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THE λ-NUMBER OF THE CARTESIAN PRODUCT OF A COMPLETE GRAPH AND A CYCLE

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Abstract. An $L(j,k)$-labeling of a graph G is a vertex labeling such that the difference of the labels of any adjacent vertices is at least j and that of any vertices of distance two is at least k for given j and k. The minimum span of all $L(2,1)$-labelings of G is called the λ-number of G and is denoted by $\lambda(G)$.

In this paper, we find a lower bound of the λ-number of the Cartesian product $K_m \Box C_n$ of the complete graph K_m of order m and the cycle C_n of order n. In fact, we show that when $n \geq 3$, $\lambda(K_4 \Box C_n) \geq 7$ and the equality holds if and only if n is a multiple of 8. Moreover when $m \geq 5$, $\lambda(K_m \Box C_n) \geq 2m-1$ and the equality holds if and only if n is even.

1. Introduction

In a wireless communication network, a channel assignment problem(CAP) involves assigning channels which are represented by nonnegative integers, to the radio transmitters in the network. There must be some restrictions to minimize interference between the channels assigned to the transmitters that are close or very close. The restrictions would

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be as follows: For a pair of positive integers $j, k (j \geq k)$, we assign channels with difference at least j to transmitters which are very close and at least k to those which are close.

CAP is first placed in a graph theoretic context in a more generalized form by Hale[8] in the following way. Vertices of a graph represent the transmitters and two vertices are adjacent if the corresponding transmitters are very close.

A distance two labeling or $L(j,k)$-labeling of a graph $G = (V_G, E_G)$ is a function $f : V_G \to [0, \infty)$ such that

$$|f(u) - f(v)| \geq \begin{cases} j & \text{if } \mathrm{dist}_G(u,v) = 1 \\ k & \text{if } \mathrm{dist}_G(u,v) = 2. \end{cases}$$

We call the elements of the image of f the labels and the difference between the maximum and the minimum of the labels assigned by f the span of f. Usually we assume that the minimum label zero, and hence we regard the maximum label of f as the span of f. Furthermore the minimum span over all labelings is called the λ_k^j-number of G and is denoted by $\lambda_k^j(G)$. Griggs and Yeh[6] proposed a problem of determining $\lambda_k^j(G)$, especially $\lambda_1^2(G)$, and proved that for a graph G of maximum degree Δ, $\lambda_1^2(G) \leq \Delta^2 + 2\Delta$. They also conjectured that $\lambda_1^2(G) \leq \Delta^2$. After their work, $\lambda_1^2(G)$ is called the λ-number of G and denoted by $\lambda(G)$. The λ-number of some classes of graphs have been studied in many literatures[1, 13, 12, 4, 10]. See [2, 3, 5] for extensive surveys.

For two graphs G and H, the Cartesian product of G and H, which is denoted by $G \square H$ is the graph with vertex set $V_G \times V_H$ where two vertices (x, y) and (x', y') are adjacent if (i) $x = x'$ and $(y, y') \in E_H$, or (ii) $y = y'$ and $(x, x') \in E_G$.

There are results on the λ_k^j-number of the Cartesian product of various simple graphs [19, 7, 11, 18, 16, 9, 15, 14].

In this paper, we find a lower bound of the λ-number of the Cartesian product $K_m \square C_n$ of the complete graph K_m of order m and the cycle C_n of order n. In fact, we show that when $n \geq 3$, $\lambda(K_4 \square C_n) \geq 7$ and the equality holds if and only if n is a multiple of 8. We also show that when $m \geq 5$, $\lambda(K_m \square C_n) \geq 2m - 1$ and the equality holds if and only if n is even.

2. The λ-number of $K_m \Box C_n$

Let $G = K_m \Box C_n = (V, E)$ be the Cartesian product of the complete graph K_m on m vertices and the cycle C_n on n vertices. See Figure 1 for $G = K_5 \Box C_6$ as an example.

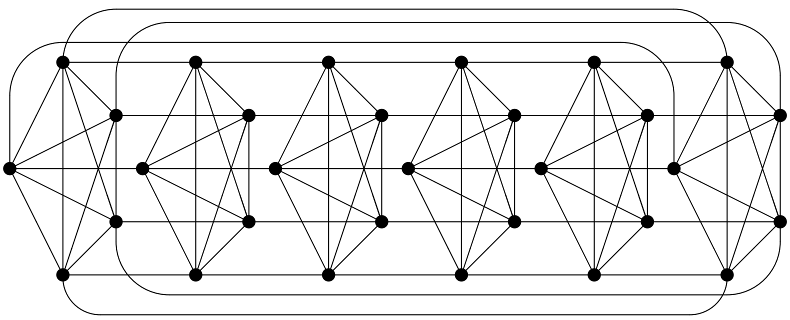

Figure 1. $K_5 \Box C_6$

For each $x \in \mathbb{Z}$ and $M \in \mathbb{Z}^+$, the residue of x modulo M is denoted by $x \pmod{M}$. Note that a function f from V to $[0, N]$ for some positive integer N is an $L(2,1)$-labeling of G if and only if it satisfies

$$\begin{aligned}
\text{(i)}|\mathrm{f(i,j) - f(i',j')}| \geq 2 \text{ if either } & i \neq i' \text{ and } j = j', \\
\text{or } & i = i' \text{ and } j = j' \pm 1 \pmod{n}, \\
\text{(ii)}|\mathrm{f(i,j) - f(i',j')}| \geq 1 \text{ if either } & i \neq i' \text{ and } j = j' \pm 1, \\
\text{or } & i = i' \text{ and } j = j' \pm 2 \pmod{n}.
\end{aligned}$$

The following lemma is easily proved.

LEMMA 1. *Let $G = K_m \Box C_n = (V, E)$ and $f : V \to [0, N]$ be a function. Extend f to all $i \in [0, m-1]$ and $j \in \mathbb{Z}$ by $f(i,j) = f(i,\tilde{j})$ where $j = \tilde{j} \pmod{n}$. Then f is an $L(2,1)$-labeling of G if and only if the extended f satisfies*

$$\begin{aligned}
\text{(i)}|\mathrm{f(i,j) - f(i',j')}| \geq 2 \textit{ if either } & i \neq i' \textit{ and } j = j' \pmod{n}, \\
\textit{or } & i = i' \textit{ and } j = j' \pm 1 \pmod{n}, \\
\text{(ii)}|\mathrm{f(i,j) - f(i',j')}| \geq 1 \textit{ if either } & i \neq i' \textit{ and } j = j' \pm 1 \pmod{n}, \\
\textit{or } & i = i' \textit{ and } j = j' \pm 2 \pmod{n}.
\end{aligned}$$

Schwartz and Troxell[17] proved that

$$\lambda(K_3\Box C_n) = \begin{cases} 7, & \text{if } n \text{ is even with } n \neq 4, 10, \\ 8, & \text{otherwise.} \end{cases}$$

Theorem 2. *For $m \geq 4$, we have the following lower bound of the λ-number of $K_m\Box C_n$ which are extreme for some n.*

(i) For each $n \geq 3$, $\lambda(K_4\Box C_n) \geq 7$ and the equality holds if and only if n is a multiple of 8.

(ii) For each $m \geq 5$ and $n \geq 3$, $\lambda(K_m\Box C_n) \geq 2m-1$ and the equality holds if and only if n is even.

Proof. Throughout this proof, let $G = K_m\Box C_n = (V, E)$.

(i) Let $f : V \to [0, N]$ be an $L(2,1)$-labeling of G. We consider the extension of f defined on $[0, m-1] \times \mathbb{Z}$ in Lemma 1. For each nonnegative integer j, we define $V_j = \{(i, j)|0 \leq i \leq 3\}$. Since for each j any two elements of $V_j \cup V_{j+1}$ are of distance at most two, $f(V_j \cup V_{j+1}) = \{f(v)|v \in V_j \cup V_{j+1}\}$ has eight elements. Thus $8 \leq N+1$ so that $N \geq 7$. It follows that $\lambda(G) \geq 7$.

We show that the equality holds if and only if n is a multiple of 8. Assume $N = 7$. For each j, since $f(V_j \cup V_{j+1})$ has eight elements, $f(V_j \cup V_{j+1}) = [0, 7]$. We may assume that $0 \in f(V_0)$ and hence $f(V_1) \subset [1, 7]$. Considering that any two labels of distinct elements of V_1 are apart by at least two, we have $f(V_1) = \{1, 3, 5, 7\}$. Therefore $f(V_0) = \{0, 2, 4, 6\}$. Similarly,

$$f(V_j) = \begin{cases} \{0, 2, 4, 6\}, & \text{if } j \text{ is even,} \\ \{1, 3, 5, 7\}, & \text{if } j \text{ is odd.} \end{cases}$$

We may assume that $f(i, 0) = 2i$ for $0 \leq i \leq 3$. There is i_1 such that $f(i_1, 1) = 1$. Since $|f(i_1, 0) - f(i_1, 1)| = |2i_1 - 1| \geq 2$, i_1 is 2 or 3.

If $i_1 = 2$, then we want to prove that

$$f(i, j) = 2i + 5j \pmod 8 \tag{1}$$

for all i, j. Since $f(3, 1) \in \{1, 3, 5, 7\}$, $|f(3, 1) - f(3, 0)| = |f(3, 1) - 6| \geq 2$ and $f(3, 1) \neq f(2, 1) = 1$, we have $f(3, 1) = 3$. Since $f(2, 2) \in \{0, 2, 4, 6\}$, $|f(2, 2) - f(2, 1)| = |f(2, 2) - 1| \geq 2$ and $f(2, 2) \neq f(2, 0) = 4$, we have $f(2, 2) = 6$. Since $|f(3, 2) - f(3, 1)| = |f(3, 2) - 3| \geq 2$ and $f(3, 2) \neq f(3, 0) = 6$, we have $f(3, 2) = 0$. Since $f(1, 2) \in \{0, 2, 4, 6\}$ and $f(1, 2)$ is different from $f(1, 0) = 2, f(2, 2) = 6, f(3, 2) = 0$, we have $f(1, 2) = 4$. Thus $f(0, 2) = 2$. Since $f(1, 1) \in \{5, 7\}$ and $|f(1, 1) - f(1, 2)| = |f(1, 1) - 4| \geq 2$, we have $f(1, 1) = 7$. Thus $f(0, 1) = 5$. Note that $f(i,1) = 2i + 5 \pmod 8$ and $f(i,2) = 2i + 2 \pmod 8 = 2i + 2 \cdot 5 \pmod 8$. We just proved that if equation (1) holds for $j = 0$ and $(i,j) = (2,1)$, then equation (1) holds for $j = 1, 2$ and $(i,j) = (1,3)$. By the same method, we can prove that for any t, if equation (1) holds for $j = 2t$ and $(i,j) = (s, 2t+1)$ where $s = 2 - t \pmod 4$, then equation (1) holds for $j = 2t+1, 2t+2$ and $(i,j) = (s', 2t+3)$ where $s' = 2 - (t+1) \pmod 4$. Thus by induction, equation (1) is true for all j. Since

$$0 = f(0,0) = f(0,n) = 5n \pmod 8,$$

n is a multiple of 8.

If $i_1 = 3$, then by a similar method as above, we can prove

$$f(i,j) = 2i + 3j \pmod 8 \tag{2}$$

holds for all i, j. Since

$$0 = f(0,0) = f(0,n) = 3n \pmod 8,$$

n is a multiple of 8.

Thus for any i_1, n is a multiple of 8. It follows that $\lambda(G) = 7$ only if n is a multiple of 8.

Conversely assume n is a multiple of 8. Then since

$$f(i,j) = 2i + 5j \pmod 8 = 2i + 5j + 5n \pmod 8 = f(i, j+n)$$

and

$$f(i,j) = 2i + 3j \pmod 8 = 2i + 3j + 3n \pmod 8 = f(i, j+n),$$

two labelings defined by equation (1) and equation (2) are well-defined. These labelings are shown in Table 1. Thus $\lambda(G) = 7$ when n is a multiple of 8.

(ii) Let $f : V \to [0, N]$ be an $L(2,1)$-labeling of G. We consider the extension of f defined on $[0, m-1] \times \mathbb{Z}$ in Lemma 1. For each $j \in [0, n-1]$, define $V_j = \{(i,j) | 0 \le i \le m-1\}$. Since for each j, any two elements of $V_j \cup V_{j+1}$ are of distance at most two, $f(V_j \cup V_{j+1})$ has $2m$ elements. Thus $2m \le N + 1$ so that $N \ge 2m - 1$. It follows that $\lambda(G) \ge 2m - 1$.

We show that the equality holds if and only if n is even. Assume $N = 2m - 1$. Since for each j, $f(V_j \cup V_{j+1})$ has $2m$ elements, $f(V_j \cup V_{j+1}) = [0, 2m-1]$. We may assume that $0 \in f(V_j)$. Then $f(V_1) \subset [1, 2m-1]$ and hence we have $f(V_1) = \{1, 3, \cdots, 2m-1\}$ considering that any two labels of distinct elements of V_1 are apart by at least two. Since $f(V_0 \cup V_1) = f(V_0) \cup f(V_1) = [0, 2m-1]$, we have $f(V_0) = \{0, 2, \cdots, 2m-2\}$. Similarly,

$$f(V_j) = \begin{cases} \{0, 2, \cdots, 2m-2\}, & \text{if } j \text{ is even,} \\ \{1, 3, \cdots, 2m-1\}, & \text{if } j \text{ is odd.} \end{cases}$$

Since $V_n = V_0$, n is even. Thus $\lambda(G) = 2m - 1$ only if n is even.

Conversely assume n is even. If $m = 5$, then there is an $L(2,1)$-labeling

$$f_1(i,j) = 2i + a_j \pmod{2m} \tag{3}$$

for $(a_0, a_1, a_2, a_3) = (0, 3, 8, 5)$ of $K_5 \Box C_4$ and an $L(2,1)$-labeling

$$f_2(i,j) = 2i + 3j \pmod{2m} \tag{4}$$

of $K_5 \Box C_6$, which are shown in Table 2. Since n is even and at least 4, $n = 4x + 6y$ for some non-negative integer x, y. Therefore

$$f(i,j) = \begin{cases} f_1(i,j), & \text{if } 0 \le j < 4x, \\ f_2(i, j-4x), & \text{if } 4x \le j \le n-1. \end{cases}$$

is an $L(2,1)$-labeling of $G = K_5 \Box C_n$ with its span 9. Thus $\lambda(G) = 9$.

If $m \ge 6$, then

$$f_3(i,j) = \begin{cases} 2i + j \pmod{2m}, & \text{if } j \text{ is even,} \\ 2i + j + 5 \pmod{2m}, & \text{if } j \text{ is odd.} \end{cases}$$

is an $L(2,1)$-labeling of $K_m \Box C_4$ and also an $L(2,1)$-labeling of $K_m \Box C_6$. They are shown in Table 3. Since n is even and at least 4, $n = 4x + 6y$ for some non-negative integer x, y. Therefore

$$f(i,j) = \begin{cases} f_3(i,j), & \text{if } 0 \le j < 4x, \\ f_3(i, j-4x), & \text{if } 4x \le j \le n-1. \end{cases}$$

is an $L(2,1)$-labeling of $G = K_m \Box C_n$ with its span $2m-1$. Thus $\lambda(G) = 2m - 1$.

$i \backslash j$	0	1	2	3	4	5	6	7
0	0	5	2	7	4	1	6	3
1	2	7	4	1	6	3	0	5
2	4	1	6	3	0	5	2	7
3	6	3	0	5	2	7	4	1

$i \backslash j$	0	1	2	3	4	5	6	7
0	0	3	6	1	4	7	2	5
1	2	5	0	3	6	1	4	7
2	4	7	2	5	0	3	6	1
3	6	1	4	7	2	5	0	3

TABLE 1. Two $L(2,1)$-labelings of $K_4 \Box C_8$

$i \backslash j$	0	1	2	3
0	0	3	8	5
1	2	5	0	7
2	4	7	2	9
3	6	9	4	1
4	8	1	6	3

$i \backslash j$	0	1	2	3	4	5
0	0	3	6	9	2	5
1	2	5	8	1	4	7
2	4	7	0	3	6	9
3	6	9	2	5	8	1
4	8	1	4	7	0	3

TABLE 2. $L(2,1)$-labelings of $K_5 \square C_4$ and $K_5 \square C_6$

$i \backslash j$	0	1	2	3
0	0	5	2	7
1	2	7	4	9
2	4	9	6	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	$2m-2$	3	0	5

$i \backslash j$	0	1	2	3	4	5
0	0	5	2	7	4	9
1	2	7	4	9	6	11
2	4	9	6	11	8	$\begin{cases} 13, & \text{if } m \geq 7, \\ 1, & \text{if } m = 6. \end{cases}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	$2m-2$	3	0	5	2	7

TABLE 3. $L(2,1)$-labelings of $K_m \square C_4$ and $K_m \square C_6$ for $m \geq 6$

$\square$

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