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NONTRIVIAL SOLUTION FOR THE BIHARMONIC BOUNDARY VALUE PROBLEM WITH SOME NONLINEAR TERM

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Abstract. We investigate the existence of weak solutions for the biharmonic boundary value problem with nonlinear term decaying at the origin. We get a theorem which shows the existence of nontrivial solutions for the biharmonic boundary value problem with nonlinear term decaying at the origin. We obtain this result by reducing the biharmonic problem with nonlinear term to the biharmonic problem with bounded nonlinear term and then approaching the variational method and using the mountain pass geometry for the reduced biharmonic problem with bounded nonlinear term.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let $p : \overline{\Omega} \times R \to R$ be a C^1 function. In this paper we investigate the number of weak solutions for the following biharmonic problem with

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Dirichlet boundary condition

$$\begin{aligned} &\Delta^2 u + c\Delta u = p(x, u(x)) \qquad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

We assume that p satisfies the followings:
($p1$) $p \in C^1(\overline{\Omega} \times R, R)$,
($p2$) $p(x, 0) = 0$, $p(x, \xi) = o(|\xi|)$ uniformly with respect to $x \in \overline{\Omega}$,
($p3$) there exists $\xi \geq 0$ such that $p(x, \xi) \leq 0$ $\forall x \in \overline{\Omega}$,
($p4$) there exist a constant $r > 0$ and an element $e \in H$ such that $\|e\| = r$, $e < \xi$ and $\frac{1}{2}r^2 - \int_\Omega P(x, e) < 0$, where H is introduced in Section 2.
The eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u = 0 \qquad &\text{in } \Omega, \\ u = 0 \qquad &\text{on } \partial\Omega \end{aligned}$$

has infinitely many eigenvalues λ_j, $j \geq 1$ which is repeated as often as its multiplicity, and the corresponding eigenfunctions ϕ_j, $j \geq 1$ suitably normalized with respect to $L^2(\Omega)$ inner product. The eigenvalue problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = \Lambda u \qquad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

has also infinitely many eigenvalues $\Lambda_j = \lambda_j(\lambda_j - c)$, $j \geq 1$ and the corresponding eigenfunctions ϕ_j, $j \geq 1$. We note that

$$\Lambda_1 < \Lambda_2 \leq \Lambda_3 \ldots, \qquad \Lambda_j \to +\infty.$$

Jung and Choi [4] showed the existence of at least two solutions, one of which is bounded solution and the other is large norm solution of (1.1) when $p(u)$ is polynomial growth or exponential growth nonlinear term. They proved this result by the variational method and the mountain pass theorem. The authors [1,6] also investigate the multiple solutions of nonlinear boundary value problems. For the constant coefficient nonlinear case Choi and Jung [3] showed that the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1.3}$$

has at least two nontrivial solutions when $c < \lambda_1$, $\Lambda_1 < b < \Lambda_2$ and $s < 0$) or when $\lambda_1 < c < \lambda_2$, $b < \Lambda_1$ and $s > 0$. The authors obtained these results by use of the variational reduction method. The authors [5] also proved that when $c < \lambda_1$, $\Lambda_1 < b < \Lambda_2$ and $s < 0$, (1.3) has at least three nontrivial solutions by use of the degree theory. Tarantello [9] also studied the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = b((u+1)^+ - 1) \qquad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

She showed that if $c < \lambda_1$ and $b \geq \Lambda_1$, then (1.4) has a negative solution. She obtained this result by the degree theory. Micheletti and Pistoia [7] also proved that if $c < \lambda_1$ and $b \geq \Lambda_2$, then (1.4) has at least three solutions by the vatiational linking theorem and Leray-Schauder degree theory. In this paper we are trying to find weak solutions of (1.1), that is,

$$\int_\Omega [\Delta^2 u \cdot v + c\Delta u \cdot v - p(x,u)v]dx = 0, \qquad \forall v \in H,$$

where H is introduced in Section 2.

We consider the associated functional of (1.1)

$$I(u) = \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - P(x,u)]dx,$$

where $P(x,s) = \int_0^s p(x,\tau)d\tau$. By $(p1)$, I is well defined.

Our main result is the following.

THEOREM 1.1. *1.1 Assume that $c < \lambda_1$ and p satisfies the conditions $(p1) - (p4)$. Then (1.1) has at least one nontrivial weak solution.*

We prove Theorem 1.1 by reducing the biharmonic problem (1.1) to the biharmonic problem with bounded nonlinear term and then approaching the variational method and using the mountain pass geometry for the reduced biharmonic problem with bounded nonlinear term. The outline of the proof of Theorem 1.1 is as follows: In Section 2, we prove that the functional $I(u) \in C^1$ and the reduced functional $\hat{I}$ of I satisfies the Palais Smale condition. In Section 3, we show that the reduced functional $\hat{I}$ satisfies the mountain pass geometry and so prove that $\hat{I}$ has at least one nontrivial critical point, from which we prove Theorem 1.1.

2. Bounded nonlinear term

Let $L^2(\Omega)$ be a square integrable function space defined on Ω. Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \phi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) | \sum |\Lambda_k| h_k^2 < \infty\}. \tag{2.1}$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum |\Lambda_k| h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have
(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$ for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$,
which is proved in [2].

By the following Proposition 2.1, the weak solutions of (1.1) coincide with the critical points of the associated functional $I(u)$.

Proposition 2.1. *Assume that $c < \lambda_1$ and p satisfies the conditions $(p1) - (p4)$. Then $I(u)$ is continuous and Fréchet differentiable in H with Fréchet derivative*

$$\nabla I(u)h = \int_\Omega [\Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - p(x,u)h] dx.$$

If we set

$$F(u) = \frac{1}{2}\int_\Omega P(x,u)dx,$$

where $P(x,s) = \int_0^s p(x,\tau)d\tau$, then $F'(u)$ is continuous with respect to weak convergence, $F'(u)$ is compact, and

$$F'(u)h = \int_\Omega p(x,u)h dx \qquad \text{for all } h \in H,$$

this implies that $I \in C^1(H,R)$ and $F(u)$ is weakly continuous.

The proof of Proposition 2.1 has the similar process to that of the proof in Appendix B in [8].

Now we shall reduce the problem (1.1) to the problem with the bounded nonlinear term.

LEMMA 2.1. *Assume that $c < \lambda_1$ and p satisfies the conditions $(p1)-(p4)$. Let*

$$\hat{p}(x,t) = \begin{cases} p(x,\xi) & \text{if } t > \xi, \\ p(x,t) & \text{if } t \leq \xi. \end{cases}$$

Assume that $u \in H$ is a solution of the equation

$$\begin{cases} \Delta^2 u + c\Delta u = \hat{p}(x,u(x)) & x \in \Omega, \\ u = 0, \qquad \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

Then $u \leq \xi$, so u is a solution of (1.1).

Proof. By the standard regularity theorem, $u \in W_p^2(\Omega)$ $\forall p < +\infty$, and hence $u \in C^1(\overline{\Omega})$. Let us set

$$C = \{x \in \Omega |\ u(x) > \xi\}.$$

Then we have

$$\begin{cases} -\Delta(-\Delta - c)u(x) \leq 0 & \forall x \in C, \\ u(x)|_{\partial C} \leq \xi \end{cases} \tag{2.3}$$

For $c < \lambda_1$ the operator $-\Delta(-\Delta - c)$ is a positive operator. It follows from (2.3) that $u(x) \leq 0$ in C. By (2.3), $u(x)|_{\partial C} \leq \xi$. By the maximum principle,

$$u(x) < \xi \qquad \text{in } C,$$

and hence $C = \emptyset$. Thus $u(x) \leq \xi$, so it is a solution of (1.1). □

By Lemma 2.1, it suffices to investigate the multiplicity of solutions of (2.2) with bounded nonlinear term for the multiplicity results of solutions of (1.1). Now we shall show that (2.2) has at least one nontrivial solution by approaching the variational method and applying mountain pass theorem in the critical point theory.

Let us consider the associated functional of (2.2)

$$\hat{I}(u) = \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \hat{P}(x,u)]dx, \tag{2.4}$$

where $\hat{P}(x,s) = \int_0^s \hat{p}(x,\tau)d\tau$. By the same process as the proof of Proposition 2.1, we can show that

$$\hat{I}(u) \in C^1(H,R).$$

Now we shall show that $\hat{I}(u)$ satisfies Palais-Smale condition.

LEMMA 2.2. *Assume that $c < \lambda_1$ and p satisfies the conditions $(p1)-(p4)$. Then the functional $\hat{I}$ satisfies Palais-Smale condition: Any sequence (u_m) in H for which $|\hat{I}(u_m)| \leq M$ and $\hat{I}'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.*

Proof. Let us choose $u \in H$. By $p \in C^1$, $\hat{P}(x,u)$ is bounded. Then we have

$$\begin{aligned}\hat{I}(u) &= \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \hat{P}(x,u)]dx \\ &\geq \frac{1}{2}\{\lambda_1(\lambda_1 - c)\}\|u\|^2_{L^2(\Omega)} - \int_\Omega \hat{P}(x,u)dx.\end{aligned}$$

Since $\lambda_1(\lambda_1 - c) > 0$, $u \in H$ and $\int_\Omega \hat{P}(x,u)dx$ is bounded, $\hat{I}(u)$ is bounded from below. Thus $\hat{I}$ satisfies the (PS) condition. □

3. Proof of Theorem 1.1

We shall show that the functional $\hat{I}$ satisfies the mountain pass geometry.

LEMMA 3.1. *Assume that $c < \lambda_1$ and p satisfies the conditions $(p1)-(p4)$. Let*

$$B_\rho = \{u \in H| \ \|u\| \leq \rho\}$$

and

$$S_\rho = \{u \in H| \ \|u\| = \rho\}.$$

Then
(i) there is an $e \in H$ with $\|e\| = r$ such that

$$\hat{I}(e) \leq 0,$$

and
(ii) there exist constants ρ, $\alpha > 0$ such that $\rho < r$ (r is a constant in $(p4)$) and

$$\inf_{u \in S_\rho} \hat{I}(u) \geq \alpha.$$

Proof. (i) By (p4), there exist a constant $r > 0$ and an element $e \in H$ such that $\|e\| = r$, $e < \xi$ and $\frac{1}{2}r^2 - \int_\Omega P(x,e) < 0$. Then we have

$$\tilde{I}(e) = \int_\Omega [\frac{1}{2}|\Delta e|^2 - \frac{c}{2}|\nabla e|^2 - \hat{P}(x,e)]dx = \frac{1}{2}r^2 - \int_\Omega P(x,e) < 0.$$

Thus (i) is proved.
(ii) Let $r > 0$ be a constant in (i). Let $u \in H$. By $(p2)$ and $(p4)$, there exists a small number $a > 0$, $\rho > 0$ and a sphere S_ρ with the radius ρ such that $\rho < r$, $a < \frac{1}{2}\lambda_1(\lambda_1 - c)$ and $|\hat{P}(x,u)| \le a\|u\|^2$ if $u \in S_\rho$. If we choose $u \in S_\rho$, then there exists a constant $\alpha > 0$ such that

$$\begin{aligned}
\hat{I}(u) &= \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \hat{P}(x,u)]dx \\
&\ge \frac{1}{2}\lambda_1(\lambda_1 - c)\|u\|^2_\Omega - a\|u\|^{L^2(\Omega)} \\
&= (\frac{1}{2}\lambda_1(\lambda_1 - c) - a)\|u\|^2 \ge \alpha.
\end{aligned}$$

□

Proof of Theorem 1.1
We will show that $\hat{I}(u)$ has a nontrivial critical point by the mountain pass theorem. By Proposition 2.1, $\hat{I}(u)$ is continuous and *Fréchet* differentiable in H. By Lemma 2.2, the functional $\hat{I}$ satisfies (PS) condition. We note that $I(0) = 0$. By Lemma 3.1, there exist constants $\rho > 0$, $r > 0$, $\alpha > 0$ and $e \in H$ with $\|e\| = r$ such that $\rho < r$, $\inf_{u\in S_\rho} \hat{I}(u) \ge \alpha$, and $\hat{I}(e) < 0$. Let us set

$$\Gamma = \{\gamma \in C([0,1], H)|\ \gamma(0) = 0, \gamma(1) = e\}.$$

By the mountain pass theorem, $\hat{I}$ possesses a critical value $c \ge \alpha$ characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma} I(u).$$

Thus we prove that $\hat{I}$ has at least one nontrivial critical point, so (2.2) has at least one nontrivial weak solution. By Lemma 2.1, this solution of (2.2) is also a weak solution of (1.1). Thus (1.1) has at least one nontrivial weak solution, and hence we prove Theorem 1.1.

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