation shows that T_φ is hyponormal if and only if

$$\Big\langle (H^*_{\overline{f}} H_{\overline{f}} - H^*_{\overline{g}} H_{\overline{g}})(k_0(z) + k_1(z)), (k_0(z) + k_1(z)) \Big\rangle_\alpha \geq 0 \quad \text{for all } k_i \in K_i \ (i = 0, 1). \tag{2.1}$$

Also we have that

$$\begin{aligned} &\Big\langle H^*_{\overline{f}} H_{\overline{f}}(k_0(z) + k_1(z)), (k_0(z) + k_1(z)) \Big\rangle_\alpha \\ &= \Big\langle H_{\overline{f}} k_0(z),\ H_{\overline{f}} k_0(z) \Big\rangle_\alpha + \Big\langle H_{\overline{f}} k_0(z),\ H_{\overline{f}} k_1(z) \Big\rangle_\alpha \\ &+ \Big\langle H_{\overline{f}} k_1(z),\ H_{\overline{f}} k_0(z) \Big\rangle_\alpha + \Big\langle H_{\overline{f}} k_1(z),\ H_{\overline{f}} k_1(z) \Big\rangle_\alpha, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\Big\langle H^*_{\overline{g}} H_{\overline{g}}(k_0(z) + k_1(z)), (k_0(z) + k_1(z)) \Big\rangle_\alpha \\ &= \Big\langle H_{\overline{g}} k_0(z),\ H_{\overline{g}} k_0(z) \Big\rangle_\alpha + \Big\langle H_{\overline{g}} k_0(z), H_{\overline{g}} k_1(z) \Big\rangle_\alpha \\ &+ \Big\langle H_{\overline{g}} k_1(z),\ H_{\overline{g}} k_0(z) \Big\rangle_\alpha + \Big\langle H_{\overline{g}} k_1(z), H_{\overline{g}} k_1(z) \Big\rangle_\alpha. \end{aligned} \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1), it follows from Lemma 2 that

$$\begin{aligned} &T_\varphi : \text{hyponormal} \\ &\iff \Big\langle (H^*_{\overline{f}} H_{\overline{f}} - H^*_{\overline{g}} H_{\overline{g}})(k_0(z) + k_1(z)), (k_0(z) + k_1(z)) \Big\rangle_\alpha \geq 0 \\ &\iff \sum_{i=0}^{1} \Big(||\overline{f} k_i||^2_\alpha - ||\overline{g} k_i||^2_\alpha + ||P_\alpha(\overline{g} k_i)||^2_\alpha - ||P_\alpha(\overline{f} k_i)||^2_\alpha \Big) \geq 0. \end{aligned}$$

Therefore it follows from Lemma 2 that T_φ is hyponormal if and only if

$$
\begin{aligned}
&(|a_1|^2-|a_{-1}|^2)\Bigg\{\frac{1}{\Gamma(\alpha+3)}|c_0|^2+\sum_{n=1}^{\infty}\bigg(\frac{\Gamma(2n+2)}{\Gamma(2n+\alpha+3)}\\
&-\frac{(2n)^2\Gamma(2n)}{(2n+\alpha+1)^2\Gamma(2n+\alpha+1)}\bigg)|c_{2n}|^2+\sum_{n=0}^{\infty}\bigg(\frac{\Gamma(2n+3)}{\Gamma(2n+\alpha+4)}\\
&-\frac{(2n+1)^2\Gamma(2n+1)}{(2n+\alpha+2)^2\Gamma(2n+\alpha+2)}\bigg)|c_{2n+1}|^2\Bigg\}+(|a_2|^2-|a_{-2}|^2)\Bigg\{\frac{2}{\Gamma(\alpha+4)}|c_0|^2\\
&+\sum_{n=1}^{\infty}\bigg(\frac{\Gamma(2n+3)}{\Gamma(2n+\alpha+4)}-\frac{(2n-1)^2(2n)^2\Gamma(2n-1)}{(2n+\alpha+1)^2(2n+\alpha)^2\Gamma(2n+\alpha)}\bigg)|c_{2n}|^2\\
&+\frac{6}{\Gamma(\alpha+5)}|c_1|^2+\sum_{n=1}^{\infty}\bigg(\frac{\Gamma(2n+4)}{\Gamma(2n+\alpha+5)}\\
&-\frac{(2n)^2(2n+1)^2\Gamma(2n)}{(2n+\alpha+1)^2(2n+\alpha+2)^2\Gamma(2n+\alpha+1)}\bigg)|c_{2n+1}|^2\Bigg\}\geq 0,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&(|a_1|^2-|a_{-1}|^2)\Bigg\{\frac{1}{\Gamma(\alpha+3)}|c_0|^2+\sum_{n=1}^{\infty}\bigg(\frac{\Gamma(n+2)}{\Gamma(n+\alpha+3)}\\
&-\frac{n^2\Gamma(n)}{(n+\alpha+1)^2\Gamma(n+\alpha+1)}\bigg)|c_n|^2\Bigg\}+(|a_2|^2-|a_{-2}|^2)\\
&\times\Bigg\{\sum_{n=0}^{1}\frac{\Gamma(n+3)}{\Gamma(n+\alpha+4)}|c_n|^2+\sum_{n=2}^{\infty}\bigg(\frac{\Gamma(n+3)}{\Gamma(n+\alpha+4)}\\
&-\frac{(n-1)^2n^2\Gamma(n-1)}{(n+\alpha)^2(n+\alpha+1)^2\Gamma(n+\alpha)}\bigg)|c_n|^2\Bigg\}\ \geq 0.
\end{aligned}
\tag{2.4}
$$

For $n\in\mathbb{N}$, define ζ_α by

$$\zeta_\alpha(n):=\frac{\frac{\Gamma(n+2)}{\Gamma(n+\alpha+3)}-\frac{n^2\Gamma(n)}{(n+\alpha+1)^2\Gamma(n+\alpha+1)}}{\frac{\Gamma(n+3)}{\Gamma(n+\alpha+4)}-\frac{(n-1)^2n^2\Gamma(n-1)}{(n+\alpha)^2(n+\alpha+1)^2\Gamma(n+\alpha)}}.$$

A direct calculation gives that

$$\zeta_\alpha(n)=\frac{(n+\alpha)(n+\alpha+1)(\alpha+1)}{4n^2+4(\alpha+2)n+2\alpha}.$$

Write

$$\zeta_\alpha(x) := \frac{(x+\alpha)(x+\alpha+1)}{4x^2+4(\alpha+2)x+2\alpha} \qquad (x \in \mathbb{R}^+).$$

Then we have that

$$\zeta'_\alpha(x) = -\frac{(\alpha+1)x^2+(2\alpha^2+5\alpha)x+(\alpha^3+4\alpha^2+\frac{9}{2}\alpha)}{(2x^2+2(\alpha+2)x+\alpha)^2},$$

which implies that $\zeta_\alpha(x)$ is strictly decreasing function when $\alpha \geq 0$. But $\zeta_\alpha(x)$ has a relative maximum $x = \frac{-2\alpha^2-5\alpha+\sqrt{-9\alpha^2-18\alpha}}{2(\alpha+1)}$ when $-1 < \alpha < 0$. Put $k_\alpha := \frac{-2\alpha^2-5\alpha+\sqrt{-9\alpha^2-18\alpha}}{2(\alpha+1)}$, then the maximum value of $\zeta_\alpha(n)$ is $\max\{\zeta_\alpha([k_\alpha]), \zeta_\alpha([k_\alpha]+1)\}$. If $0 < k_\alpha < 1$, then

$$\max\{\zeta_\alpha(0), \zeta_\alpha(1)\} = \max\left\{\frac{\alpha+3}{2}, \frac{(\alpha+1)(\alpha+4)}{8\alpha+12}\right\} = \frac{\alpha+3}{2}.$$

Let $k_\alpha \geq 1$. Since $\zeta_\alpha(k_\alpha)$ is relative maximum,

$$\zeta_\alpha(k_\alpha) \geq \max\{\zeta_\alpha([k_\alpha]), \zeta_\alpha([k_\alpha]+1)\}.$$

It follows from $-1 < \alpha < 0$ that

$$k_\alpha^2 + k_\alpha - \alpha^2 - 2\alpha \geq 0.$$

Therefore

$$\zeta_\alpha(k_\alpha) = \frac{(k_\alpha+\alpha)(k_\alpha+\alpha+3)}{4k_\alpha^2+4(\alpha+2)k_\alpha+2\alpha} \leq \frac{1}{2} \leq \frac{\alpha+3}{2}.$$

Hence

$$\max\{\zeta_\alpha([k_\alpha]), \zeta_\alpha([k_\alpha]+1)\} \leq \frac{\alpha+3}{2}. \tag{2.5}$$

Let $|a_{-2}| \leq |a_2|$ and hence $|a_1| \leq |a_{-1}|$. Observe that

$$\lim_{n\to\infty} \zeta_\alpha(n) = \frac{1}{4} \tag{2.6}$$

and

$$\frac{\alpha+3}{2} \geq \frac{(\alpha+1)(\alpha+4)}{6(\alpha+2)}. \tag{2.7}$$

Therefore (2.4), (2.5) and (2.7) give that T_φ is hyponormal if and only if

$$2(|a_2|^2 - |a_{-2}|^2) \geq (\alpha+3)(|a_{-1}|^2 - |a_1|^2).$$

Let $|a_2| \leq |a_{-2}|$ and hence $|a_{-1}| \leq |a_1|$. Since $\zeta_\alpha(n) \geq \frac{1}{4}$, it follows from (2.4), (2.5), (2.6) and (2.7) that T_φ is hyponormal if and only if

$$4(|a_{-2}|^2 - |a_2|^2) \leq (|a_1|^2 - |a_{-1}|^2).$$

This completes the proof. □

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