J. Appl. Math. & Informatics Vol. **31**(2013), No. 1 - 2, pp. 221 - 228
Website: http://www.kcam.biz

THREE-POINT BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

RAHMAT ALI KHAN*

Abstract. The method of upper and lower solutions and the generalized quasilinearization technique is developed for the existence and approximation of solutions to boundary value problems for higher order fractional differential equations of the type

$$ {}^{c}\mathcal{D}^{q}u(t) + f(t, u(t)) = 0, \quad t \in (0,1), q \in (n-1, n],\ n \geq 2$$
$$u'(0) = 0, u''(0) = 0, ..., u^{n-1}(0) = 0, u(1) = \xi u(\eta),$$

where ξ, $\eta \in (0,1)$, the nonlinear function f is assumed to be continuous and ${}^{c}\mathcal{D}^{q}$ is the fractional derivative in the sense of Caputo. Existence of solution is established via the upper and lower solutions method and approximation of solutions uses the generalized quasilinearization technique.

AMS Mathematics Subject Classification : 34A08, 34A45.
Key words and phrases : Boundary value problems, Fractional differential equations, Three-point boundary conditions, Upper and lower solutions, Generalized quasilinearization.

1. Introduction

Fractional differential equations is rapidly growing area of differential equations both theoretically and in practical point of view to real world problems. The theory of existence of solutions to nonlinear boundary value problems corresponding to fractional differential equations have recently been attracted the attention of many researchers, see for example [4, 5, 6, 7, 10, 11, 15, 17, 18, 19, 21] and the references therein. In these cited references, sufficient conditions for existence of solutions are established via the classical tools of functional analysis and fixed point theory. To estimate the exact solution of a nonlinear problem and to develop algorithms for approximating the exact solutions, the method of upper and lower solutions plays a fundamental role. The method of upper

Received January 3, 2012. Revised March 26, 2012. Accepted April 6, 2012. *The author is grateful the referee for constructive comments which lead to improvement of the manuscript.

and lower solutions for existence and multiplicity results are well studied for boundary value problems involving integer order derivatives. But, the upper and lower solutions method for the existence of solutions to boundary value problems corresponding to fractional differential equations is not well developed and as for as I know, only few results can be found in the literature dealing with the upper and lower solutions method [14, 20]. The quasilinearization method is developed for initial value problems corresponding to fractional differential equations [8, 9, 13, 16] but results dealing with quasilinearization to boundary value problems for higher order fractional differential equations can hardly be seen in the literature. For nice contribution to the literature of boundary value problems for fractional differential equations, the problem we study is a strong candidate. The motivation of this paper is to develop comparison result and the upper and lower solutions method for the existence of solution to a class of multi-point boundary value problems (BVPs) for higher order fractional differential equations

$$\begin{aligned} &{}^c\mathcal{D}^q u(t) + f(t, u(t)) = 0, \quad t \in (0,1), q \in (n-1, n], \, n \geq 2 \\ &u'(0) = 0, u''(0) = 0, ..., u^{n-1}(0) = 0, u(1) = \xi u(\eta), \end{aligned} \tag{1}$$

where ξ, $\eta \in (0,1)$, the function $f : [0,1] \times R \to R$ is continuous and may be nonlinear.

We develop the quasilinearization method for approximating the solution of the problem as sequence of solutions of linear problems. I am grateful to the reviewer who directed my attentions to the recent interested work studied in [1, 2, 3]. In these references, the authors studied BVPs for higher order fractional differential equations too via the classical tools functional analysis using fixed point theorems. But not only our problem is different from the problems they studied but also our objectives are different. To the best of my knowledge, the upper and lower solutions method and the quasilinearization technique for higher order fractional differential equations subject to three-point boundary conditions have never been studied previously.

2. Preliminaries

We recall some basic definitions and lemmas from fractional calculus [12].

Definition 2.1. The fractional integral of order $q > 0$ of a function $u : (0, \infty) \to R$ is defined by

$$\mathcal{I}^q u(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{u(s)}{(t-s)^{1-q}} ds,$$

provided the integral converges.

Definition 2.2. The Caputo fractional derivative of order $q > 0$ of a function $u \in AC^n[0,1]$ is defined by

$${}^c\mathcal{D}^q_{0+} u(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{u^n(s)}{(t-s)^{q-n+1}} ds, \text{ where } n = \lceil q \rceil,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([5]). *For $q > 0$, $u \in C(0,1) \cap L(0,1)$, the homogenous functional differential equation ${}^c\mathcal{D}^q_{0+}u(t) = 0$, has a solution $u(t) = c_1 + c_2t + c_3t^2 + ... + c_nt^{n-1}$, where $c_i \in R$ and $n = \lceil q \rceil + 1$.*

Lemma 2.2. *Assume that $u \in C(0,1) \cap L(0,1)$ with derivatives of order n that belong to $u \in C(0,1) \cap L(0,1)$, then*

$$I^{q\,c}\mathcal{D}^q u(t) = u(t) + c_1 + c_2t + c_3t^2 + ... + c_nt^{n-1},$$

where, $c_i \in R$ and $n = \lceil q \rceil + 1$.

For the purpose of comparison result, consider the following boundary value problem for fractional differential equation

$$\begin{aligned} &{}^c\mathcal{D}^q u(t) + h(t) = 0, \quad t \in (0,1), n-1 < q \leq n,\ n \geq 2 \\ &u'(0) = a_1, u''(0) = a_2, ..., u^{n-1}(0) = a_{n-1}, u(1) - \xi u(\eta) = b, \end{aligned} \tag{2}$$

where $h \in C[0,1]$, b, $a_j \in R$, $j = 1, 2, ..., n-1$. The boundary value problem (2) is equivalent to the following integral equation

$$u(t) = \frac{b}{(1-\xi)} - \frac{1}{(1-\xi)} \sum_{j=1}^{n-1} \frac{a_j}{j!}\psi_j + \int_0^1 k(t,s) f(s, u(s)) ds,\ t \in [0,1], \tag{3}$$

where $\psi_j(t) = (1-t^j) - \xi(\eta^j - t^j)$ and the Green function $k(t,s)$ is given by

$$k(t,s) = \begin{cases} \frac{(1-s)^{q-1} - (1-\xi)(t-s)^{q-1} - \xi(\eta-s)^{q-1}}{(1-\xi)\Gamma(q)}, & s \leq t \leq 1,\ \eta \geq s, \\ \frac{(1-s)^{q-1} - (1-\xi)(t-s)^{q-1}}{(1-\xi)\Gamma(q)}, & \eta \leq s \leq t \leq 1, \\ \frac{(1-s)^{q-1} - \xi(\eta-s)^{q-1}}{(1-\xi)\Gamma(q)}, & 0 \leq t \leq s \leq \eta < 1, \\ \frac{(1-s)^{q-1}}{(1-\xi)\Gamma(q)}, & 0 \leq t \leq s \leq 1,\ \eta \leq s. \end{cases}$$

Clearly, $\psi_j(t) > 0$ on $(0,1)$ and from the expression of $k(t,s)$, it follows that $k(t,s) \geq 0$ on $(0,1) \times (0,1)$. Hence, if $b \geq 0$, $a_j \leq 0$, $j = 1, 2, ...n-1$ and $h(t) \geq 0$, for $t \in [0,1]$, then $u(t) \geq 0$. On the other hand, if $b \leq 0$, $a_j \geq 0$, $j = 1, 2, ...n-1$ and $h(t) \leq 0$, for $t \in [0,1]$, then $u(t) \leq 0$ Thus, we have the following comparison result.

Comparison results: (i) If $u'(0) \leq 0, u''(0) \leq 0, ..., u^{n-1}(0) \leq 0, u(1) \geq \xi u(\eta)$ and ${}^c\mathcal{D}^q u(t) \leq 0$ on $(0,1)$ for $q \in (n-1, n]$, then any solution u of $-{}^c\mathcal{D}^q u(t) = h(t)$ is such that $u \geq 0$ on $(0,1)$. (ii) If $u'(0) \geq 0, u''(0) \geq 0, ..., u^{n-1}(0) \geq 0, u(1) \leq \xi u(\eta)$ and ${}^c\mathcal{D}^q u(t) \geq 0$ on $(0,1)$ for $q \in (n-1, n]$, then any solution u of $-{}^c\mathcal{D}^q u(t) = h(t)$ is such that $u \leq 0$ on $(0,1)$.

Hence we introduce the definition of upper and lower solutions corresponding to the BVP (1) as follows:

Definition 2.3. A function α is called a lower solution of the BVP (1), if $\alpha \in C^{n-1}[0,1]$ and satisfies

$$\begin{aligned} &-{}^c\mathcal{D}^q\alpha(t) \leq f(t, \alpha(t)),\ q \in (n-1, n],\ t \in (0,1), \\ &\alpha'(0) \geq 0,\ \alpha''(0) \geq 0, ..., \alpha^{n-1}(0) \geq 0,\ \alpha(1) \leq \xi\alpha(\eta). \end{aligned}$$

An upper solution $\beta \in C^{n-1}[0,1]$ of the BVP (1) is defined similarly by reversing the inequality.

Define $\bar{\beta} = \max\{\beta(t) : t \in [0,1]\}$ and $\bar{\alpha} = \min\{\alpha(t) : t \in [0,1]\}$.

3. Main results

Theorem 3.1. *Assume that there exist lower and upper solutions $\alpha, \beta \in C[0,1]$ of the BVP (1) such that $\alpha \leq \beta$ on $[0,1]$. Assume that $f : [0,1] \times R \to (0,\infty)$ is continuous and non-decreasing with respect to u on $[0,1]$. Then the BVP (1) has $C[0,1]$ positive solution u such that $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in [0,1]$.*

Proof. Define the following modification of f

$$F(t,u) = \begin{cases} f(t,\beta(t)), & \text{if } u \geq \beta(t), \\ f(t,u(t)), & \text{if } \alpha(t) \leq u \leq \beta(t), \\ f(t,\alpha(t)), & \text{if } \ u \leq \alpha(t) \end{cases} \tag{4}$$

and consider the modified BVP for fractional differential equations

$$\begin{aligned} -{}^c\mathcal{D}^q u(t) &= F(t,u(t)),\ q \in (n-1,n],\ n \geq 2,\ t \in (0,1), \\ u'(0) &= 0, u''(0) = 0, ..., u^{n-1}(0) = 0, u(1) = \xi u(\eta), \end{aligned} \tag{5}$$

which is equivalent to the integral equation

$$u(t) = \int_0^1 k(t,s)F(s,u(s))ds,\ t \in [0,1].$$

By a solution of the BVP (5) we mean a solution of the integral equation, that is, a fixed point of the operator equation $(I - A)u(t) = 0$, where I is the identity operator and $Au(t) = \int_0^1 k(t,s)F(s,u(s))ds$, $t \in [0,1]$.

Further, if u is a solution of the modified problem (5) such that $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in [0,1]$, then u is a solution of the BVP (1). Since F is continuous and bounded on $[0,1] \times R$, it follows by Schauder's fixed point theorem that the integral equations and hence the BVP has a solutions.

We only need to show that $\alpha(t) \leq u \leq \beta(t)$, $t \in [0,1]$, where u is solution of the BVP (5). For each fixed $t \in [0,1]$, the non-decreasing property of $f(t,u)$ with respect to u implies that $F(t,u)$ is non-decreasing with respect to u on $[\bar{\alpha}, \bar{\beta}]$ and

$$f(t,\alpha(t)) \leq F(t,u) \leq f(t,\beta(t)),\ (t,u) \in [0,1] \times R. \tag{6}$$

Define $m(t) = \alpha(t) - u(t)$, $t \in [0,1]$, where u is solution of the BVP (5), then in view of the boundary conditions, we obtain $m'(0) \geq 0$, $m''(0) \geq 0$, ..., $m^{n-1}(0) \geq 0$, $m(1) \leq \xi m(\eta)$. Using the definition of lower solution and (6), for each $t \in [0,1]$, we obtain

$$-{}^c\mathcal{D}^q m(t) = -\mathcal{D}^q \alpha(t) + \mathcal{D}^q u(t) \leq f(t,\alpha(t)) - F(t,u(t)) \leq 0,\ q \in (n-1,n].$$

Hence, by comparison result $m(t) \leq 0$, $t \in [0,1]$. Similarly, we can show that $u(t) \leq \beta(t)$, $t \in [0,1]$. □

To approximate the solutions of the BVP (1), we develop the generalized quasilinearization technique.

Theorem 3.2. *Under the hypothesis of Theorem 3.1, there exists a bounded monotone sequence of solutions of linear problems converging uniformly and quadratically to a solution of the problem* (1)*.*

Proof. Choose a function $\phi(t,u)$ with ϕ, $\phi_u, \phi_{uu} \in C([0,1] \times R)$ such that

$$\frac{\partial^2}{\partial u^2}[f(t,u) + \phi(t,u)] \geq 0 \text{ on } [0,1] \times [\bar{\alpha}, \bar{\beta}]. \tag{7}$$

Define $F^* : [0,1] \times R \to R$ by $F^*(t,u) = f(t,u) + \phi(t,u)$. Note that $F^* \in C([0,1] \times R)$ and

$$\frac{\partial^2}{\partial u^2} F^*(t,u) \geq 0 \text{ on } [0,1] \times [\bar{\alpha}, \bar{\beta}], \tag{8}$$

which implies that

$$f(t,u) \geq f(t,y) + F_u^*(t,y)(u-y) - [\phi(t,u) - \phi(t,y)],\ t \in [0,1], \tag{9}$$

where u, $y \in [\bar{\alpha}, \bar{\beta}]$. Using the non decreasing property of ϕ_u with respect to u on $[\bar{\alpha}, \bar{\beta}]$ for each $t \in [0,1]$, we obtain

$$\phi(t,u) - \phi(t,y) = \phi_u(t,c)(u-y) \leq \phi_u(t,\bar{\beta})(u-y) \text{ for } u \geq y, \tag{10}$$

where u, $y \in [\bar{\alpha}, \bar{\beta}]$ such that $y \leq c \leq u$. Substituting in (9), for $u \geq y$, we have

$$f(t,u) \geq f(t,y) + [F_u^*(t,y) - \phi_u(t,\bar{\beta})](u-y) \geq f(t,y) + \lambda(u-y), \tag{11}$$

where $\lambda = \min\{0, \min\{F_u^*(t,\bar{\alpha}) - \phi_u(t,\bar{\beta}) : t \in [0,1]\}\}$. We note that $\lambda \leq F_u^*(t,z) - \phi_u(t,\bar{\beta}) \leq f_u(t,\bar{\beta}) : t \in [0,1]$.

Define $g : [0,1] \times R \times R \to R$ by

$$g(t,u,y) = f(t,y) + \lambda(u-y). \tag{12}$$

We note that $g(t,u,y)$ is continuous on $[0,1] \times R \times R$ and for $u, y \in [\bar{\alpha}, \bar{\beta}]$, using (11) and (12), we have

$$\begin{cases} f(t,u) \geq g(t,u,y), \text{for } u \geq y, \\ f(t,u) = g(t,u,u). \end{cases} \tag{13}$$

Now, we develop the iterative scheme to approximate the solution. As an initial approximation, we choose $w_0 = \alpha$ and consider the linear problem

$$\begin{gathered} -{}^c\mathcal{D}^q u(t) = g(t,u(t),w_0(t)),\ q \in (n-1,n],\ t \in [0,1] \\ u'(0) = 0, u''(0) = 0, ..., u^{n-1}(0) = 0, u(1) = \xi u(\eta). \end{gathered} \tag{14}$$

The definition of lower and upper solutions and (13) imply that

$$g(t,w_0(t),w_0(t)) = f(t,w_0(t)) \geq -{}^c\mathcal{D}^q w_0(t),\ q \in (n-1,n],\ t \in [0,1]$$
$$g(t,\beta(t),w_0(t)) \leq f(t,\beta(t)) \leq -{}^c\mathcal{D}^q \beta(t),\ q \in (n-1,n],\ t \in [0,1],$$

which imply that w_0 and β are lower and upper solutions of (14) and since $w_0 = \alpha \leq \beta$ on $[0,1]$. Hence by Theorem 3.1, there exists a solution $w_1 \in C[0,1]$

of (14) such that $w_0 \leq w_1 \leq \beta$ on $[0,1]$. Again, from (13) and the fact that w_1 is a solution of (14), we obtain

$$-{}^c\mathcal{D}^q w_1(t) = g(t, w_1(t), w_0(t)) \leq f(t, w_1(t)),\ q \in (n-1, n],\ t \in [0,1], \tag{15}$$

which implies that w_1 is a lower solution of (1).

Similarly, we can show that w_1 and β are lower and upper solutions of the linear problem

$$\begin{aligned} -{}^c\mathcal{D}^q u(t) &= g(t, u(t), w_1(t)),\ q \in (n-1, n],\ t \in [0,1] \\ u'(0) = 0, u''(0) &= 0, ..., u^{n-1}(0) = 0, u(1) = \xi u(\eta). \end{aligned} \tag{16}$$

Hence by Theorem 3.1, there exists a solution $w_2 \in C[0,1]$ of (16) such that $w_1 \leq w_2 \leq \beta$ on $[0,1]$. Continuing in the above fashion, we obtain a bounded monotone sequence $\{w_n\}$ of solutions of linear problems satisfying

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq ... \leq w_n \leq \beta \text{ on } [0,1], \tag{17}$$

where the element w_n of the sequence is a solution of the linear problem

$$\begin{aligned} -{}^c\mathcal{D}^q u(t) &= g(t, u(t), w_0(t)),\ q \in (n-1, n],\ t \in [0,1] \\ u'(0) = 0, u''(0) &= 0, ..., u^{n-1}(0) = 0, u(1) = \xi u(\eta) \end{aligned}$$

and is given by

$$w_n(t) = \int_0^1 G(t,s) g(s, w_n(s), w_{n-1}(s)) ds,\ t \in [0,1]. \tag{18}$$

The monotonicity and uniform boundedness of the sequence $\{w_n\}$ implies the existence of a pointwise limit w on $[0,1]$ such that $w_n \to w$ uniformly. From the dominated convergence theorem, it follows that for any $t \in [0,1]$,

$$\int_0^1 G(t,s) g(s, w_n(s), w_{n-1}(s)) ds \to \int_0^1 G(t,s) f(s, w(s)) ds.$$

Passing to the limit as $n \to \infty$, (18) yields $w(t) = \int_0^1 G(t,s) f(s, w(s)) ds$, $t \in [0,1]$, which is integral representation of the BVP (1), implies that w is a solution of (1).

Now, we show that the convergence is quadratic, set $e_n(t) = w(t) - w_n(t)$, $t \in [0,1]$, where w is a solution of (1). Then, $e_n(t) \geq 0$ on $[0,1]$ and from the boundary conditions, $e_n'(0) = 0$, $e_n''(0) = 0, ..., e_n^{n-1}(0) = 0, e_n(1) = \xi e_n(\eta)$. For every $t \in [0,1]$, we have

$$-{}^c\mathcal{D}^q e_n(t) = F^*(t, w(t)) - \phi(t, w(t)) - f(t, w_{n-1}(t)) - \lambda(w_n(t) - w_{n-1}(t)). \tag{19}$$

Using the mean value theorem and the nondecreasing property of ϕ_u, that is, $\phi_{uu} \geq 0$ on $[0,1] \times [\bar{\alpha}, \bar{\beta}]$, we obtain,

$$\begin{aligned} \phi(t, w(t)) &\geq \phi(t, w_{n-1}(t)) + \phi_u(t, w_{n-1}(t))(w(t) - w_{n-1}(t)) \\ &\geq \phi(t, w_{n-1}(t)) + \phi_u(t, \bar{\alpha})(w(t) - w_{n-1}(t)), \end{aligned}$$

$$F(t,w(t)) = F(t,w_{n-1}(t)) + F_u(t,w_{n-1}(t))(w(t)-w_{n-1}(t))$$
$$+\frac{F_{uu}(t,\delta)}{2}(w(t)-w_{n-1}(t))^2$$
$$\leq F(t,w_{n-1}(t)) + F_u(t,\bar{\beta})(w(t)-w_{n-1}(t)) + \frac{F_{uu}(t,\delta)}{2}(w(t)-w_{n-1}(t))^2,$$

where $w_{n-1} \leq \delta \leq w$. Hence,

$$F(t,w(t)) - \phi(t,w(t)) \leq f(t,w_{n-1}(t)) + [F_u(t,\bar{\beta}) - \phi_u(t,\bar{\alpha})](w(t)-w_{n-1}(t))$$
$$+\frac{F_{uu}(x,\delta)}{2}(w(t)-w_{n-1}(t))^2.$$

Hence the equation (19) can be rewritten as

$$\begin{aligned}
-{}^c\mathcal{D}^q e_n(t) &\leq [F_u(t,\bar{\beta}) - \phi_u(t,\bar{\alpha})]e_{n-1}(t) + \frac{F_{uu}(t,\delta)}{2}(e_{n-1}(t))^2 - \\
&\lambda(e_{n-1}(t) - e_n(t)) \\
&\leq [F_u(t,\bar{\beta}) - \phi_u(t,\bar{\alpha}) - \lambda]e_{n-1}(t) + \lambda e_n(t) + \frac{F_{uu}(t,\delta)}{2}(e_{n-1}(t))^2 \\
&\leq [F_u(t,\bar{\beta}) - \phi_u(t,\bar{\alpha}) - \lambda]e_{n-1}(t) + \frac{F_{uu}(t,\delta)}{2}(e_{n-1}(t))^2 \\
&\leq \rho e_n(t) + d\|e_{n-1}\|^2,
\end{aligned} \tag{20}$$

where $\rho = \max\{F_u(t,\bar{\beta}) - \phi_u(t,\bar{\alpha}) - \lambda :\ t \in [0,1]\} \geq 0$ and $d = \max\{\frac{F_{uu}(t,y)}{2} : y \in [\bar{\alpha},\bar{\beta}]\}$. By comparison result, $e_n(t) \leq z(t)$, $t \in [0,1]$, where $z(t)$ is a unique solution of the linear BVP

$$\begin{gathered}
-{}^c\mathcal{D}^q_{0+} z(t) - \rho z(t) = d\|e_{n-1}\|^2,\ q \in (n-1,n] \\
z'(0) = 0,\ z''(0) = 0, ..., z^{n-1}(0) = 0,\ z(1) = \xi z(\eta),
\end{gathered} \tag{21}$$

and is given by

$$e_n(t) \leq z(t) = \int_0^1 k_1(t,s) d\|e_{n-1}\|^2 \leq A\|e_{n-1}\|^2, \tag{22}$$

where $A = \max\{d\int_0^1 k_1(t,s)\}$, $k(t,s)$ is the Green's function corresponding to the homogenous problem $-{}^c\mathcal{D}^q u(t) - \rho u(t) = 0$, $u'(0) = 0$, $u''(0) = 0, ..., u^{n-1}(0) = 0$, $u(1) = \xi u(\eta)$. Hence the convergence is quadratic. □

References

1. B. Ahmad and J. J Nieto, *Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations*, Abst. Applied Anal., (2009), pages 9, ID 494720.
2. B.Ahmad and A. Alsaedi, *Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fractional differential equations*, Fixed Point Theory Appl., (2010), pages 17, ID 364560.

3. B. Ahmad, A. Alsaedi, J. J Nieto and M. El-Shahed, *A study of nonlinear Langevin equation involving two fractional orders in different intervals*, Nonlinear Anal.(RWA), **13** (2012), 599-606.
4. B. Ahmad and J. J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comp. Math. Appl., **58** (2009), 1838–1843.
5. Z. Bai and H. Lu, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Analy. Appl., **311**(2005),495-505.
6. Z. Bai, *On positive solutions of a nonlocal fractional boudary value problem*, Nonlinear Anal.,**72** (2010), 916-924.
7. M. Benchohra, S. Hamani and S. K. Ntouyas,*Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal.,**71**(2009) 2391-2396.
8. J. V. Devi and Ch. Suseela, *Quasilinearization for fractional differential equations*, Commmmu. Appl. Anal.,**12**(2008), 407–418.
9. J. V. Devi, F. A. McRae and Z. Drici, *Generalized quasilinearization for fractional differential equations*, Comput. Math. Appl.,**12**(2010), 1057–1062.
10. C. S. Goodrich, *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett.,**23**(2010),1050-1055.
11. R. A. Khan and M. Rehman, *Existence of Multiple Positive Solutions for a General System of Fractional Differential Equations*, Commun. Appl. Nonlinear Anal.,**18**(2011), 25–35.
12. A. A. Kilbas, H. M. Srivastava and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science, Amsterdam, 2006.
13. V. Lakshmikantham and A. S. Vatsala, *General uniqueness and monotone iterative technique for fractional differential equations*, Commmmu. Appl. Anal.,**12**(2008), 399–406.
14. S. Liang and J. Zhang, *Positive solutions for boundary value problems of nonlinear fractional differential equation*, Nonlinear Anal.,**71** (2009) 5545–5550.
15. C. F. Li, X. N. Luo and Y. Zhou, *Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations*, Comput. Math. Appl.,**59**(2010)1363-1375.
16. F. A. McRae, *Monotone iterative technique and existence results for fractional differential equations*, Nonlinear Anal.,**12**(2009), 6093–6096.
17. M. Rehman and R. A. Khan, *Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations*, Appl. Math. Lett.,**23**(2010), 1038-1044.
18. M. Rehman, R. A. Khan and N. Asif, *Three point boundary value problems for nonlinear fractional differential equations*, Acta Math. Sci.,**31**(2011), 1–10.
19. Z. Shuqin, *Existence of Solution for Boundary Value Problem of Fractional Order*, Acta Math. Sci.,**26**(2006), 220–228.
20. J. Wang and H. Xiang, *Upper and Lower Solutions Method for a Class of Singular Fractional Boundary Value Problems with p-Laplacian Operator*, Abst. Appl. Analy.,(2010), doi:10.1155/2010/971824.
21. W. Zhong, *Positive Solutions for Multipoint Boundary Value Problem of Fractional Differential Equations*, Abst. Appl. Analy., (2010),doi:10.1155/2010/601492.

R. A. Khan received M.Phil. from Quaid-i-Azam University Islamabad and Ph.D at University of Glasgow UK. Since 2010 he has been at the University of Malakand as Professor and Dean of Sciences. His research interests include Boundary value problems for differential equations.

Department of Mathematics, University of Malakand, Chakadara Dir(L), Khyber Pakhtunkhwa, Pakistan.
e-mail: rahmat_alipk@yahoo.com