GCR-LIGHTLIKE SUBMANIFOLDS OF INDEFINITE NEARLY KAEHLER MANIFOLDS

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Abstract. We introduce *CR*, *SCR* and *GCR*-lightlike submanifolds of indefinite nearly Kaehler manifolds and obtain their existence in indefinite nearly Kaehler manifolds of constant holomorphic sectional curvature c and of constant type α. We also prove characterization theorems on the existence of totally umbilical and minimal *GCR*-lightlike submanifolds of indefinite nearly Kaehler manifolds.

1. Introduction

The geometry of *CR*-submanifolds of Kaehler manifolds was initiated by Bejancu [2], as a generalization of totally real and complex submanifolds and has been further developed by many others [3, 4, 5, 6, 7]. The study of *CR*-submanifolds of nearly Kaehler manifolds was initiated by Deshmukh et al. [8] and further developed by [16, 17]. The *CR* structures on real hypersurfaces of complex manifolds have interesting applications to relativity. Duggal studied geometry of *CR* submanifolds with Lorentzian metric and obtained their interaction with relativity [9, 10]. Duggal and Bejancu [11] introduced a new class called *CR*-lightlike submanifolds of indefinite Kaehler manifolds, which excludes the complex and totally real cases. Then Duggal and Sahin [13] introduced Screen Cauchy-Riemann (*SCR*)-lightlike submanifolds of indefinite Kaehler manifolds, which contains complex and screen real sub-cases. But there was no inclusion relation between *SCR* and *CR* cases. So to obtain the desired relationship, Duggal and Sahin [14] introduced Generalized Cauchy-Riemann (*GCR*)-lightlike submanifolds of indefinite Kaehler manifolds and further developed by [18, 19, 20]. The theory of lightlike submanifolds has interaction with some results on Killing horizon, electromagnetic and radiation fields and asymptotically flat spacetimes. Thus the significant applications of *CR* structures in relativity and growing importance of lightlike submanifolds

in mathematical physics and very limited information available motivated the present authors to work on it.

In present paper, we introduce the notion of CR-lightlike submanifolds of indefinite nearly Kaehler manifolds and obtain the existence theorem for this class of submanifolds. We conclude that CR-lightlike submanifolds do not include invariant (complex) and totally real lightlike submanifolds. Thus we introduce a class SCR-lightlike submanifolds of indefinite nearly Kaehler manifolds, and derive the existence theorem for this class. We further conclude that there is no inclusion relation between CR and SCR subcases. Therefore we introduce a new class called GCR-lightlike submanifolds of indefinite nearly Kaehler manifolds and show that this class of submanifolds contains CR and SCR-lightlike submanifolds as subcases. We obtain the existence of this class and the non-existence of totally umbilical GCR-lightlike submanifolds of indefinite nearly Kaehler manifolds with constant holomorphic sectional curvature c and of constant type α. We also study minimal GCR-lightlike submanifolds and give some characterization theorems on minimal GCR-lightlike submanifolds.

2. Lightlike submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m + n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m + n - 1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$, for detail see [11]. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\mathrm{Rad}T_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $\mathrm{Rad}TM : x \in M \longrightarrow \mathrm{Rad}T_xM$, defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $\mathrm{Rad}TM$ is called the radical distribution on M. Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM therefore

$$(1) \qquad\qquad TM = \mathrm{Rad}TM \perp S(TM)$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $\mathrm{Rad}TM$ in $TM^{\perp}$. Let $\mathrm{tr}(TM)$ and $\mathrm{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $\mathrm{Rad}TM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$(2) \qquad\qquad \mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp S(TM^{\perp}).$$

$$(3) \quad T\bar{M}\mid_M = TM \oplus \mathrm{tr}(TM) = (\mathrm{Rad}TM \oplus \mathrm{ltr}(TM)) \perp S(TM) \perp S(TM^{\perp}).$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\}$, where $\{\xi_1, \ldots, \xi_r\}, \{N_1, \ldots, N_r\}$ are local lightlike

bases of $\Gamma(\mathrm{Rad}TM \mid_u)$, $\Gamma(\mathrm{ltr}(TM) \mid_u)$ and $\{W_{r+1}, \ldots, W_n\}, \{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp}) \mid_u)$ and $\Gamma(S(TM) \mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have:

Theorem 2.1 ([11]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $\mathrm{ltr}(TM)$ of $\mathrm{Rad}TM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(\mathrm{ltr}(TM) \mid_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp} \mid_u$, where u is a coordinate neighborhood of M such that*

$$(4) \qquad \bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0 \quad \text{for any} \ \ i, j \in \{1, 2, \ldots, r\},$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ then according to the decomposition (3), the Gauss and Weingarten formulas are given by

$$(5) \qquad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(\mathrm{tr}(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear a operator on M and known as shape operator.

According to (2) considering the projection morphisms L and S of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$ respectively, then (5) become

$$(6) \ \ \bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U,$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$(7) \qquad \begin{aligned} \bar{\nabla}_X N &= -A_N X + \nabla^l_X N + D^s(X, N), \\ \bar{\nabla}_X W &= -A_W X + \nabla^s_X W + D^l(X, W), \end{aligned}$$

where $X \in \Gamma(TM), N \in \Gamma(\mathrm{ltr}(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (6) and (7) we obtain

$$(8) \qquad \bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y)$$

for any $W \in \Gamma(S(TM^{\perp}))$. Let P be the projection morphism of TM on $S(TM)$ then using (1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$(9) \qquad \nabla_X PY = \nabla^*_X PY + h^*(X, PY), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}TM)$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X, PY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\mathrm{Rad}TM)$ respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $\mathrm{Rad}TM$

respectively. h^* and A^* are $\Gamma(\text{Rad}TM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and are called as second fundamental forms of distributions $S(TM)$ and $\text{Rad}TM$ respectively.

Using (6) and (9), we obtain

(10) $\bar{g}(h^l(X, PY), \xi) = g(A_\xi^* X, PY), \quad \bar{g}(h^*(X, PY), N) = \bar{g}(A_N X, PY)$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(\text{Rad}(TM))$ and $N \in \Gamma(\text{ltr}(TM))$.

In general, the induced connection ∇ on M is not a metric connection. Since $\bar{\nabla}$ is a metric connection, by using (6) we get

(11) $(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y).$

However, it is important to note that ∇^* is a metric connection on $S(TM)$.

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively then by straightforward calculations ([11]), we have

$$\bar{R}(X, Y)Z = R(X, Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y$$
$$- A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z)$$
$$+ D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) + (\nabla_X h^s)(Y, Z)$$
(12) $$- (\nabla_Y h^s)(X, Z) + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)).$$

Gray [15], defined nearly Kaehler manifolds as:

Definition 2.2. Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite nearly Kaehler manifold if

(13) $(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_Y \bar{J})X = 0, \quad \forall \quad X, Y \in \Gamma(T\bar{M}).$

It is well known that every Kaehler manifold is a nearly Kaehler manifold but converse is not true. S^6 with its canonical almost complex structure is a nearly Kaehler manifold but not a Kaehler manifold. Due to rich geometric and topological properties, the study of nearly Kaehler manifolds is as important as that of Kaehler manifolds. Therefore we study the geometry of CR, SCR and GCR-lightlike submanifolds of an indefinite nearly Kaehler manifold.

Nearly Kaehler manifold of constant holomorphic curvature c is denoted by $\bar{M}(c)$ and its curvature tensor field $\bar{R}$ is given by, [21]
(14)
$$\bar{R}(X, Y, Z, W) = \frac{c}{4}\{\bar{g}(X, W)\bar{g}(Y, Z) - \bar{g}(X, Z)\bar{g}(Y, W) + \bar{g}(X, \bar{J}W)\bar{g}(Y, \bar{J}Z)$$
$$- \bar{g}(X, \bar{J}Z)\bar{g}(Y, \bar{J}W) - 2\bar{g}(X, \bar{J}Y)\bar{g}(Z, \bar{J}W)\}$$
$$+ \frac{1}{4}\{\bar{g}((\bar{\nabla}_X \bar{J})(W), (\bar{\nabla}_Y \bar{J})(Z))$$
$$- \bar{g}((\bar{\nabla}_X \bar{J})(Z), (\bar{\nabla}_Y \bar{J})(W)) - 2\bar{g}((\bar{\nabla}_X \bar{J})(Y), (\bar{\nabla}_Z \bar{J})(W))\}$$

and the sectional curvature is given by

$$\bar{R}(X, Y, X, Y) = \frac{c}{4}\{\bar{g}(X, Y)^2 - \bar{g}(X, X)\bar{g}(Y, Y) - 3\bar{g}(X, \bar{J}Y)^2\}$$

$$(15) \qquad\qquad -\frac{3}{4}\|(\bar{\nabla}_X\bar{J})(Y)\|^2.$$

A nearly Kaehler manifold is said to be of constant type α [15], if there exists a real valued C^∞ function α on $\bar{M}$ such that

$$(16) \qquad \|(\bar{\nabla}_X\bar{J})(Y)\|^2 = \alpha\{\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(X,\bar{J}Y)^2\}.$$

3. Cauchy Riemann lightlike submanifolds

Definition 3.1. A submanifold $(M, g, S(TM))$ of an indefinite nearly Kaehler manifold $(\bar{M}, \bar{g}, \bar{J})$ is said to be a Cauchy-Riemann (CR)-lightlike submanifold if and only if following conditions are satisfied

  (A) $\bar{J}(\mathrm{Rad}(TM))$ is a distribution on M such that $\mathrm{Rad}(TM) \cap \bar{J}\mathrm{Rad}(TM) = \{0\}$.
  (B) There exist vector bundles $S(TM)$, $S(TM^\perp)$, $\mathrm{ltr}(TM)$, D_0 and D' over M such that

$$S(TM) = \{\bar{J}(\mathrm{Rad}(TM)) \oplus D'\}\perp D_0, \quad \bar{J}(D_0) = D_0, \quad \bar{J}(D') = L_1 \perp L_2,$$

where D_0 is a non degenerate distribution on M, L_1 and L_2 are vector bundles of $\mathrm{ltr}(TM)$ and $S(TM)^\perp$ respectively. Then the tangent bundle TM of M is decomposed as $TM = D \perp D'$, $D = \mathrm{Rad}(TM) \oplus D_0 \oplus \bar{J}\mathrm{Rad}(TM)$.

Example 1. Let M be a submanifold of (R_2^8, g) given by the equations $x_3 = x_8$ and $x_5 = \sqrt{1 - x_6^2}$, where g is of signature $(+, +, -, +, +, -, +, +)$ with respect to a basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8)$. Then the tangent bundle of M is spanned by

$$Z_1 = \partial x_1, \quad Z_2 = \partial x_2, \quad Z_3 = \partial x_3 + \partial x_8,$$
$$Z_4 = \partial x_4, \quad Z_5 = -x_6\partial x_5 + x_5\partial x_6, \quad Z_6 = \partial x_7.$$

Clearly M is a 1- lightlike submanifold with $\mathrm{Rad}(TM) = \mathrm{Span}\{Z_3\}$ and $\bar{J}Z_3 = Z_4 - Z_6 \in \Gamma(S(TM))$. Moreover, $\bar{J}Z_1 = Z_2$ and $\bar{J}Z_2 = -Z_1$ and therefore $D_0 = \mathrm{Span}\{Z_1, Z_2\}$. By direct calculations, we get $S(TM^\perp) = \mathrm{Span}\{W = x_5\partial x_5 + x_6\partial x_6\}$. Thus, $\bar{J}W = Z_5$ and hence $L_2 = S(TM^\perp)$. On the other hand, the lightlike transversal bundle is spanned by $N = \frac{1}{2}(-\partial x_3 + \partial x_8)$. Then $\bar{J}N = -\frac{1}{2}(\partial x_4 + \partial x_7) = -\frac{1}{2}(Z_4 + Z_6)$, hence $L_1 = \mathrm{Span}\{N\}$ and $D' = \{\bar{J}N, \bar{J}W\}$. Thus M is a proper CR-lightlike submanifold of R_2^8.

Theorem 3.2 (Existence Theorem). *A lightlike submanifold M of an indefinite nearly Kaehler manifold $\bar{M}(c)$ of constant type α and of constant holomorphic sectional curvature c such that $c = -3\alpha$, where $\alpha \neq 0$ is a CR-lightlike submanifold with $D_0 \neq 0$, if and only if*

  (i) *The maximal complex subspaces of $T_pM, p \in M$ define a distribution*

$$D = \mathrm{Rad}(TM) \oplus \bar{J}(\mathrm{Rad}(TM)) \oplus D_0,$$

  where D_0 is a non-degenerate complex distribution.

(ii) *There exists a lightlike transversal vector bundle* $\mathrm{ltr}(TM)$ *such that*

$$\bar{g}(\bar{R}(\xi, N)\xi, N) = 0, \quad \forall \xi \in \Gamma(\mathrm{Rad}(TM)), N \in \Gamma(\mathrm{ltr}(TM)).$$

(iii) *There exists a vector subbundle* M_2 *on* M *such that*

$$\bar{g}(\bar{R}(W, W')W, W') = 0 \quad \forall W, W' \in \Gamma(M_2),$$

where M_2 *is orthogonal to* D *and* $\bar{R}$ *be curvature tensor of* $\bar{M}(c)$.

Proof. Suppose M is a CR-lightlike submanifold of $\bar{M}(c)$ such that $c \neq 0$. Then $D = \mathrm{Rad}(TM) \oplus \bar{J}(\mathrm{Rad}(TM)) \oplus D_0$ is a maximal subspace. Thus (i) is satisfied. Using (14) and (16), we have $\bar{g}(\bar{R}(\xi, N)\xi, N) = 3\alpha\bar{g}(\xi, \bar{J}N)^2$ for $\xi \in \Gamma(\mathrm{Rad}(TM))$, $N \in \Gamma(\mathrm{ltr}(TM))$. Since by the definition of CR-lightlike submanifolds $\bar{g}(\xi, \bar{J}N) = 0$, hence (ii) holds. Similarly (14) and (16) imply $\bar{g}(\bar{R}(W, W')W, W') = 3\alpha\bar{g}(W, \bar{J}W')^2$ for $W, W' \in \Gamma(M_2)$, by definition of CR-lightlike submanifolds $\bar{g}(W, \bar{J}W') = 0$, hence (iii) holds.

Conversely from (i), we see that $\mathrm{Rad}(TM)$ is a distribution on M such that $\bar{J}(\mathrm{Rad}(TM)) \cap \mathrm{Rad}(TM) = \{0\}$. Thus condition (A) of the definition of CR-lightlike submanifold is satisfied. Therefore we can choose a screen distribution containing $\bar{J}\mathrm{Rad}(TM)$ and D_0 (since D_0 is non-degenerate). Since $\mathrm{ltr}(TM)$ is orthogonal to $S(TM)$ therefore $\bar{g}(\xi, \bar{J}N) = -\bar{g}(\bar{J}\xi, N) = 0$ for $\xi \in \Gamma(\mathrm{Rad}(TM))$. Hence we conclude that some part of $\mathrm{ltr}(TM)$ defines a distribution on M, say M_1. On the other hand, from (ii), we derive $3\alpha\bar{g}(\xi, \bar{J}N) = 0$ for $\xi \in \Gamma(\mathrm{Rad}(TM))$, $N \in \Gamma(\mathrm{ltr}(TM))$. Since $\alpha \neq 0$ therefore we have $\bar{g}(\xi, \bar{J}N) = 0$, that is, $\bar{J}\mathrm{ltr}(TM) \cap \mathrm{Rad}(TM) = \{0\}$. Moreover, since $\bar{g}(N, \xi) = 1$ for $\xi \in \Gamma(\mathrm{Rad}(TM))$ and $N \in \Gamma(\bar{J}M_1)$ therefore we have $\bar{g}(\bar{J}N, \bar{J}\xi) = 1$. This shows that M_1 is not orthogonal to $\bar{J}\mathrm{Rad}(TM)$ and hence not orthogonal to D. Now, consider a distribution M_2 which is orthogonal to D such that $M_2 \cap M_1 = \{0\}$ and orthogonal to M_1. Then from (iii), we have $3\alpha\bar{g}(W, \bar{J}W') = 0$ for all $W, W' \in \Gamma(M_2)$. Since $\alpha \neq 0$, we have $\bar{g}(W, \bar{J}W') = 0$, which implies $M_2 \perp \bar{J}M_2$. On the other hand, since M_2 is orthogonal to D, we obtain $g(\bar{J}W, X) = -g(W, \bar{J}X) = 0$ for all $X \in \Gamma(D)$ and $W \in \Gamma(M_2)$. Hence $\bar{J}M_2 \perp D$. Thus $\bar{J}M_2 \perp D$, $\bar{J}M_2 \perp M_1$ and $\bar{J}M_2 \perp M_2$ imply that $\bar{J}M_2 \subset S(TM^\perp)$, hence the proof. □

Remark 1. Let M be a complex lightlike submanifold, that is, $\bar{J}(TM) = TM$. Then easily we can show that $\bar{J}(\mathrm{Rad}(TM)) = \mathrm{Rad}(TM)$. Hence M is not a CR-lightlike submanifold. If M is totally real lightlike submanifold, that is, $\bar{J}(TM) \subset TM^\perp$, then from the condition that $\bar{J}(\mathrm{Rad}(TM))$ is a distribution on M we can derive $\bar{J}(\mathrm{Rad}(TM)) = \mathrm{Rad}(TM)$. Thus M is not a CR-lightlike submanifold. Thus to include complex and totally real submanifolds, we introduce a new class, called Screen Cauchy Riemann (SCR)-lightlike submanifolds of indefinite nearly Kaehler manifolds as below.

4. Screen Cauchy Riemann lightlike submanifolds

Definition 4.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite nearly Kaehler manifold $(\bar{M}, \bar{g}, \bar{J})$ then M is called a Screen Cauchy-Riemann (SCR)-lightlike submanifold if the following conditions are satisfied

(A) There exists a real non-null distribution $D \subseteq S(TM)$ such that

$$S(TM) = D \oplus D^\perp, \quad \bar{J}D^\perp \subset S(TM^\perp), \quad D \cap D^\perp = \{0\},$$

where $D^\perp$ is orthogonal complementary to D in $S(TM)$.

(B) $\mathrm{Rad}TM$ is invariant with respect to $\bar{J}$.

It follows that D and $\mathrm{ltr}(TM)$ are invariant with respect to $\bar{J}$, that is, $\bar{J}D = D$, $\bar{J}\mathrm{ltr}(TM) = \mathrm{ltr}(TM)$, $TM = D' \oplus D^\perp$ and $D' = D \perp \mathrm{Rad}(TM)$. Denote the orthogonal complement to $\bar{J}D^\perp$ in $S(TM^\perp)$ by μ. Then $\mathrm{tr}(TM) = \mathrm{ltr}(TM) \perp \bar{J}D^\perp \perp \mu$. We say that M is a proper SCR-lightlike submanifold of $\bar{M}$ if $D \neq \{0\}$ and $D^\perp \neq \{0\}$.

Example 2. Let M be a submanifold of R_2^8 given by the equations

$$x_1 = u_1 - u_2, \quad x_2 = u_1 + u_2, \quad x_3 = u_4, \quad x_4 = u_5,$$

$$x_5 = -u_2 - u_3, \quad x_6 = x_7 = u_1, \quad x_8 = u_2 - u_3,$$

where g is of signature $(-, -, +, +, +, +, +, +)$ with respect to a basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8)$. Then TM is spanned by $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$ where

$$Z_1 = \partial x_1 + \partial x_2 + \partial x_6 + \partial x_7, \quad Z_2 = -\partial x_1 + \partial x_2 - \partial x_5 + \partial x_8,$$

$$Z_3 = -\partial x_5 - \partial x_8, \quad Z_4 = \partial x_3, \quad Z_5 = \partial x_4.$$

Thus M is a 2-lightlike submanifold with $\mathrm{Rad}TM = \mathrm{Span}\{Z_1, Z_2\}$ such that $\bar{J}Z_1 = Z_2$, therefore $\mathrm{Rad}(TM)$ is invariant with respect to $\bar{J}$. Since $\bar{J}Z_4 = Z_5$ therefore $D = \mathrm{Span}\{Z_4, Z_5\}$ is also invariant with respect to $\bar{J}$. Moreover, $S(TM^\perp)$ is spanned by $W = -\partial x_6 + \partial x_7 = \bar{J}Z_3$. Hence a lightlike transversal vector bundle $\mathrm{ltr}(TM)$ is spanned by

$$N_1 = \frac{1}{4}(-\partial x_1 - \partial x_2 + \partial x_6 + \partial x_7), \quad N_2 = \frac{1}{4}(\partial x_1 - \partial x_2 - \partial x_5 + \partial x_8),$$

which is invariant with respect to $\bar{J}$. Thus M is a proper SCR-lightlike submanifold of R_2^8, with $D' = \mathrm{Span}\{Z_1, Z_2, Z_4, Z_5\}$ and $D^\perp = \mathrm{Span}\{Z_3\}$.

Theorem 4.2 (Existence Theorem)**.** *Let M be a lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}(c)$ of constant type α and of constant holomorphic sectional curvature c such that $c = -3\alpha$, where $\alpha \neq 0$. Then M is a SCR-lightlike submanifold with $D \neq 0$, if and only if*

(i) *The maximal complex subspaces of $T_pM, p \in M$ define a distribution $D' = \mathrm{Rad}(TM) \perp D$, where D is an almost complex distribution.*

(ii) *$\bar{g}(\bar{R}(W, W')W, W') = 0$ for all $W, W' \in \Gamma(D^\perp)$, where $D^\perp$ is an orthogonal complementary distribution to D in $S(TM)$ and $\bar{R}$ be curvature tensor of $\bar{M}(c)$.*

Proof. Let M be a SCR-lightlike submanifold of an indefinite nearly Kaehler manifold such that $c = -3\alpha$. Then $D' = \text{Rad}TM \perp D$ is a maximal subspace. Using (14) and (16), we have $\bar{g}(\bar{R}(W, W')W, W') = 3\alpha\bar{g}(W, \bar{J}W')^2$ for $W, W' \in \Gamma(D^\perp)$, since $\bar{J}D^\perp \subset S(TM^\perp)$, therefore $\bar{g}(W, \bar{J}W') = 0$ and hence $\bar{g}(\bar{R}(W, W')W, W') = 0$.

Conversely, since D is an almost complex distribution, therefore $\text{Rad}TM \cap D = \{0\}$ and D' is invariant implies $\text{Rad}TM$ is invariant with respect to $\bar{J}$. Using (14) and (16) with (ii), we obtain $\bar{g}(W, \bar{J}W') = 0$, for all $W, W' \in \Gamma(D^\perp)$, that is, $D^\perp \perp \bar{J}D^\perp$. On the other hand, since $D^\perp$ is orthogonal to D', we obtain $\bar{g}(\bar{J}W, X) = -g(W, \bar{J}X) = 0$ for all $X \in \Gamma(D)$ and $W \in \Gamma(D^\perp)$, this implies $\bar{J}D^\perp \perp D'$. Moreover, since $\text{ltr}(TM)$ is invariant with respect to $\bar{J}$, we obtain $\bar{J}D^\perp \in \Gamma(S(TM^\perp))$. $\qquad\square$

Remark 2. From the definition of SCR-lightlike submanifolds, it is clear that SCR-lightlike submanifolds include complex and totally real submanifolds as subcases. But there does not exist any inclusion relation between SCR-lightlike submanifolds and CR-lightlike submanifolds. Thus we need new class of submanifolds which acts as an umbrella over the CR and SCR-lightlike submanifolds.

5. Generalized Cauchy-Riemann lightlike submanifolds

Definition 5.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite nearly Kaehler manifold $(\bar{M}, \bar{g}, \bar{J})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $\text{Rad}(TM)$ such that

$$\text{Rad}(TM) = D_1 \oplus D_2, \quad \bar{J}(D_1) = D_1, \quad \bar{J}(D_2) \subset S(TM).$$

(B) There exist two subbundles D_0 and D' of $S(TM)$ such that

$$S(TM) = \{\bar{J}D_2 \oplus D'\} \perp D_0, \quad \bar{J}(D_0) = D_0, \quad \bar{J}(D') = L_1 \perp L_2,$$

where D_0 is a non degenerate distribution on M, L_1 and L_2 are vector subbundles of $\text{ltr}(TM)$ and $S(TM)^\perp$ respectively.

Then the tangent bundle TM of M is decomposed as $TM = D \perp D'$ and $D = \text{Rad}(TM) \oplus D_0 \oplus \bar{J}D_2$. M is called a proper GCR-lightlike submanifold if $D_1 \neq \{0\}, D_2 \neq \{0\}, D_0 \neq \{0\}$ and $L_2 \neq \{0\}$, which has the following features

1. The condition (A) implies that $\dim(\text{Rad}(TM)) \geq 3$.
2. The condition (B) implies that $\dim(D) = 2s \geq 6$, $\dim(D') \geq 2$ and $\dim(D_2) = \dim(L_1)$. Thus $\dim(M) \geq 8$ and $\dim(\bar{M}) \geq 12$.
3. Any proper 8-dimensional GCR-lightlike submanifold is 3-lightlike.

Proposition 5.2. *A GCR-lightlike submanifold M of an indefinite nearly Kaehler manifold $\bar{M}$, is a CR-(respectively SCR-) lightlike submanifold if and only if $D_1 = \{0\}$ (respectively, $D_2 = \{0\}$).*

Proof. Let M be a CR-lightlike submanifold of $\bar{M}$. Then $\bar{J}\mathrm{Rad}(TM)$ is a distribution on M such that $\bar{J}\mathrm{Rad}(TM) \cap \mathrm{Rad}(TM) = \{0\}$. Hence $D_2 = \mathrm{Rad}(TM)$ and $D_1 = \{0\}$. Conversely, assume that M is a GCR-lightlike submanifold such that $D_1 = \{0\}$. Then $D_2 = \mathrm{Rad}(TM)$ and hence $\bar{J}\mathrm{Rad}(TM) \cap \mathrm{Rad}(TM) = \{0\}$, that is, $\bar{J}\mathrm{Rad}(TM)$ is a vector subbundle of $S(TM)$. Thus M is a CR-lightlike submanifold. Similarly, we can prove the other assertion. $\square$

Remark 3. Since any lightlike real hypersurface of an indefinite Hermitian manifold is a CR-lightlike submanifold, [11]. Also invariant and screen real lightlike submanifolds of $\bar{M}$ are sub cases of SCR-lightlike submanifolds, [13]. Thus using above proposition, we conclude that class of GCR-lightlike submanifolds is an **umbrella** of real hypersurfaces, invariant, screen real, CR and SCR-lightlike submanifolds.

Example 3. Let M be a submanifold of R_4^{14} given by the equations

$$x_1 = x_{14}, \quad x_2 = -x_{13}, \quad x_3 = x_{12}, \quad x_7 = \sqrt{1 - x_8^2}.$$

Then TM is spanned by $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}$, where

$$Z_1 = \partial x_1 + \partial x_{14}, \quad Z_2 = \partial x_2 - \partial x_{13}, \quad Z_3 = \partial x_3 + \partial x_{12},$$

$$Z_4 = \partial x_4, \quad Z_5 = \partial x_5, \quad Z_6 = \partial x_6, \quad Z_7 = -x_8 \partial x_7 + x_7 \partial x_8,$$

$$Z_8 = \partial x_9, \quad Z_9 = \partial x_{10}, \quad Z_{10} = \partial x_{11}.$$

Clearly M is 3-lightlike with $\mathrm{Rad}(TM) = \mathrm{Span}\{Z_1, Z_2, Z_3\}$ and $\bar{J}Z_1 = Z_2$. Therefore $D_1 = \mathrm{Span}\{Z_1, Z_2\}$. On the other hand, $\bar{J}Z_3 = Z_4 - Z_{10} \in \Gamma(S(TM))$ implies that $D_2 = \mathrm{Span}\{Z_3\}$. Since $\bar{J}Z_5 = Z_6$ and $\bar{J}Z_8 = Z_9$ therefore $D_0 = \mathrm{Span}\{Z_5, Z_6, Z_8, Z_9\}$. By direct calculations, $S(TM^\perp) = \mathrm{Span}\{W = x_7 \partial x_7 + x_8 \partial x_8\}$. Thus $\bar{J}Z_7 = -W$. Hence $L_2 = S(TM^\perp)$. On the other hand, the lightlike transversal bundle $\mathrm{ltr}(TM)$ is spanned by

$$\{N_1 = \frac{1}{2}(-\partial x_1 + \partial x_{14}), N_2 = \frac{1}{2}(-\partial x_2 - \partial x_{13}), N_3 = \frac{1}{2}(-\partial x_3 + \partial x_{12})\},$$

therefore $\mathrm{Span}\{N_1, N_2\}$ is invariant with respect to $\bar{J}$ and $\bar{J}N_3 = -\frac{1}{2}Z_4 - \frac{1}{2}Z_{10}$. Hence $L_1 = \mathrm{Span}\{N_3\}$ and $D' = \mathrm{Span}\{\bar{J}N_3, \bar{J}W\}$. Thus M is a proper GCR-lightlike submanifold of R_4^{14}.

Let Q, P_1 and P_2 be the projections on $D, \bar{J}(L_1) = M_1$ and $\bar{J}(L_2) = M_2$, respectively. Then for any $X \in \Gamma(TM)$ we have

(17) $$X = QX + P_1 X + P_2 X,$$

applying $\bar{J}$ to (17), we obtain

(18) $$\bar{J}X = TX + wP_1 X + wP_2 X,$$

and we can write the equation (18) as

(19) $$\bar{J}X = TX + wX,$$

where TX and wX are the tangential and transversal components of $\bar{J}X$ respectively. Similarly

$$(20) \qquad \bar{J}V = BV + CV$$

for any $V \in \Gamma(\text{tr}(TM))$, where BV and CV are the sections of TM and $\text{tr}(TM)$ respectively. Applying $\bar{J}$ to (19) and (20), we get $T^2 = -I - B\omega$ and $C^2 = -I - \omega B$. Differentiating (18) and using (6), (7) and (20), we have

$$(21) \qquad \begin{aligned} (\nabla_X T)Y + (\nabla_Y T)X &= A_{wP_1X}Y + A_{wP_2X}Y + A_{wP_1Y}X \\ &\quad + A_{wP_2Y}X + 2Bh(X,Y). \end{aligned}$$

$$(22) \qquad \begin{aligned} D^s(X, wP_1Y) + D^s(Y, wP_1X) = &-\nabla^s_X wP_2Y - \nabla^s_Y wP_2X \\ &+ wP_2\nabla_X Y + wP_2\nabla_Y X \\ &- h^s(X,TY) - h^s(TX,Y) \\ &+ 2Ch^s(X,Y). \end{aligned}$$

$$(23) \qquad \begin{aligned} D^l(X, wP_2Y) + D^l(Y, wP_2X) = &-\nabla^l_X wP_1Y - \nabla^l_Y wP_1X \\ &+ wP_1\nabla_X Y + wP_1\nabla_Y X \\ &- h^l(X,TY) - h^l(TX,Y) \\ &+ 2Ch^l(X,Y). \end{aligned}$$

Using nearly Kaehlerian property of $\bar{\nabla}$ with (7), we have the following lemma.

Lemma 5.3. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then we have*

$$(24) \qquad (\nabla_X T)Y + (\nabla_Y T)X = A_{wY}X + A_{wX}Y + 2Bh(X,Y),$$

and

$$(25) \qquad (\nabla^t_X w)Y + (\nabla^t_Y w)X = 2Ch(X,Y) - h(X,TY) - h(TX,Y)$$

for any $X, Y \in \Gamma(TM)$, where

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (\nabla^t_X w)Y = \nabla^t_X wY - w\nabla_X Y.$$

Theorem 5.4. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then the induced connection is a metric connection if and only if the following hold*

$$A^*_{\bar{J}Y}X - \nabla_{\bar{J}Y}X - \nabla^{*t}_X \bar{J}Y \in \Gamma(\bar{J}D_2 \bot D_1), \text{ when } Y \in \Gamma(D_1),$$

$$\nabla^*_{\bar{J}Y}X + \nabla_{\bar{J}Y}X + h^*(X, \bar{J}Y) \in \Gamma(\bar{J}D_2 \bot D_1), \text{ when } Y \in \Gamma(D_2),$$

$$\nabla_{\bar{J}Y}TX - A_{wX}\bar{J}Y \in \Gamma(\text{Rad}(TM)), \text{ and } Bh(X, \bar{J}Y) = 0, \text{ when } Y \in \Gamma(\text{Rad}(TM)).$$

Proof. Since $\bar{J}$ is the almost complex structure of $\bar{M}$ therefore we have $\bar{\nabla}_X Y = -\bar{\nabla}_X \bar{J}^2 Y$ for any $Y \in \Gamma(\text{Rad}(TM))$ and $X \in \Gamma(TM)$. Then from (13), we obtain $\bar{\nabla}_X Y = -\bar{J}\bar{\nabla}_X \bar{J}Y + \bar{\nabla}_{\bar{J}Y}\bar{J}X - \bar{J}\bar{\nabla}_{\bar{J}Y}X$ and using (6) and (19), we get

$$\nabla_X Y + h(X, Y) = -\bar{J}(\nabla_X \bar{J}Y + \nabla_{\bar{J}Y} X) + \nabla_{\bar{J}Y} TX + h(TX, \bar{J}Y)$$
$$- A_{wX}\bar{J}Y + \nabla^t_{\bar{J}Y} wX - 2Bh(X, \bar{J}Y) - 2Ch(X, \bar{J}Y).$$

Since $\text{Rad}(TM) = D_1 \oplus D_2$ therefore using (9), (19) and (20) and then equating the tangential part for any $Y \in \Gamma(D_1)$, we obtain

$$(26) \quad \nabla_X Y = T(A^*_{\bar{J}Y} X - \nabla^{*t}_X \bar{J}Y + \nabla_{\bar{J}Y} X) + \nabla_{\bar{J}Y} TX - A_{wX}\bar{J}Y - 2Bh(X, \bar{J}Y),$$

and for any $Y \in \Gamma(D_2)$, we obtain

$$(27)$$
$$\nabla_X Y = -T(\nabla^*_X \bar{J}Y + h^*(X, \bar{J}Y) + \nabla_{\bar{J}Y} X) + \nabla_{\bar{J}Y} TX - A_{wX}\bar{J}Y - 2Bh(X, \bar{J}Y).$$

Thus from (26), $\nabla_X Y \in \Gamma(\text{Rad}(TM))$, if and only if

$$T(A^*_{\bar{J}Y} X - \nabla^{*t}_X \bar{J}Y - \nabla_{\bar{J}Y} X) \in \Gamma(\bar{J}D_2 \perp D_1),$$
$$(28) \quad \nabla_{\bar{J}Y} TX - A_{wX}\bar{J}Y \in \Gamma(\text{Rad}(TM)), \quad Bh(X, \bar{J}Y) = 0.$$

From (27), $\nabla_X Y \in \Gamma(\text{Rad}(TM))$, if and only if

$$T(\nabla^*_X \bar{J}Y + h^*(X, \bar{J}Y) + \nabla_{\bar{J}Y} X) \in \Gamma(\bar{J}D_2 \perp D_1),$$
$$(29) \quad \nabla_{\bar{J}Y} TX - A_{wX}\bar{J}Y \in \Gamma(\text{Rad}(TM)), \quad Bh(X, \bar{J}Y) = 0.$$

Thus the assertion follows from (28) and (29). □

Theorem 5.5 (Existence Theorem)**.** *A lightlike submanifold M of an indefinite nearly Kaehler manifold $\bar{M}(c)$ of constant type α and of constant holomorphic sectional curvature c such that $c = -3\alpha$, where $\alpha \neq 0$ is a GCR-lightlike submanifold with $D_0 \neq 0$, if and only if*

(i) *The maximal complex subspaces of $T_p M, p \in M$ define a distribution $D = D_1 \perp D_2 \perp \bar{J}D_2 \perp D_0$ where $\text{Rad}(TM) = D_1 \oplus D_2$, D_0 is a non degenerate complex distribution.*

(ii) *There exists a lightlike transversal vector bundle $\text{ltr}(TM)$ such that $\bar{g}(\bar{R}(\xi, N)\xi, N) = 0$ for any $\xi \in \Gamma(\text{Rad}(TM))$ and $N \in \Gamma(L_1)$.*

(iii) *There exists a vector subbundle M_2 on M such that $\bar{g}(\bar{R}(W, W')W, W') = 0$ for any $W, W' \in \Gamma(M_2)$, where M_2 is orthogonal to D and $\bar{R}$ be curvature tensor of $\bar{M}(c)$.*

Proof. Suppose that M is a *GCR*-lightlike submanifold of $\bar{M}(c)$ such that $c = -3\alpha$ and $c \neq 0$. Then by the definition of *GCR*-lightlike submanifolds, $D = D_1 \perp D_2 \perp \bar{J}D_2 \perp D_0$ is a maximal subspace. Next from (14) and (16), we have

$$(30) \qquad\qquad \bar{g}(\bar{R}(\xi, N)\xi, N) = 3\alpha\bar{g}(\xi, \bar{J}N)^2$$

for any $\xi \in \Gamma(\text{Rad}(TM))$ and $N \in \Gamma(L_1)$. Since $\bar{g}(\xi, \bar{J}N) = 0$, by the definition of *GCR*-lightlike submanifolds therefore we have $\bar{g}(\bar{R}(\xi, N)\xi, N) = 0$. Similarly, from (14) and (16) we have $\bar{g}(\bar{R}(W, W')W, W') = 3\alpha\bar{g}(W, \bar{J}W')^2 = 0$ for $W, W' \in \Gamma(M_2)$, which proves (iii).

Conversely, from (i) it is clear that a part D_2 of $\mathrm{Rad}(TM)$ is a distribution on M such that $\bar{J}D_2 \cap \mathrm{Rad}(TM) = \{0\}$, this implies that other part of $\mathrm{Rad}(TM)$ is invariant. Thus (A) of the definition of GCR-lightlike submanifold is satisfied. Therefore we can choose a screen distribution containing $\bar{J}D_2$ and D_0. Since $\mathrm{ltr}(TM)$ is orthogonal to $S(TM)$ this implies that $\bar{g}(\xi, \bar{J}N) = -\bar{g}(\bar{J}\xi, N) = 0$ for $\xi \in \Gamma(D_2)$. Hence we conclude that some part of $\bar{J}\mathrm{ltr}(TM)$ defines a distribution on M, say M_1. Also from (ii) and (30), we have $3\alpha\bar{g}(\xi, \bar{J}N)^2 = 0$ for $\xi \in \Gamma(\mathrm{Rad}(TM)), N \in \Gamma(\mathrm{ltr}(TM))$. Since $\alpha \neq 0$ we conclude that $\bar{g}(\xi, JN) = 0$ that is $\bar{J}\mathrm{ltr}(TM) \cap \mathrm{Rad}(TM) = \{0\}$. Moreover, $\bar{g}(N, \xi) = 1$ for $\xi \in \Gamma(D_2) \subset \Gamma(\mathrm{Rad}(TM))$ and $N \in \Gamma(\bar{J}M_1) \subset \Gamma(\mathrm{ltr}(TM))$ implies that $\bar{g}(\bar{J}N, \bar{J}\xi) = 1$, this shows that M_1 is not orthogonal to D_2 and hence not orthogonal to D. Now, consider a distribution M_2 which is orthogonal to D then $M_2 \cap M_1 = \{0\}$ and orthogonal to M_1. From (iii) we have $\bar{g}(W, \bar{J}W') = 0$ for $W, W' \in \Gamma(M_2)$, this implies that $M_2 \perp \bar{J}M_2$. Since M_2 is orthogonal to D, we obtain $g(\bar{J}W, X) = -g(W, \bar{J}X) = 0$ for $X \in \Gamma(D)$ and $W \in \Gamma(M_2)$, hence $\bar{J}M_2 \perp D$. Thus $\bar{J}M_2 \perp D$, $\bar{J}M_2 \perp M_1$ and $\bar{J}M_2 \perp M_2$ imply that $\bar{J}M_2 \subset S(TM^\perp)$, this completes the proof. $\qquad\square$

Lemma 5.6 ([21]). *If $\bar{M}$ is a nearly Kaehler manifold, then*

$$(31) \qquad (\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_{\bar{J}X}\bar{J})\bar{J}Y = 0, \quad N(X,Y) = -4\bar{J}((\bar{\nabla}_X\bar{J})(Y))$$

for any $X, Y \in \Gamma(T(\bar{M}))$, where $N(X,Y)$ is the Nijenhuis tensor and given by

$$(32) \qquad N(X,Y) = [\bar{J}X, \bar{J}Y] - \bar{J}[X, \bar{J}Y] - \bar{J}[\bar{J}X, Y] - [X, Y].$$

Theorem 5.7. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. If D is integrable, then $h(X, \bar{J}Y) = h(\bar{J}X, Y)$ for $X, Y \in \Gamma(D)$.*

Proof. Let $X, Y \in \Gamma(D)$. Then using (5) and (31), we obtain

$$\bar{J}N(X,Y) = 2(\nabla_X \bar{J}Y - \nabla_Y \bar{J}X) + 2(h(X, \bar{J}Y) - h(\bar{J}X, Y)) - 2\bar{J}[X, Y].$$

Since D is integrable then using (32), it follows that $\bar{J}N(X,Y) \in \Gamma(D)$ and $\bar{J}[X, Y] \in \Gamma(D)$. Hence by equating the transversal components, the result follows. $\qquad\square$

Theorem 5.8. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$ and D-defines a totally geodesic foliation in M. Then*

$$(33) \qquad h(X, \bar{J}Y) = h(\bar{J}X, Y) = \bar{J}h(X, Y), \forall X, Y \in \Gamma(D).$$

Proof. Assume D defines a totally geodesic foliation in M then clearly D is integrable. Therefore the first equality of (33) follows from Theorem 5.7. Let $X, Y \in \Gamma(D)$. Then using the hypothesis that D defines totally geodesic foliation in M with (22) and (23), we have $h(X, \bar{J}Y) = Ch(X, Y)$, using $\bar{J}h(X, Y) = Bh(X, Y) + Ch(X, Y)$, we get $h(\bar{J}X, Y) = \bar{J}h(X, Y) - Bh(X, Y)$. Since $X, Y \in \Gamma(D)$ therefore from (21), we have $(\nabla_X T)Y + (\nabla_Y T)X =$

$Bh(X, Y)$. Using D defines a totally geodesic foliation in M with (13), we obtain $Bh(X, Y) = 0$ and hence $h(\bar{J}X, Y) = \bar{J}h(X, Y)$. $\square$

Definition 5.9. A GCR-lightlike submanifold of an indefinite nearly Kaehler manifold is called D geodesic (respectively, mixed geodesic) GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$ (respectively, $X \in \Gamma(D)$ and $Y \in \Gamma(D')$).

Theorem 5.10. *Let M be a GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. If D defines a totally geodesic foliation in $\bar{M}$, then M is D geodesic.*

Proof. Let D defines a totally geodesic foliation in $\bar{M}$. Then $\bar{\nabla}_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Then using (6) for any $\xi \in \Gamma(\mathrm{Rad}(TM))$ and $W \in \Gamma(S(TM^\perp))$, we obtain

$$\bar{g}(h^l(X, Y), \xi) = \bar{g}(\bar{\nabla}_X Y, \xi) = 0, \quad \bar{g}(h^s(X, Y), W) = \bar{g}(\bar{\nabla}_X Y, W) = 0,$$

Hence $h^l(X, Y) = h^s(X, Y) = 0$ and the assertion follows. $\square$

Theorem 5.11. *Let M be a mixed geodesic proper GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}(c)$ of constant type α with constant holomorphic sectional curvature c. If the distribution D_0 defines a totally geodesic foliation in $\bar{M}$, then it is necessary that $c = \alpha$.*

Proof. From (14) and Lemma 5.6 together with the fact that $(\bar{\nabla}_X \bar{J})(\bar{J}Z) = -\bar{J}(\bar{\nabla}_X \bar{J})(Z)$, we obtain

$$(34) \qquad \bar{g}(\bar{R}(X, \bar{J}X)Z, \bar{J}Z) = -\frac{c}{2}g(X, X)g(Z, Z) + \frac{1}{2}\|(\bar{\nabla}_X \bar{J})(Z)\|^2$$

for any $X \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}L_2)$. On the other hand, using M is mixed geodesic and (12), we get

$$(35) \qquad \bar{g}(\bar{R}(X, \bar{J}X)Z, \bar{J}Z) = \bar{g}((\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z), \bar{J}Z)$$

for $X \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}L_2)$. Using the mixed geodesic property of M along with the hypothesis, we obtain

$$(\nabla_X h^s)(\bar{J}X, Z) = -h^s(\nabla_X \bar{J}X, Z) - h^s(\bar{J}X, \nabla_X Z),$$

and

$$(\nabla_{\bar{J}X} h^s)(X, Z) = -h^s(\nabla_{\bar{J}X} X, Z) - h^s(X, \nabla_{\bar{J}X} Z).$$

Hence

$$(\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z)$$
$$= h^s([\bar{J}X, X], Z) - h^s(\bar{J}X, \nabla_X Z) + h^s(X, \nabla_{\bar{J}X} Z).$$

Let $X, Y \in \Gamma(D_0)$ and $Z \in \Gamma(M_2)$ then using that D_0 defines a totally geodesic foliation in $\bar{M}$, we obtain $g(T\nabla_X Z, Y) = -g(\nabla_X Z, TY) = -g(\bar{\nabla}_X Z, TY) =$

$g(Z, \bar{\nabla}_X TY) = 0$. Hence from the non-degeneracy of D_0, we obtain $\nabla_X Z \in \Gamma(D')$. Thus from above equation, we have

$$(\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z) = 0.$$

Thus from (34) and (35), we have

(36) $$cg(X, X)g(Z, Z) = \|(\bar{\nabla}_X \bar{J})(Z)\|^2.$$

Since $\bar{M}$ is of constant type α therefore from (16) and (36), we obtain

$$cg(X, X)g(Z, Z) = \alpha\{g(X, X)g(Z, Z) - g(X, Z)^2 - g(X, \bar{J}Z)^2\}.$$

Since $X \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}L_2)$ therefore we get $(c - \alpha)g(X, X)g(Z, Z) = 0$, then non-degeneracy of D_0 and $\bar{J}L_2$ gives the result. $\qquad \square$

6. Totally umbilical GCR-lightlike submanifolds

Definition 6.1 ([12])**.** A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(\mathrm{tr}(TM))$ on M, called the transversal curvature vector field of M, such that, for $X, Y \in \Gamma(TM)$

(37) $$h(X, Y) = H\bar{g}(X, Y).$$

Using (7), it is clear that M is a totally umbilical, if and only if, on each coordinate neighborhood u there exist smooth vector fields $H^l \in \Gamma(\mathrm{ltr}(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ such that

(38) $$h^l(X, Y) = H^l g(X, Y), \quad h^s(X, Y) = H^s g(X, Y), \quad D^l(X, W) = 0$$

for $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$. M is called totally geodesic if $H = 0$, that is, if $h(X, Y) = 0$.

Theorem 6.2. *Let M be a totally umbilical proper GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. If D_0 defines a totally geodesic foliation in M, then the induced connection ∇ is a metric connection. Moreover, $h^s = 0$.*

Proof. Let $X, Y \in \Gamma(D_0)$. Then using (23), we obtain $wP_1\nabla_X Y + wP_1\nabla_Y X = h^l(X, \bar{J}Y) + h^l(\bar{J}X, Y) - 2Ch^l(X, Y)$. Using D_0 defines a totally geodesic foliation in M and Theorem 5.8, we obtain $h^l(X, \bar{J}Y) = Ch^l(X, Y)$. Since M is a totally umbilical GCR-lightlike submanifold therefore we have $H^l g(X, \bar{J}Y) = CH^l g(X, Y)$. For $X = \bar{J}Y$ and using the non-degeneracy of D_0, we have $H^l = 0$. Thus from (38), we have $h^l = 0$. Hence from (11), the induced connection ∇ is a metric connection. Similarly using (22), we can prove that $h^s = 0$. $\qquad \square$

Lemma 6.3. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then $\nabla_X X \in \Gamma(D)$ for any $X \in \Gamma(D)$.*

Proof. Since $D' = \bar{J}(L_1 \perp L_2)$, therefore $\nabla_X X \in \Gamma(D)$, if and only if, $g(\nabla_X X, \bar{J}\xi) = 0$ and $g(\nabla_X X, \bar{J}W) = 0$, for any $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$, respectively. Using M is a totally umbilical GCR-lightlike submanifold, we obtain

$$g(\nabla_X X, \bar{J}\xi) = -\bar{g}(\bar{\nabla}_X \bar{J}X, \xi)$$
$$= -\bar{g}(h^l(X, \bar{J}X), \xi)$$
$$= -\bar{g}(H^l, \xi)g(X, \bar{J}X) = 0,$$
$$g(\nabla_X X, \bar{J}W) = -\bar{g}(\bar{\nabla}_X \bar{J}X, W)$$
$$= -\bar{g}(h^s(X, \bar{J}X), W)$$
$$= -\bar{g}(H^s, W)g(X, \bar{J}X) = 0.$$

Hence the result follows. $\qquad\square$

Theorem 6.4. *Let M be a totally umbilical proper GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then one of the following holds*

(a) *M is totally geodesic, if D_0 defines a totally geodesic foliation in M.*
(b) *$h^s = 0$ or $dim(L_2) = 1$, if D_0 does not define a totally geodesic foliation in M.*

Proof. Let D_0 defines a totally geodesic foliation in M. Then from Theorem 6.2, we obtain that $h^l = h^s = 0$, thus (a) follows. Now suppose D_0 does not define totally geodesic foliation in M then using (6), (7), (19), (20) and then taking tangential part, we have

$$-A_{\bar{J}W}Z - A_{\bar{J}Z}W = T\nabla_Z W + T\nabla_W Z + 2Bh(Z,W)$$

for any $Z, W \in \Gamma(\bar{J}L_2)$. Taking inner product with Z and hence using (8) and (20), we get

$$\bar{g}(h^s(Z,Z), \bar{J}W) = \bar{g}(h^s(Z,W), \bar{J}Z).$$

Since M is totally umbilical, therefore we have

$$(39) \qquad \bar{g}(H^s, \bar{J}W)g(Z,Z) = \bar{g}(H^s, \bar{J}Z)g(Z,W).$$

Interchanging the role of Z and W in above equation, we obtain

$$(40) \qquad \bar{g}(H^s, \bar{J}Z)g(W,W) = \bar{g}(H^s, \bar{J}W)g(Z,W).$$

Thus from (39) and (40), we obtain

$$(41) \qquad \bar{g}(H^s, \bar{J}Z) = \frac{g(Z,W)^2}{g(Z,Z)g(W,W)}\bar{g}(H^s, \bar{J}Z).$$

Let $X \in \Gamma(D_0)$. Then using (22) with Lemma 6.3, we obtain $h^s(X, \bar{J}X) = Ch^s(X,X)$. Since M is totally umbilical therefore we get $g(X,X)CH^s = 0$, then non-degeneracy of D_0 implies that $CH^s = 0$, that is, $H^s \in \Gamma(L_2)$. Since $\bar{J}L_2$ is also non-degenerate, thus choosing non-null vector fields Z and W in (41), we conclude that either $H^s = 0$ or Z and W are linearly dependent, which proves (b). $\qquad\square$

Theorem 6.5. *There exist no totally umbilical proper GCR-lightlike subman-ifold of an indefinite nearly Kaehler manifold $\bar{M}(c)$ of constant type α with constant holomorphic sectional curvature c, such that $c \neq \alpha$.*

Proof. Let M be a totally umbilical GCR-lightlike submanifold of $\bar{M}(c)$ such that $c \neq \alpha$. Then from (14), we obtain

$$(42) \qquad \bar{g}(\bar{R}(X, \bar{J}X)Z, \bar{J}Z) = -\frac{c}{2}g(X, X)g(Z, Z) + \frac{1}{2}\|(\bar{\nabla}_X \bar{J})(Z)\|^2$$

for any $X \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}L_2)$. On the other hand, from (12) and (38) we have

$$(43) \qquad \bar{g}(\bar{R}(X, \bar{J}X)Z, \bar{J}Z) = \bar{g}((\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z), \bar{J}Z)$$

for any $X \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}L_2)$. Now from (42) and (43), we obtain

$$(44) \qquad \begin{aligned} & -\frac{c}{2}g(X, X)g(Z, Z) + \frac{1}{2}\|(\bar{\nabla}_X \bar{J})(Z)\|^2 \\ & = \bar{g}((\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z), \bar{J}Z). \end{aligned}$$

Since M is totally umbilical therefore using (38), we have

$$(\nabla_X h^s)(\bar{J}X, Z) = -g(\nabla_X \bar{J}X, Z)H^s - g(\bar{J}X, \nabla_X Z)H^s.$$

Since $\bar{g}(\bar{J}X, Z) = 0$ for any $X \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}L_2)$, differentiating this with respect to X, we get $g(\nabla_X \bar{J}X, Z) = -g(\bar{J}X, \nabla_X Z)$, therefore

$$(\nabla_X h^s)(\bar{J}X, Z) = 0.$$

Similarly $(\nabla_{\bar{J}X} h^s)(X, Z) = 0$. Hence (44) becomes

$$\frac{c}{2}g(X, X)g(Z, Z) = \frac{1}{2}\|(\bar{\nabla}_X \bar{J})(Z)\|^2.$$

Since $\bar{M}$ is of constant type α, therefore using (16), we obtain

$$cg(X, X)g(Z, Z) = \alpha\{g(X, X)g(Z, Z) - g(X, Z)^2 - g(X, \bar{J}Z)^2\}.$$

Since $X \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}L_2)$ therefore we have $(c - \alpha)g(X, X)g(Z, Z) = 0$, then using non-degeneracy of D_0 and $\bar{J}L_2$, we obtain $c = \alpha$. Hence this contradiction completes the proof. $\qquad \square$

7. Minimal GCR-lightlike submanifolds

Definition 7.1 ([1])**.** A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be minimal if

(i) $h^s = 0$ on $\text{Rad}(TM)$ and
(ii) trace $h = 0$, where trace is written with respect to g restricted to $S(TM)$.

As in the semi-Riemannian case, any lightlike totally geodesic M is minimal. Therefore from Theorem 6.4, a totally umbilical proper GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$ with D_0 defines a totally geodesic foliation in M is minimal.

Theorem 7.2. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then M is minimal if and only if M is totally geodesic.*

Proof. Suppose M is minimal then $h^s(X,Y) = 0$ for any $X, Y \in \Gamma(\mathrm{Rad}(TM))$. Since M is totally umbilical therefore $h^l(X,Y) = H^l g(X,Y) = 0$ for any $X, Y \in \Gamma(\mathrm{Rad}(TM))$. Now, choose an orthonormal basis $\{e_1, e_2, \ldots, e_{m-r}\}$ of $S(TM)$ then from (38), we obtain

$$\mathrm{trace}h(e_i, e_i) = \sum_{i=1}^{m-r} \epsilon_i g(e_i, e_i) H^l + \epsilon_i g(e_i, e_i) H^s = (m-r)H^l + (m-r)H^s.$$

Since M is minimal and $\mathrm{ltr}(TM) \cap S(TM^\perp) = \{0\}$, we get $H^l = 0$ and $H^s = 0$. Hence M is totally geodesic. Converse follows directly. $\square$

Theorem 7.3. *A totally umbilical proper GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$ is minimal if and only if*

$$\mathrm{trace}A_{W_p} = 0 \quad and \quad \mathrm{trace}A^*_{\xi_k} = 0 \quad on \quad D_0 \perp \bar{J}L_2$$

for $W_p \in \Gamma(S(TM^\perp))$, where $k \in \{1, 2, \ldots, r\}$ and $p \in \{1, 2, \ldots, n-r\}$.

Proof. Using (37), it is clear that $h^s(X,Y) = 0$ on $\mathrm{Rad}(TM)$. Using the definition of a *GCR*-lightlike submanifold, we have

$$\mathrm{trace}h|_{S(TM)}$$
$$= \sum_{i=1}^a h(Z_i, Z_i) + \sum_{j=1}^b h(\bar{J}\xi_j, \bar{J}\xi_j) + \sum_{j=1}^b h(\bar{J}N_j, N_j) + \sum_{l=1}^c h(\bar{J}W_l, \bar{J}W_l),$$

where $a = \dim(D_0)$, $b = \dim(D_2)$ and $c = \dim(L_2)$. Since M is totally umbilical therefore from (37), we have $h(\bar{J}\xi_j, \bar{J}\xi_j) = h(\bar{J}N_j, N_j) = 0$. Thus above equation becomes
(45)
$$\mathrm{trace}h|_{S(TM)}$$
$$= \sum_{i=1}^a h(Z_i, Z_i) + \sum_{l=1}^c h(\bar{J}W_l, \bar{J}W_l)$$
$$= \sum_{i=1}^a \frac{1}{r} \sum_{k=1}^r \bar{g}(h^l(Z_i, Z_i), \xi_k)N_k + \sum_{i=1}^a \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(h^s(Z_i, Z_i), W_p)W_p$$
$$+ \sum_{l=1}^c \frac{1}{r} \sum_{k=1}^r \bar{g}(h^l(\bar{J}W_l, \bar{J}W_l), \xi_k)N_k + \sum_{l=1}^c \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(h^s(\bar{J}W_l, \bar{J}W_l), W_p)W_p$$

where $\{W_1, W_2, \ldots, W_{n-r}\}$ is an orthonormal basis of $S(TM^\perp)$. Using (8) and (10) in (45), we obtain

(46)

$$\text{trace} h|_{S(TM)}$$

$$= \sum_{i=1}^{a} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A_{\xi_k}^* Z_i, Z_i) N_k + \sum_{i=1}^{a} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(A_{W_p} Z_i, Z_i) W_p$$

$$+ \sum_{l=1}^{c} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A_{\xi_k}^* \bar{J} W_l, \bar{J} W_l) N_k + \sum_{l=1}^{c} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(A_{W_p} \bar{J} W_l, \bar{J} W_l) W_p.$$

Thus trace $h|_{S(TM)} = 0$ if and only if trace $A_{W_p} = 0$ and trace $A_{\xi_k^*} = 0$ on $D_0 \perp \bar{J} L_2$. Hence the result follows. $\qquad\square$

Definition 7.4. A lightlike submanifold M of a semi-Riemannian manifold is said to be an irrotational submanifold if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma\text{Rad}(TM)$. Thus M is an irrotational lightlike submanifold if and only if $h^l(X, \xi) = 0$ and $h^s(X, \xi) = 0$.

Theorem 7.5. *Let M be an irrotational lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then M is minimal if and only if $\text{trace} A_{\xi_k}^*|_{S(TM)} = 0$ and $\text{trace} A_{W_j}|_{S(TM)} = 0$, where $W_j \in \Gamma(S(TM^\perp))$, where $k \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, n - r\}$.*

Proof. M is irrotational implies $h^s(X, \xi) = 0$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad}(TM))$ therefore $h^s = 0$ on $\text{Rad}(TM)$. Also

$$\text{trace} h|_{S(TM)}$$

$$= \sum_{i=1}^{m-r} \epsilon_i (h^l(e_i, e_i) + h^s(e_i, e_i))$$

(47)

$$= \sum_{i=1}^{m-r} \epsilon_i \{ \frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(e_i, e_i), \xi_k) N_k + \frac{1}{n-r} \sum_{j=1}^{n-r} \bar{g}(h^s(e_i, e_i), W_j) W_j \}$$

$$= \sum_{i=1}^{m-r} \epsilon_i \{ \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A_{\xi_k}^* e_i, e_i) N_k + \frac{1}{n-r} \sum_{j=1}^{n-r} \bar{g}(A_{W_j} e_i, e_i) W_j \}.$$

Hence the theorem follows. $\qquad\square$

Theorem 7.6. *Let M be an irrotational GCR-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. If D is integrable, then M is minimal if and only if $\text{trace} A_\xi^*|_{\bar{J} D_2 \oplus D'} = 0$ and $\text{trace} A_W|_{\bar{J} D_2 \oplus D'} = 0$.*

Proof. Since M is irrotational therefore we have $h^s = 0$ on $\text{Rad}(TM)$. The integrality of D implies that $h(X, \bar{J} Y) = h(\bar{J} X, Y)$ for $X, Y \in \Gamma(D)$, which further implies $h(\bar{J} X, \bar{J} Y) = -h(X, Y)$. Choose an orthonormal basis $\{e_1, e_2, \ldots, e_p,$

$\bar{J}e_1, \bar{J}e_2, \ldots, \bar{J}e_p\}$ of D_0 therefore

$$\text{trace} h|_{D_0} = \sum_{i=1}^{2p} \epsilon_i h(e_i, e_i) = \sum_{i=1}^{2p} \epsilon_i (h(e_i, e_i) + h(\bar{J}e_i, \bar{J}e_i)) = 0.$$

Thus M is minimal if and only if

$$(48) \qquad \sum_{j=1}^{b} h(\bar{J}\xi_j, \bar{J}\xi_j) = \sum_{j=1}^{b} h(\bar{J}N_j, N_j) = \sum_{l=1}^{c} h(\bar{J}W_l, \bar{J}W_l) = 0,$$

where $b = \dim(D_2)$ and $c = \dim(L_2)$. Clearly using (8) and (10) in (48), the assertion follows. $\qquad\square$

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