

# On the Stability of Critical Point for Positive Systems and Its Applications to Biological Systems

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**Abstract** – The coexistence and extinction of species are important concepts for biological systems and can be distinguished by an investigation of stability. When determining local stability of nonlinear systems, Lyapunov indirect method based on the Jacobian linearization has been widely employed due to its simplicity. Despite such popularity, it is not applicable to singular systems whose Jacobian has at least one eigenvalue that is equal to zero. In such singular cases, an appropriate Lyapunov function should be sought to determine the stability of systems, which is rather difficult and quite involved. In this paper, we seek for a simple criterion to determine stability of the equilibrium that is located at the boundary of the positive orthant, when one of eigenvalues of the Jacobian is zero. The goal of the paper is to present a generalized condition for the equilibrium to attract all trajectories that starting from initial condition in the positive orthant and near the equilibrium. Unlike the Lyapunov direct method, the proposed method requires just a simple algebraic computation for checking the stability of the critical point. Our approach is applied to various biological systems to show the effectiveness of the proposed method.

**Keywords:** Prey-predator model, Positive system, Center manifold, Jacobian linearization, Lyapunov function

## 1. Introduction

Stability is an important concept that can provide powerful insight into qualitative behavior of various biological systems [1, 2] such as predator-prey systems, viral and immune systems, epidemic systems and so on. Stability analysis tells us whether a solution trajectory near an equilibrium will converge toward or move away from it. For predator-prey system [3-15], an investigation of stability distinguishes the coexistence of predators and prey from the extinction of some species. In addition, stability analysis may determine the basin of attraction of each equilibrium, that is, the region of coexistence and the region of extinction. Aside from predator-prey systems, some research works on virus infection of CD4<sup>+</sup> cells have been performed in [16-20]. In their works, mathematical models describing infection dynamics have infection-free equilibrium and chronic-infection equilibrium. The infection will die out if the former is stable, while the infection will persist if the latter is stable. For the epidemic model with some vaccination strategy [21-23], an examination of stability can predict whether the vaccination successfully leads to disease eradication. An epidemic outbreak is produced if an endemic equilibrium is stable, whereas the disease will die out if a disease-free equilibrium is stable.

In order to check the local stability of a nonlinear system at its certain equilibrium, Lyapunov indirect method based on the Jacobian linearization has been widely employed due to its simplicity. Despite such popularity, the method cannot be applied to determine stability when one eigenvalue of the Jacobian matrix is equal to zero. In fact, although the research works in [2-12, 15-18] have used the Jacobian matrix to determine stability, most of them have not provided any conclusion on stability when the Jacobian matrix has an eigenvalue at the origin of the complex plane. While the extinction of some species can be predicted by investigating the stability of equilibria with some zero component, stability of such equilibria has not been determined in [2-12, 15-18] since the Jacobian has an eigenvalue at the origin.

In order to analyze stability for such cases, an appropriate Lyapunov function should be constructed as in [24], or some other nonlinear stability theory needs to be employed [25]. Even though Lyapunov stability theory and LaSalle's invariance principle can be employed to determine stability [26], it is rather difficult and quite involved to construct an appropriate Lyapunov function. An alternative method may be to study the stability of reduced system by invoking the center manifold theory [26, 27]. However, the verification procedure is not convenient in general, because a solution to the partial differential equation for the center manifold needs to be obtained. To avoid the difficulty in solving the partial differential equation, the authors have presented in [28] an alternative technique under a certain assumption of the Jacobian

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structure.

In this paper, we extend the result of [28], and, from the extension, a variety of ecological systems are investigated about their stability. The motivation of this research lies in the observation that there might exist a boundary equilibrium that attracts all trajectories starting in the positive orthant although it is not stable in the usual sense. A condition is presented for the equilibrium to attract all trajectories that reside in the positive orthant and near the equilibrium (even though it may not attract those trajectories outside the positive orthant). The proposed condition can deal with the case to which the technique of [28] is not applicable (motivated by the models in [6, 9]). We also extend the applicability of the previous technique by relaxing the assumption on the Jacobian structure. To show the effectiveness of the proposed method, we apply the proposed condition to inspect the stability of several different predator-prey models [6, 8, 9] and an HIV infection model [17]. Although the authors of [6, 8, 9, 17] have discussed the stability of various equilibria, they have not determined the stability of boundary equilibria when one of eigenvalues of the Jacobian is zero. Moreover, the result of [28] cannot be used for [6, 9], either. In contrast to those previous results, the proposed method can determine the stability in such singular cases.

The paper is organized as follows: Section II recalls the concept of stability *with respect to* the positive orthant and presents a simple condition for checking the stability of boundary equilibrium. In Section III, we apply the proposed method to several predator-prey models and an HIV infection model to show the effectiveness of the proposed method. Finally, some concluding remarks are given in Section IV.

**Notations:** A function is said to be of class  $C^k$  if it is continuously differentiable  $k$  times. For a vector  $x$  and a matrix  $A$ , the  $i$ -th component of  $x$  and the  $i$ -th row of  $A$  are denoted by  $x_{(i)}$  and  $A_{(i)}$ , respectively, and when there is no confusion,  $x_{(i)}$  is abbreviated to  $x_i$ . For a matrix  $A$ ,  $A^T$  denotes a transpose of the matrix  $A$ . We denote by  $e_k$  the column vector  $[0 \ 0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0]^T$  with the entry 1 in the  $k$ -th place. The elementary matrix obtained by interchanging the first and  $k$ -th rows of identity matrix is denoted by  $E_k$ . The  $n \times n$  identity matrix and the  $r \times 1$  zero vector are denoted by  $I_n$  and  $0_r$ , respectively, and when there is no confusion,  $0_r$  is abbreviated to  $0$ . When all eigenvalues of a matrix  $A$  have negative real parts,  $A$  is called a *Hurwitz* matrix. The maximum and the minimum eigenvalue of matrix  $A$  are represented by  $\lambda_{\max}(A)$  and  $\lambda_{\min}(A)$ , respectively. For some  $r > 0$  and  $x_0 \in \mathbb{R}^n$ ,  $B(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ , where  $\|x\|$  stands for the Euclidean norm of a vector  $x$ . For a vector  $x$ , we write

$x \gg 0$  and  $x \geq 0$  to indicate that every component of  $x$  is positive and nonnegative, respectively. Let  $\bar{\mathbb{R}}_+^n = \{x \in \mathbb{R}^n : x \geq 0\}$  and  $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x \gg 0\}$ . The order of magnitude notation  $\mathcal{O}$  is used as follows: we say  $f(x) = \mathcal{O}(g(x))$  if, for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x)| \leq \varepsilon |g(x)|$  for  $|x| < \delta$ . A set  $M$  is said to be a *positively invariant set* for the system  $\dot{x} = f(x)$  if the solution  $x(t)$  satisfies  $x(0) \in M \Rightarrow x(t) \in M, \forall t \geq 0$ .

## 2. Stability Condition for Positive Nonlinear Systems

In this paper, we consider a class of nonlinear system in the absence of an input  $u$ , that is, so-called unforced system

$$\dot{x} = f(x), \quad x \in \mathbb{R}_+^n \tag{1}$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is assumed to be  $C^3$  and  $\mathbb{R}_+^n$  is assumed to be a positively invariant set for system (1). We also assume that there exists an isolated equilibrium  $x^*$  that is located on the boundary of  $\mathbb{R}_+^n$  and satisfies the following assumption.

**Assumption JS** (Jacobian Structure): Suppose that the Jacobian matrix at  $x^*$ ,  $A := \frac{\partial f}{\partial x}(x^*)$ , has one eigenvalue at the origin and all others with negative real parts. Moreover, there exist a nonsingular matrix  $M$  and an integer  $k$  ( $1 \leq k \leq n$ ) such that  $M_{(k)} \geq 0$  and

$$(Mx^*)_{(k)} = 0 \tag{2}$$

and

$$(MAM^{-1})_{(k)} = 0_n^T. \tag{3}$$

**Remark 1:** Usually,  $M$  is chosen as an identity matrix, for which (2) and (3) become

$$\begin{aligned} x_k^* &= 0, \\ \frac{\partial f_k}{\partial x}(x^*) &= 0_n^T. \end{aligned} \tag{4}$$

However, an introduction of the matrix  $M$  allows a larger class of systems to satisfy Assumption JS. In fact, as will be illustrated in Section 3.4., the HIV infection model of [17] does not satisfy (4) but assumption JS (if  $M$  is chosen as an appropriate matrix).

Although Assumption JS seems to be quite restrictive at

first glance, it is not. In fact, the class of systems under consideration includes the following typical form [2-11, 14, 15, 19, 20], which often arises in various biological systems:

$$\begin{aligned} \dot{x}_i &= f_i(x), \quad 1 \leq i \leq k-1, \\ &\vdots \\ \dot{x}_k &= f_k(x) = x_k \bar{f}_k(x), \\ &\vdots \\ \dot{x}_i &= f_i(x), \quad k+1 \leq i \leq n. \end{aligned} \tag{5}$$

Here, we assume that the equilibrium of our interest satisfies that

$$x_k^* = 0 \quad \text{and} \quad \bar{f}_k(x^*) = 0. \tag{6}$$

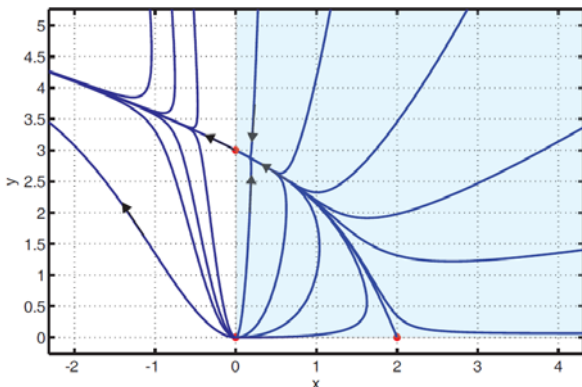
It is easily seen from (6) that the class of systems (5) satisfies Assumption JS. However, it should be noted that not all systems satisfying Assumption JS can be written as in (5). (As a matter of fact, the HIV infection model of [17] is not of the form (5), but satisfies Assumption JS.)

To illustrate the basic concept of the proposed method, consider the basic 2-species Lotka-Volterra competition model [2]:

$$\begin{aligned} \dot{x} &= x(1-x-y), \\ \dot{y} &= y(1-y-x). \end{aligned} \tag{7}$$

The system (7) has several equilibria, but we are interested in the stability of equilibrium (0, 1). The Jacobian matrix at (0, 1) is computed as

$$A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix},$$



**Fig. 1.** Phase portrait of system (7). In this figure, solid red circles indicate equilibria of system (7). Although phase trajectories starting from initial conditions outside  $\mathbb{R}_+^2$  do not converge to the equilibrium (0, 1), those trajectories starting from  $\mathbb{R}_+^2$  (shaded region) converge to (0, 1).

which has eigenvalues 0 and 1, and stability of (0, 1) is not determined. In order to see what is going on in this case, we check the phase plane<sup>1)</sup> in Fig. 1. It is seen that all trajectories starting from initial conditions in  $\mathbb{R}_+^2$  converge to the equilibrium (0, 1), while those starting outside  $\mathbb{R}_+^2$  do not converge to (0, 1). Thus, this equilibrium is not stable in the usual sense. However, if we restrict our interest to initial conditions in  $\mathbb{R}_+^2$  (the shaded region in Fig. 1), we may say that the equilibrium (0, 1) is asymptotically stable with respect to  $\mathbb{R}_+^2$  since it is attracting all trajectories in  $\mathbb{R}_+^2$ .

**Definition 1:** The equilibrium point  $x^*$  is locally stable with respect to (*w.r.t.*) the set  $\mathbb{R}_+^n$  if, for each  $\varepsilon < 0$ , there exists  $\delta(\varepsilon) > 0$  such that

$$x(0) \in B(x^*, \delta) \cap \mathbb{R}_+^n \Rightarrow x(t) \in B(x^*, \varepsilon) \cap \mathbb{R}_+^n.$$

Moreover, it is locally asymptotically stable *w.r.t.*  $\mathbb{R}_+^n$  if it is stable *w.r.t.*  $\mathbb{R}_+^n$  and  $\delta$  can be chosen such that

$$x(0) \in B(x^*, \delta) \cap \mathbb{R}_+^n \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0.$$

Now, we provide a simple condition for the stability *w.r.t.*  $\mathbb{R}_+^n$  that does not require the existence of an appropriate Lyapunov function. From Assumption JS, we obtain

$$(MAM^{-1})_{(k)} = [0 \ 0 \ \dots \ 0],$$

which implies

$$E_k (MAM^{-1}) E_k^{-1} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_2 \end{bmatrix}, \tag{8}$$

for some matrix  $A_{21} \in \mathbb{R}^{(n-1) \times 1}$  and  $A_2 \in \mathbb{R}^{(n-1) \times (n-1)}$ . (Recall that premultiplication and postmultiplication by an elementary matrix results in elementary row and column operation, respectively.) Note that  $A_2$  is a Hurwitz matrix because of Assumption JS.

**Lemma 1:** Under Assumption JS, the matrix

$$T := \begin{bmatrix} 1 & 0 \\ A_2^{-1} A_{21} & I \end{bmatrix} E_k M$$

Satisfies

<sup>1)</sup> In this paper, all the phase portraits were calculated using the software PPlane6 [29].

$$TAT^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}. \tag{9}$$

**Proof:** Let

$$\bar{T} = \begin{bmatrix} 1 & 0 \\ A^{-1}A_{21} & I \end{bmatrix}.$$

Then, we obtain  $T = \bar{T}E_kM$  and

$$\bar{T}^{-1} = \begin{bmatrix} 1 & 0 \\ -A^{-1}A_{21} & I \end{bmatrix}. \tag{10}$$

Moreover, it follows from (8) and (10) that

$$(E_kMAM^{-1}E_k^{-1})\bar{T}^{-1} = \bar{T}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix},$$

which results in

$$TAT^{-1} = (\bar{T}E_k)MAM^{-1}(\bar{T}E_k)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}.$$

With matrix  $T$  of Lemma 1, let  $g_1 : \mathbb{R}^n \rightarrow \mathbb{R}^1$  and  $g_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  be functions

$$\begin{bmatrix} g_1(z_1, z_2) \\ g_2(z_1, z_2) \end{bmatrix} := T \cdot f \left( T^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + x^* \right) - \begin{bmatrix} 0 \\ A_2 z_2 \end{bmatrix}, \tag{11}$$

where  $z_2 = [z_2, z_3, \dots, z_n]^T$ . In addition, for nonnegative integers  $m, k$  and  $2 \leq i_1, \dots, i_k \leq n$ , let

$$P_{i_1, \dots, i_k}^{[m,k]} = \begin{cases} \frac{\partial^m g_1}{\partial z_1^m}(0, 0_{n-1}), & k = 0, \\ \frac{\partial^{m+k} g_1}{\partial z_1^m \partial z_{i_1} \cdots \partial z_{i_k}}(0, 0_{n-1}), & k = 1, 2, \dots. \end{cases} \tag{12}$$

**Theorem 1:** For an equilibrium  $x^*$  of system (1), suppose that  $x^*$  and (1) satisfy Assumption JS. Then, we have the followings:

**(Case 1)**  $x^*$  is locally asymptotically stable w. r. t.  $\mathbb{R}_+^n$  (respectively, unstable) if  $P^{[2,0]} < 0$  (respectively,  $P^{[2,0]} > 0$ ).

**(Case 2)** Suppose that  $P^{[2,0]} = 0$ . Then,  $x^*$  is locally asymptotically stable (respectively, unstable) if

$$P^{[3,0]} + 6 \sum_{i=2}^n P_i^{[1,1]} \pi_i^{[2]} < 0$$

(respectively,  $P^{[3,0]} + 6 \sum_{i=2}^n P_i^{[1,1]} \pi_i^{[2]} > 0$ ), where  $\pi^{[2]} = [\pi_2^{[2]}, \dots, \pi_n^{[2]}]^T = -\frac{1}{2}A_2^{-1} \left[ \frac{\partial^2 g_2}{\partial z_1^2}(0, 0) \right]$ .

**Remark 2:** At first glance, it may seem rather complicated to apply Theorem 1. But, as will be illustrated in Section 3, the application of the theorem is not difficult since it only requires numerical computations. Moreover, when  $M = I$ , it is convenient to use  $P^{[m,0]} = \frac{\partial^m \psi}{\partial s^m}$ , where

$$\psi(s) := f_{(k)} \left( E_k \begin{bmatrix} 1 \\ -A_2^{-1}A_{21} \end{bmatrix} s + x^* \right). \tag{13}$$

**Proof: (Case 1):** Let  $x \in \mathbb{R}_+^n$  and

$$\bar{x} := x - x^*.$$

Then, we have

$$\dot{\bar{x}} = f(\bar{x} + x^*) =: \bar{f}(\bar{x}) \tag{14}$$

and

$$\frac{\partial \bar{f}}{\partial \bar{x}}(0) = \frac{\partial f}{\partial x}(x^*) = A. \tag{15}$$

Since  $\bar{f}(\bar{x})$  is  $C^3$  and  $\bar{f}(0) = 0$ , (14) can be represented as

$$\dot{\bar{x}} = \frac{\partial \bar{f}}{\partial \bar{x}}(0)\bar{x} + \left[ \bar{f}(\bar{x}) - \frac{\partial \bar{f}}{\partial \bar{x}}(0)\bar{x} \right] = \frac{\partial \bar{f}}{\partial \bar{x}}(0)\bar{x} + \bar{f}_{H.O.}(\bar{x})$$

where

$$\bar{f}_{H.O.}(\bar{x}) := \bar{f}(\bar{x}) - \frac{\partial \bar{f}}{\partial \bar{x}}(0)\bar{x}$$

is  $C^3$  and

$$\bar{f}_{H.O.}(0) = 0, \quad \frac{\partial \bar{f}_{H.O.}}{\partial \bar{x}}(0) = 0.$$

With matrix  $T$  of Lemma 1, the change of variables

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := T\bar{x}; \quad z_1 \in \mathbb{R}^1; \quad z_2 \in \mathbb{R}^{n-1}$$

transforms (14) into

$$\dot{z} = T \bar{f}(T^{-1}z) = TAT^{-1}z + g(z),$$

where

$$g(z) := T \bar{f}_{H,O}(T^{-1}z).$$

From (9), the system is rewritten as

$$\begin{aligned} \dot{z}_1 &= g_1(z_1, z_2), \\ \dot{z}_2 &= A_2 z_2 + g_2(z_1, z_2), \end{aligned} \quad (16)$$

where  $g_1$  and  $g_2$  are  $C^3$  functions such that  $g = [g_1, g_2^T]^T$  and, for  $i = 1, 2$ ,

$$g_i(0,0) = 0; \frac{\partial g_i}{\partial z_1}(0,0) = 0; \frac{\partial g_i}{\partial z_2}(0,0) = 0. \quad (17)$$

Thus,  $g_1(z_1, z_2) = g_1(z_1, \dots, z_n)$  can be written as

$$\begin{aligned} &g_1(z_1, \dots, z_n) \\ &= \frac{1}{2!} \sum_{i=1}^n \sum_{j=1}^n \left[ \frac{\partial^2 g_1}{\partial z_i \partial z_j}(0) \right] z_i z_j \\ &\quad + \frac{1}{3!} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \left[ \frac{\partial^3 g_1}{\partial z_i \partial z_j \partial z_k}(0) \right] z_i z_j z_k + o(\|z_1, z_2\|^3) \\ &= \frac{1}{2} \left( P^{[2,0]} z_1^2 + 2 \sum_{i=2}^n P_i^{[1,1]} z_1 z_i + \sum_{i=2}^n \sum_{j=2}^n P_{i,j}^{[0,2]} z_i z_j \right) \\ &\quad + \frac{1}{6} \left( P^{[3,0]} z_1^3 + 3 \sum_{i=2}^n P_i^{[2,1]} z_1^2 z_i + 3 \sum_{i=2}^n \sum_{i_2=2}^n P_{i,i_2}^{[1,2]} z_1 z_{i_2} z_i \right. \\ &\quad \left. + \sum_{i_1=2}^n \sum_{i_2=2}^n \sum_{i_3=2}^n P_{i_1,i_2,i_3}^{[0,3]} z_{i_1} z_{i_2} z_{i_3} \right) + o(\|z_1, z_2\|^3). \end{aligned}$$

Moreover, according to center manifold theorem (e.g., see Theorem 8.1 of [26]), there exists  $\delta > 0$  and a  $C^1$  function  $\pi: \mathbb{R}^1 \rightarrow \mathbb{R}^{n-1}$ , defined for all  $|z_1| < \delta_1$ , such that  $z_2 = \pi(z_1) := [\pi_2(z_1), \dots, \pi_n(z_1)]^T$  is a center manifold for (16), i.e.,

$$\begin{aligned} (\pi(z_1)) &:= A_2 \pi(z_1) + g_2(z_1, \pi(z_1)) \\ &\quad - \frac{\partial \pi}{\partial z_1} g_1(z_1, \pi(z_1)) = 0 \end{aligned} \quad (18)$$

and

$$\pi(0) = 0; \frac{\partial \pi}{\partial z_1}(0) = 0. \quad (19)$$

From (19),  $\pi$  can be written as

$$\pi(z_1) = \bar{\pi} z_1^2 + o(|z_1|^2), \quad (20)$$

for some  $\bar{\pi} = [\bar{\pi}_2, \dots, \bar{\pi}_n]^T \in \mathbb{R}^{(n-1) \times 1}$ . Therefore, we get

$$g_1(z_1, \pi_2(z_1), \dots, \pi_n(z_1)) = -c_2 z_1^2 - c_3 z_1^3 + \tilde{g}_1(z_1), \quad (21)$$

where  $c_2 := -\frac{1}{2} P^{[2,0]}$ ,  $c_3 := -\frac{1}{6} (P^{[3,0]} + 6 \sum_{i=2}^n P^{[1,1]} \bar{\pi}_i)$ , and  $\tilde{g}_1(z_1) := o(|z_1|^3)$ .

Now, another change of variables

$$\begin{bmatrix} z_1 \\ w \end{bmatrix} := \begin{bmatrix} z_1 \\ z_2 - \pi(z_1) \end{bmatrix}$$

transforms (16) into

$$\dot{z}_1 = g_1(z_1, w + \pi(z_1)), \quad (22)$$

$$\begin{aligned} \dot{w} &= A_2(w + \pi(z_1)) + g_2(z_1, w + \pi(z_1)) \\ &\quad - \frac{\partial \pi}{\partial z_1} g_1(z_1, w + \pi(z_1)). \end{aligned} \quad (23)$$

Subtracting (21) into (22) yields

$$\begin{aligned} \dot{z}_1 &= g_1(z_1, w + \pi(z_1)) \\ &= g_1(z_1, \pi(z_1)) + [g_1(z_1, w + \pi(z_1)) - g_1(z_1, \pi(z_1))] \\ &= -c_2 z_1^2 - c_3 z_1^3 + \tilde{g}_1(z_1) \\ &\quad + [g_1(z_1, w + \pi(z_1)) - g_1(z_1, \pi(z_1))] \end{aligned}$$

and subtracting (18) from (23) gives

$$\begin{aligned} \dot{w} &= A_2 w + g_2(z_1, w + \pi(z_1)) - g_2(z_1, \pi(z_1)) \\ &\quad - \frac{\partial \pi}{\partial z_1} [g_1(z_1, w + \pi(z_1)) - g_1(z_1, \pi(z_1))]. \end{aligned}$$

Hence, we can rewrite (22) and (23) as

$$\begin{aligned} \dot{z}_1 &= -c_2 z_1^2 - c_3 z_1^3 + \tilde{g}_1(z_1) + N_1(z_1, w), \\ \dot{w} &= A_2 w + N_2(z_1, w), \end{aligned} \quad (24)$$

where

$$\begin{aligned} N_1(z_1, w) &= g_1(z_1, w + \pi(z_1)) - g_1(z_1, \pi(z_1)), \\ N_2(z_1, w) &= g_2(z_1, w + \pi(z_1)) - g_2(z_1, \pi(z_1)) \\ &\quad - \frac{\partial \pi}{\partial z_1}(z_1) N_1(z_1, w). \end{aligned}$$

It is easily seen that  $N_1$  and  $N_2$  are  $C^2$  and

$$N_i(z_1, 0) = 0, \quad i = 1, 2, \quad (25)$$

and, by virtue of (17),

$$\frac{\partial N_i}{\partial w}(0, 0) = 0, \quad i = 1, 2. \quad (26)$$

Thus, there exists  $\delta_2 > 0$  such that  $\| [z_1, w^T]^T \| < \delta_2$  implies

$$\| N_i(z_1, w) \| \leq k_i \| w \|, \quad i = 1, 2,$$

where  $k_1$  and  $k_2$  can be made arbitrary small by choosing  $\delta_2$  small enough.

Now, suppose that  $P^{[2,0]} < 0$  and consider a Lyapunov function candidate for full system (24)

$$V(z, w) = z_1^2 + \sqrt{w^T P_0 w},$$

where  $P_0$  is the positive definite solution of

$$P_0 A_2 + A_2^T P_0 = -I.$$

Then, there exists  $\delta_3 < \min(\delta_1, \delta_2)$  such that  $\| [z_1, w^T]^T \| < \delta_3$  implies

$$2 \| z_1 (-c_2 z_1^3 + \tilde{g}_1(z_1)) \| \leq c_2 z_1^3, \\ \frac{1}{4\sqrt{\lambda_{\max}(P_0)}} - 2k_1 \| z_1 \| - k_2 \frac{\lambda_{\max}(P_0)}{\sqrt{\lambda_{\max}(P_0)}} > 0.$$

Thus, the derivative of  $V$  along the trajectories of the system (24) is given by

$$\begin{aligned} \dot{V} &= 2z_1 [-c_2 z_1^2 - C_3 z_1^3 + \tilde{g}_1(z_1) + N_1(z_1, w)] \\ &\quad + \frac{1}{2\sqrt{w^T P_0 w}} [w^T (P_0 A_2 + A_2^T P_0) w + 2w^T P_0 N_2(z_1, w)] \\ &\leq -c_2 z_1^3 + 2k_1 \| z_1 \| \| w \| - \frac{\| w \|}{2\sqrt{\lambda_{\max}(P_0)}} \\ &\quad + k_2 \frac{\lambda_{\max}(P_0)}{\sqrt{\lambda_{\min}(P_0)}} \| w \| \\ &= -c_2 z_1^3 - \frac{\| w \|}{4\sqrt{\lambda_{\max}(P_0)}} - \left[ \frac{1}{4\sqrt{\lambda_{\max}(P_0)}} - 2k_1 \| z_1 \| \right. \\ &\quad \left. - k_2 \frac{\lambda_{\max}(P_0)}{\sqrt{\lambda_{\min}(P_0)}} \right] \| w \| \\ &\leq -c_2 z_1^3 - \frac{\| w \|}{4\sqrt{\lambda_{\max}(P_0)}}. \end{aligned}$$

On the other hand, from Lemma 1, we obtain

$$z_1 = (\bar{T}E_k)_{(1)} (Mx) = e_k^T Mx = M_{(k)} x,$$

which implies that  $z_1 \geq 0$  since  $x \in \mathbb{R}_+^n$  and  $M_{(k)} \geq 0$ . Therefore, using the standard Lyapunov stability techniques, it can be shown that  $(z_1(t), w(t)) \rightarrow 0$  as  $t \rightarrow \infty$ . Since  $\pi(0) = 0$ , it follows that  $(z_1(t), z_2(t)) \rightarrow 0$  as  $t \rightarrow \infty$  and as a consequence,  $x(t) \rightarrow x^*$  as  $t \rightarrow \infty$  for  $x(0) \in B(x^*, \delta) \cap \mathbb{R}_+^n$  for some  $\delta > 0$ .

On the contrary, suppose that  $P^{[2,0]} > 0$ . For the convenience, system (24) is rewritten as

$$\begin{aligned} \dot{z}_1 &= -c_2 z_1^2 - c_3 z_1^3 + \tilde{g}_1(z_1) + N_1(z_1, w), \\ \dot{w} &= A_2 w + N_2(z_1, w), \end{aligned} \quad (27)$$

in which the origin corresponds to the equilibrium  $x^*$  of (1). For given  $\bar{z}_0 > 0$ , let  $\bar{z}(t)$  be the solution of  $\dot{z}_1 = -c_2 z_1^2 - c_3 z_1^3 + \tilde{g}_1(z_1)$  with  $\bar{z}(0) = \bar{z}_0$ . Then,  $\bar{z}(t)$  cannot be kept within a small neighborhood of the origin because  $\dot{z}_1 = -c_2 z_1^2 + o(z_1^2)$  and  $-C_2 = P^{[2,0]} > 0$ . (In fact,  $\bar{z} > 0$  when  $\bar{z}(t)$  is positive and sufficiently small.) But, observe that  $(\bar{z}(t), 0)$  is also a solution of system (27). This implies that the origin is unstable since some initial condition  $(\bar{z}, 0)$  with  $\bar{z} > 0$  moves away from the origin. Therefore,  $x^*$  is unstable.

**(Case 2):** From (17),  $g_2(z_1, z_2)$  can be written as

$$\begin{aligned} g_2(z_1, z_2) &= \frac{1}{2} \frac{\partial^2 g_2}{\partial z_1^2}(0, 0) z_1^2 + z_1 C_{12} z_2 + z_2^T C_{22} z_2 \\ &\quad + o(\| z_1, z_2 \|^2), \end{aligned}$$

where  $C_{12}$  and  $C_{22}$  are some matrices of appropriate dimensions. Since

$$\begin{aligned} N(\bar{\pi} z_1^2) &= A_2 \bar{\pi} z_1^2 + g_2(z_1, \bar{\pi} z_1^2) - 2\bar{\pi} z_1 g_1(z_1, \bar{\pi} z_1^2) \\ &= \left[ A_2 \bar{\pi} + \frac{1}{2} \frac{\partial g_2}{\partial z_1^2}(0, 0) \right] z_1^2 + o(|z_1|^2), \end{aligned}$$

letting  $\pi^{[2]} := -\frac{1}{2} A_2^{-1} \frac{\partial^2 g_2}{\partial z_1^2}(0, 0)$  yields

$$N(\pi^{[2]} z_1^2) = o(|z_1|^2), \quad \pi(z_1) = \pi^{[2]} z_1^2 + o(|z_1|^2).$$

Thus, since  $c_2 = -\frac{1}{2} P^{[2,0]} = 0$ , we get from (21)

$$\begin{aligned} g_1(z_1, \pi_2(z_1), \dots, \pi_n(z_1)) &= -c_3 z_1^3 + \tilde{g}_1(z_1) + o(|z_1^3|) \\ &= -c_3 z_1^3 + o(|z_1^3|). \end{aligned}$$

Now, suppose that  $P^{[3,0]} + 6 \sum_{i=2}^n P_i^{[1,1]} \pi_i^{[2]} < 0$ , i.e.,  $c_3 > 0$ . Then, with the same Lyapunov function candidate as in the proof of (Case 1), there exists a  $\delta_4 > 0$  such that  $\| [z_1, w^T]^T \| < \delta_4$  leads to

$$\dot{V}(z_1, w) \leq -c_3 z_1^4 - \frac{1}{4\sqrt{\lambda_{\max}(P)}} \|w\|. \quad (28)$$

This implies that there exists a  $\delta > 0$  such that  $x(t) \rightarrow x^*$  as  $t \rightarrow \infty$  for  $x(0) \in B(x^*, \delta)$ .

### 3. Applications to Biological Systems

To show the effectiveness of the proposed method we apply our result to several predator-prey models and an HIV model in the ecological literatures.

#### 3.1 A predator-prey model with hawk and dove tactics

In this subsection, we consider the predator-prey model that incorporates individual behavior of the predators [6]. It is assumed that individual predators can use two tactics when fighting with other predators to keep a captured prey, the hawk and dove tactics. While the hawk fights in any case, the dove is never aggressive. When a hawk meets a dove, there is no fighting and the hawk keeps the prey and dove gets nothing. When two hawks encounter, they fight and both of them suffer injuries. After the fighting the winner keeps the prey. When two doves encounter, there is no fighting and they share the prey. The model consists of two parts: a fast part that represents the change of tactics of predators and a slow part that expresses the predator-prey interactions. Let  $n$ ,  $P_H$ , and  $P_D$  be the population of prey, hawk predator, and dove predator, respectively. Then, the system model is given by [6]

$$\begin{aligned} \frac{dn}{dt} &= rn \left(1 - \frac{n}{K}\right) - an(p_H + p_D), \\ \frac{dp_H}{dt} &= \frac{1}{\varepsilon} p_H (\Delta_H - \Delta) + \left[ \mu p_H + \alpha \left( \frac{G-C}{2} x + Gy \right) p_H \right], \\ \frac{dp_D}{dt} &= \frac{1}{\varepsilon} p_D (\Delta_D - \Delta) + \left[ \mu p_D + \alpha \frac{G}{2} p_D \right], \end{aligned} \quad (29)$$

where

$$x := \frac{p_H}{p_H + p_D}, \quad y := \frac{p_D}{p_H + p_D},$$

and

$$\begin{aligned} \Delta_H &:= \frac{G-C}{2} x + Gy, \quad \Delta_D := \frac{G}{2} y, \\ \Delta &:= \frac{G-C}{2} x^2 + Gxy + \frac{G}{2} y^2 \end{aligned}$$

with  $G = an$ . Taking advantage of the two-time scales, the complete system has been studied from the system model of reduced dimension. When  $n < \frac{c}{a}$ , the aggregated model is given by [6]

$$\begin{aligned} \dot{n} &= rn \left(1 - \frac{n}{K}\right) - anp, \\ \dot{p} &= -\mu p + \frac{\alpha a}{2} np - \frac{\alpha a^2}{2C} n^2 p, \end{aligned} \quad (30)$$

where  $p$  represents the total population of predators, i.e.  $p = p_H + p_D$ . The system (30) has five equilibria,  $E_1(0,0)$ ,  $E_2(k,0)$ ,  $E_3(n_1^*, p_1^*)$ ,  $E_4(n_2^*, p_2^*)$ , and  $E_5(n_3^*, n_4^*)$ , where  $n_1^* = \frac{c}{2a} - \frac{\sqrt{\alpha a^2 C (\alpha C - 8\mu)}}{2\alpha a^2}$ ,  $n_2^* = n_3^* = \frac{2\mu + \alpha C}{\alpha a}$ , and  $p_i^* = \frac{r}{a} \left(1 - \frac{n_i^*}{K}\right)$ ,  $i = 1, 2, 3$ .

We first consider the case where  $\alpha C > 8\mu$ . It has been shown in [6] that, when  $\alpha C > 8\mu$ ,  $E_2$  is stable (respectively, unstable) if  $K < n_1^*$  or  $n_2^* < K < n_3^*$  (respectively,  $n_1^* < K < n_2^*$ ). But, the stability of  $E_2$  has not been determined when  $K = n_1^*$  or  $K = n_2^*$ . Now, using the proposed theorem, we are going to determine the stability of  $E_2$  for such cases. Since the Jacobian matrix at  $E_2$  when  $K = n_1^*$  is computed as

$$A = \begin{bmatrix} -r & -aK \\ 0 & 0 \end{bmatrix},$$

Assumption JS is satisfied with  $M = I_2$  and  $k = 2$ . More-over, with  $D_1 := C\alpha a - \sqrt{\alpha a^2 C (\alpha C - 8\mu)}$ , we obtain

$$\psi(s) = -\frac{D_1^2}{8\alpha C r^2} s^3 + \left(-\frac{1}{4} + \frac{aK}{2C}\right) \frac{D_1}{r} s^2,$$

which leads to

$$P^{[2,0]} = \frac{\partial^2 \psi}{\partial s^2}(0) = \left(-\frac{1}{2} + \frac{aK}{C}\right) \frac{D_1}{r} < 0,$$

since  $K = n_1^* < \frac{c}{2a}$ . Therefore,  $E_2$  is locally asymptotically stable w.r.t.  $\mathbb{R}_+^2$  and, as a result, the predator goes to extinction when  $K = n_1^*$ . On the other hand, for the case

where  $K = n_2^*$ , we can show that

$$P^{[2,0]} = \frac{\partial^2 \psi}{\partial s^2}(0) = \left(-\frac{1}{2} + \frac{aK}{C}\right) \frac{D_1}{r} > 0,$$

where  $D_2 = C\alpha a + \sqrt{\alpha a^2 C(\alpha C - 8\mu)}$ . Therefore,  $E_2$  is un-stable, and hence predator and prey coexist when  $K = n_2^*$ .

Next, we consider the stability of  $E_2$  when  $\alpha C = 8\mu$  for which  $n_1^* = n_2^* = \frac{C}{2a}$ . Suppose that we need to check the stability of  $E_2$  when  $K = n_1^*$ . Again, the stability has not been determined in [6] since the Jacobian has one eigenvalue at the origin. It should be also noted that the result of [28] cannot be used to determine the stability, while Theorem 1 can be used. In contrast to the case where  $\alpha C > 8\mu$ , we need to rely on (Case 2) of Theorem 1 since  $\alpha C = 8\mu$  implies  $p^{[2,0]} = 0$ . Since

$$\begin{aligned} \pi^{[2]} &= -\frac{1}{2}[-r]^{-1} \left[ \frac{\partial^2 g_2}{\partial z_1^2}(0,0) \right] \\ &= \frac{C(2r + a\alpha K)(-C + 2aK)}{8r^3 K} = 0, \end{aligned}$$

And

$$\begin{aligned} P^{[3,0]} &= -\frac{3a^2 C \alpha}{4r^2}, \\ P^{[1,1]} &= \frac{a\alpha}{2} - \frac{\alpha a^2 K}{C}, \end{aligned}$$

we obtain

$$P^{[3,0]} + 6P_2^{[1,1]}\pi^{[2]} = -\frac{3a^2 C \alpha}{4r^2}. \tag{31}$$

Thus, when  $\alpha C = 8\mu$  and  $K = \frac{C}{2a}$ ,  $E_2$  is locally asymptotically stable w.r.t.  $\mathbb{R}_+^2$ .

### 3.2 A partial-dependent predator-prey system

In this subsection, we consider a partial-dependent predator-prey system that has been discussed in [9]. The model is given by

$$\begin{aligned} \dot{x} &= rx \left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{1 + \alpha_1 x}, \\ \dot{y} &= sy \left(1 - \frac{y}{L}\right) + \frac{\beta xy}{1 + \alpha_1 x}, \end{aligned} \tag{32}$$

where  $r, s, K, L, \alpha, \beta$ , and  $\alpha_1$  are some positive constants.

The state variables  $x$ , and  $y$  represent the population density of prey and predator, respectively. The system (32) has several equilibria such as  $E_o(0,0)$ ,  $E_c(K,0)$ , and so on. It has been shown in [9] that the system (32) experiences a saddle-node bifurcation at the equilibrium  $E_p$  when  $r = \alpha L$  and  $\alpha_1 Ks - s - \beta K \neq 0$ , and a pitchfork bifurcation at the equilibrium  $E_p$  when  $r = \alpha L$  and  $\alpha_1 Ks - s - \beta K = 0$ . Although it has been shown in [9] that  $E_p$  is a stable node (respectively, a saddle)  $r < \alpha L$  if (respectively,  $r > \alpha L$ ), the stability of  $E_p$  has not been determined when  $r = \alpha L$ . Now, using the proposed theorem, we are going to determine the stability of  $E_p$  when  $r = \alpha L$ . The Jacobian matrix at  $E_p$  is computed as

$$A = \begin{bmatrix} r - \alpha L & 0 \\ \beta L & -s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \beta L & -s \end{bmatrix}, \tag{33}$$

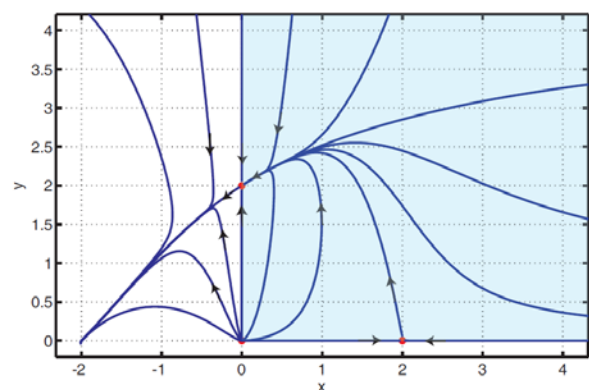
which shows that Assumption JS is satisfied with  $M = I_2$  and  $K = 1$ . Moreover, we obtain

$$\psi(u) = ru \left(1 - \frac{u}{K}\right) - \alpha u \frac{\left(\frac{u\beta L}{s} + L\right)}{(1 + \alpha_1 u)},$$

so that

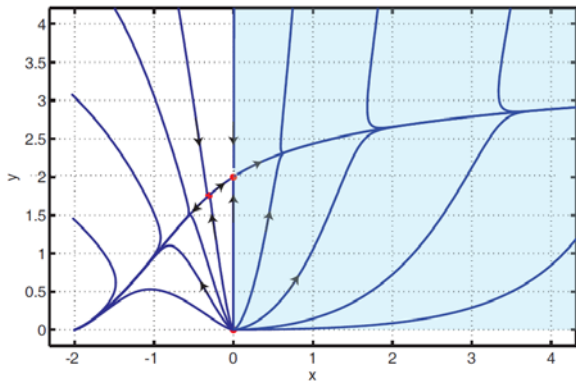
$$P^{[2,0]} = \frac{\partial^2 \psi}{\partial u^2}(0) = 2 \frac{\alpha L(\alpha_1 Ks - s - \beta K)}{Ks}. \tag{34}$$

Therefore,  $E_p$  is locally asymptotically stable (respectively, unstable) w.r.t.  $\mathbb{R}_+^2$  if  $\alpha_1 Ks - s - \beta K < 0$  (respectively,  $\alpha_1 Ks - s - \beta K > 0$ ). (See Fig. 2 and Fig. 3

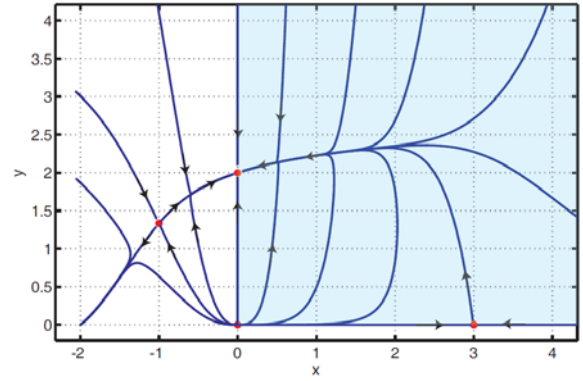


**Fig. 2.** Phase portrait of system (32) with  $\alpha = 1, K = 2, s = 3, L = 2, \beta = 1, r = 2$ , and  $\alpha_1 = 1/2$ , for which  $r = \alpha L$  and  $\alpha_1 Ks - s - \beta K < 0$ . In this figure, solid red circles indicate equilibria of system (32). Although not all trajectories move toward to  $E_p(0,2)$ , all trajectories residing in  $\mathbb{R}_+^2$  (shaded region) converge to  $E_p$ .





**Fig. 3.** Phase portrait of system (32) with  $\alpha = 1, K = 8, s = 3, L = 2, \beta = 1, r = 2,$  and  $\alpha_1 = 1/2,$  for which  $r = \alpha L$  and  $\alpha_1 K s - s - \beta K > 0.$  All trajectories starting from initial conditions in  $\mathbb{R}_+^2$  move away from  $E_p.$



**Fig. 4.** Phase portrait of system (32) with  $\alpha = 1, K = 3, s = 6, L = 2, \beta = 1, r = 2,$  and  $\alpha_1 = 1/2,$  for which  $r = \alpha L, \alpha_1 K s - s - \beta K = 0,$  and  $s < K\beta.$  All trajectories move toward to  $E_p(0, 2),$  which implies that  $E_p$  is asymptotically stable in the usual sense.

for each cases.)

When  $\alpha_1 K s - s - \beta K = 0$  as well as  $r = \alpha L,$  the determination of stability of  $E_p$  becomes more complicated. Although the stability of  $E_p$  cannot be determined by the result of [28], Theorem 1 (Case 2) can be applied. To this end, we compute

$$\begin{aligned} \pi^{[2]} &= -\frac{1}{2}[-s]^{-1} \left[ \frac{\partial^2 g_2}{\partial z_1^2}(0,0) \right] \\ &= -\frac{\beta L(\beta K + s)}{s^2 K}, \end{aligned}$$

And

$$\begin{aligned} P^{[3,0]} &= -6 \frac{\alpha L(\beta K + s)}{s K^2}, \\ P_2^{[1,1]} &= -\alpha, \end{aligned}$$

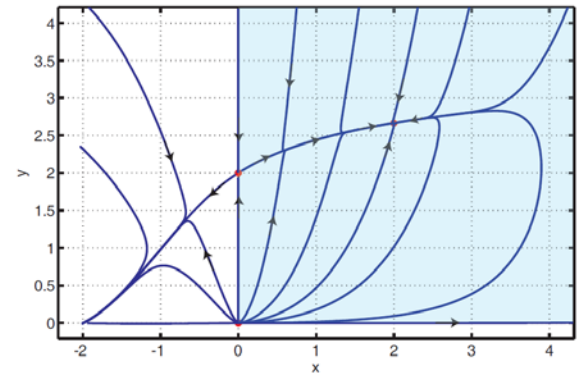
which imply that

$$P^{[3,0]} + 6P_2^{[1,1]}\pi^{[2]} = -6 \frac{\alpha L}{s^2 K^2} (s^2 - K^2 \beta^2). \quad (35)$$

Thus, when  $r = \alpha L$  and  $\alpha_1 K s - s - \beta K = 0,$   $E_p$  is locally asymptotically stable (respectively, unstable) if  $s > K\beta$  (respectively,  $s < K\beta$ ). (See Fig. 4 and Fig. 5 for each cases.)

### 3.3 A predator-prey model with habitat destruction

In this subsection, we consider the predator-prey model that has been discussed in [8], which was developed to study the consequences of habitat destruction. The model is given by



**Fig. 5.** Phase portrait of system (32) with  $\alpha = 1, K = 6, s = 3, L = 2, \beta = 1, r = 2,$  and  $\alpha_1 = 1/2,$  for which  $r = \alpha L, \alpha_1 K s - s - \beta K = 0,$  and  $s < K\beta.$  Most trajectories move away from  $E_p(0, L)$  in the direction of unstable manifold, which implies that  $E_p$  is unstable.

$$\begin{aligned} \dot{x} &= c_x x(1-x-D) - e_x x - \mu xy, \\ \dot{y} &= c_y y(1-y-D) - e_y y - \phi y(1-x), \end{aligned} \quad (36)$$

where  $c_x, e_x, \mu, c_y, e_y, \phi,$  and  $D$  are some positive constants. The state variables  $x$  and  $y$  represent the proportion of patches occupied by prey and predators, respectively.

Four possible equilibria exist for the model (36):

$$E_0(0,0), E_1\left(1-D-\frac{e_x}{c_x}, 0\right), E_2\left(0, 1-D-\frac{e_y+\phi}{c_y}\right), \text{ and } E_3(x^*, y^*),$$

where  $x^* = \frac{1}{c_x c_y + \mu \phi} [c_y(c_x - \mu)(1-D) - c_y e_x + \mu(e_y + \phi)]$  and

$$y^* = \frac{1}{c_x c_y + \mu \phi} [c_x c_y(1-D) - c_x \phi D - c_x e_y - e_x \phi]. \text{ It has}$$

been shown in [8] that  $E_1$  is stable (respectively, unstable) if  $D > D_x$  (respectively,  $D < D_x$ ), where  $D_x := \frac{c_x c_y - c_x e_y - e_x \phi}{c_x(c_y + \phi)}$ . However, when  $D = D_x$ , the stability of  $E_1$  has not been determined since the Jacobian matrix has one eigenvalue at the origin.

Now, using the proposed theorem, we are going to show that  $E_1$  is indeed stable when  $D = D_x$ . The Jacobian matrix at  $E_1$  is computed as

$$A = \begin{bmatrix} -c_x x_1 & -\mu x_1 \\ 0 & 0 \end{bmatrix}, \tag{37}$$

from which Assumption JS is satisfied with  $M = I_2$  and  $k = 2$ . Moreover, we obtain

$$E_2 A E_2^{-1} = \begin{bmatrix} 0 & 0 \\ -\mu x_1 & -c_x x_1 \end{bmatrix},$$

which implies that

$$E_2 \begin{bmatrix} 1 \\ -A_2^{-1} A_{21} \end{bmatrix} s + \begin{bmatrix} 1 - D - \frac{e_x}{c_x} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\mu}{c_x} s + 1 - D - \frac{e_x}{c_x} \\ s \end{bmatrix}.$$

Thus, we get

$$\psi(s) = -\frac{c_y c_x + \mu \phi}{c_x} s^2,$$

which leads to

$$P^{[2,0]} = \frac{\partial^2 \psi}{\partial s^2}(0) < 0.$$

Therefore,  $E_1$  is locally asymptotically stable w.r.t.  $\mathbb{R}_+^2$  when  $D = D_x$ .

### 3.4 HIV dynamics model

In this subsection, we consider the HIV model that has been discussed in [17], which depicts the interaction between T cells and virus particles. The model is given by

$$\begin{aligned} \dot{T} &= f(T) - kVT, \\ \dot{T}^* &= -\beta T^* + kVT, \\ \dot{V} &= -\gamma V + N\beta T^* - ikVT, \end{aligned} \tag{38}$$

where  $k, \beta, \gamma$ , and  $N$  are some positive constants,  $i$  can be either 1 or 0, and  $f$  is a smooth function that

satisfies

$$\begin{aligned} f(T) &> 0, \quad 0 \leq T < \bar{T}, \quad f(\bar{T}) = 0, \\ f'(\bar{T}) &< 0, \quad \text{and} \quad f(T) < 0, \quad T > \bar{T}. \end{aligned}$$

The state variables  $T, T^*$ , and  $V$  represent the population of uninfected T cells, productively infected T cells, and free virus particles, respectively. The system (38) has two equilibria,  $E_0(\bar{T}, 0, 0)$  and  $E_e(T_e, T_e^*, V_e)$ ,

where  $T_e = \frac{\gamma}{k(N-i)}$ ,  $T_e^* = \frac{\gamma V_e}{(N-i)\beta}$ , and  $V_e = \frac{f(T_e)}{kT_e}$ . It has been shown in [17] that  $E_0$  is stable (respectively, unstable) if  $R_0 := \frac{k\bar{T}(N-i)}{\gamma} < 1$  (respectively,  $R_0 > 1$ ). But,

the stability of  $E_0$  has not been determined when  $R_0 = 1$ .

Now, using the proposed theorem, we are going to determine the stability of  $E_0$  when  $R_0 = 1$ , that is,  $\gamma = k\bar{T}N - k\bar{T}i$ . The Jacobian matrix, evaluated at  $E_0$ , is

$$A = \begin{bmatrix} f'(\bar{T}) & 0 & -k\bar{T} \\ 0 & -\beta & k\bar{T} \\ 0 & N\beta & -\gamma - ik\bar{T} \end{bmatrix} = \begin{bmatrix} f'(\bar{T}) & 0 & -k\bar{T} \\ 0 & -\beta & k\bar{T} \\ 0 & N\beta & -Nk\bar{T} \end{bmatrix}, \tag{39}$$

which shows that Assumption JS is not satisfied with  $M = I_3$ . If we choose

$$M := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & N & 1 \end{bmatrix},$$

then

$$MAM^{-1} = \begin{bmatrix} f'(\bar{T}) & 0 & -k\bar{T} \\ 0 & -\beta & k\bar{T} \\ 0 & 0 & 0 \end{bmatrix},$$

from which Assumption JS is satisfied with  $k = 3$ . With some computation, we obtain

$$P^{[2,0]} = 2 \frac{\beta^2 k^2 \bar{T} (N-i)}{(\beta + k\bar{T}N)^2 f'(\bar{T})},$$

which is negative since  $f'(\bar{T}) < 0$  and  $N$  is typically large. Therefore, according to Theorem 1,  $E_0$  is locally asymptotically stable w.r.t.  $\mathbb{R}_+^3$ , which suggests that the virus population will decline and die out.

#### 4. Conclusions

We have proposed a simple test for checking local stability of an equilibrium that is located on the boundary of the positive orthant. While Lyapunov's indirect method based on the Jacobian linearization cannot draw any conclusion on the stability when the Jacobian matrix has an eigenvalue at the origin, the proposed method is able to determine the stability. Since our approach is based on the approximate solution to center manifold equation, it only guarantees the local result. To the contrary, if the global behavior needs to be studied, rather elaborate tools such as Lyapunov direct method or LaSalle's invariance principle should be resorted to. Nonetheless, we believe that our results are quite attractive in that it requires just a simple algebraic computation but may provide a clue to the prediction of global behavior. One might think that phase portrait is sufficient to determine the stability. But, the phase portrait is hardly applicable to higher dimensional systems (e.g.,  $\mathbb{R}^n$  with  $n \geq 3$ ), while the proposed method can be easily employed. Moreover, we believe that applying our result requires less time and effort than sketching a complete phase portrait.

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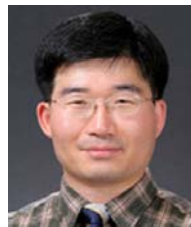
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