

Iterative Algorithm for a New System of Variational Inclusions with B -monotone Operators in Banach Spaces

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ABSTRACT. In this paper, we introduce and study a new system of variational inclusions with B -monotone operators in Banach spaces. By using the proximal mapping associated with B -monotone operator, we construct a new iterative algorithm for approximating the solution of this system of variational inclusions. We also prove the existence of solutions and the convergence of the sequences generated by the algorithm for this system of variational inclusions. The results presented in this paper extend and improve some known results in the literature.

1. Introduction

In recent years, variational inclusion theory has emerged one of the main branch of mathematical and engineering sciences. This theory provides us with a simple, natural, unified and general framework to study a wide class of unrelated problems in media, elasticity, transportation, economics, optimization, regional, physical, structural and applied science. For details see [1-9,11] and references therein.

In 2010, Luo and Huang [9] introduced and studied the following variational inclusions with B -monotone operators in Banach spaces:

Suppose that E is a reflexive Banach space with the topological dual space E^* , $A : E \rightarrow E^*$, $N : E \times E \times E \rightarrow E^*$, $f, g : E \rightarrow E$ are single-valued mappings, $S, T, W : E \rightarrow CB(E)$ are set-valued mappings and $M : E \times E \rightarrow 2^{E^*}$ is a B -monotone mapping, where $CB(E)$ denotes the family of all nonempty closed and bounded subsets of E . For any given $a \in E^*$, find $x \in E$, $s \in S(x)$, $t \in T(x)$ and

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$w \in W(x)$ such that

$$(1.1) \quad a \in A(x) + M(f(x), g(x)) - N(s, t, w).$$

Inspired and motivated by the results in Luo and Huang [9], the purpose of this paper is to introduce and study a new system of generalized variational inclusions with B -monotone operators in Banach spaces. By using the technique of the proximal mapping, we establish the equivalence between the generalized variational inclusions and the proximal mapping equations in reflexive Banach spaces. We use this equivalence and Nadler's theorem [10] to construct a new iterative algorithm for solving this system of generalized variational inclusions. And we prove the existence of solutions for this system of generalized variational inclusions. We also prove the convergence of an iterative algorithm approximating the solution for this system of generalized variational inclusions in reflexive Banach spaces. The results in this paper unify, extend and improve some results from [6,8,9,11].

2. Preliminaries

Throughout this paper we assume that E is a real Banach space with dual space E^* , $\langle \cdot, \cdot \rangle$ is the dual pair between E and E^* , 2^{E^*} denotes the family of all the nonempty subsets of E^* and $CB(E)$ is the family of all nonempty closed and bounded subsets of E .

Definition 2.1. Let $A : E \rightarrow E^*$ be a single-valued mapping. A is said to be

(i) monotone if

$$\langle A(x) - A(y), x - y \rangle \geq 0, \quad \forall x, y \in E,$$

(ii) strictly monotone if

$$\langle A(x) - A(y), x - y \rangle > 0, \quad \forall x, y \in E,$$

and equality holds if and only if $x = y$.

Definition 2.2. Let $T : E \rightarrow 2^{E^*}$ be a multi-valued mapping. T is said to be

(i) monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall x, y \in E, u \in Tx, v \in Ty;$$

(ii) strictly monotone if

$$\langle u - v, x - y \rangle > 0, \quad \forall x, y \in E, u \in Tx, v \in Ty,$$

and equality holds if and only if $x = y$;

(iii) r -strongly monotone if there exists a constant $r > 0$ such that

$$\langle u - v, x - y \rangle \geq r \|x - y\|^2, \quad \forall x, y \in E, u \in Tx, v \in Ty;$$

(iv) s -relaxed monotone if there exists a constant $s > 0$ such that

$$\langle u - v, x - y \rangle \geq -s\|x - y\|^2, \quad \forall x, y \in E, u \in Tx, v \in Ty.$$

Definition 2.3. Let $M : E \times E \rightarrow 2^{E^*}$ be a multi-valued mapping and $f, g : E \rightarrow E$ be single-valued mappings.

(i) $M(f, \cdot)$ is said to be α -strongly monotone with respect to f if there exists a constant $\alpha > 0$ such that

$$\langle u - v, x - y \rangle \geq \alpha\|x - y\|^2, \quad \forall x, y, \omega \in E, u \in M(f(x), \omega), v \in M(f(y), \omega),$$

(ii) $M(\cdot, g)$ is said to be β -relaxed monotone with respect to g if there exists a constant $\beta > 0$ such that

$$\langle u - v, x - y \rangle \geq -\beta\|x - y\|^2, \quad \forall x, y, \omega \in E, u \in M(\omega, g(x)), v \in M(\omega, g(y)),$$

(iii) $M(\cdot, \cdot)$ is said to be (α, β) -symmetric monotone with respect to f and g if $M(f, \cdot)$ is α -strongly monotone with respect to f and $M(\cdot, g)$ is β -relaxed monotone with respect to g with $\alpha \geq \beta$ and $\alpha = \beta$ if and only if $x = y$.

Definition 2.4([9]). Let E be a Banach space with the dual space E^* . Let $f, g : E \rightarrow E, B : E \rightarrow E^*$ be single-valued mappings and $M : E \times E \rightarrow 2^{E^*}$ be a multi-valued mapping. Then M is said to be B -monotone if M is (α, β) -symmetric monotone with respect to f and g and $(B + \rho M(f, g))(E) = E^*$ for every $\rho > 0$.

Definition 2.5. Let E be a reflexive Banach space with the dual space E^* . Let $f, g : E \rightarrow E$ be single-valued mappings, $B : E \rightarrow E^*$ be a strictly monotone mapping and $M : E \times E \rightarrow 2^{E^*}$ be a B -monotone mapping and $\rho > 0$ be a constant. Then a proximal mapping $R_{M(\cdot, \cdot), \rho}^B : E^* \rightarrow E$ is defined by

$$R_{M(\cdot, \cdot), \rho}^B(x^*) = (B + \rho M(f, g))^{-1}(x^*), \quad \forall x^* \in E^*.$$

Remark 2.1. If $M(f, g) = M$ and M is a general H -monotone, then the proximal mapping $R_{M(\cdot, \cdot), \rho}^B$ reduces to the proximal mapping R_M^H considered in [6,11].

Definition 2.6. A multi-valued mapping $T : E \rightarrow CB(E)$ is said to be D -Lipschitz continuous if there exists a constant $h > 0$ such that

$$D(T(x), T(y)) \leq h\|x - y\|, \quad \forall x, y \in E,$$

where $D(\cdot, \cdot)$ denotes the Hausdorff metric on $CB(E)$.

Lemma 2.1([9]). Let E be a reflexive Banach space with the dual space E^* . Let

$f, g : E \rightarrow E$ be single-valued mappings, $B : E \rightarrow E^*$ be a strictly monotone mapping and $M : E \times E \rightarrow 2^{E^*}$ be a B -monotone mapping. Then the proximal mapping $R_{M(\cdot, \cdot), \rho}^B : E^* \rightarrow E$ is Lipschitz continuous with constant $\frac{1}{\rho(\alpha - \beta)}$, i.e.,

$$\|R_{M(\cdot, \cdot), \rho}^B(x^*) - R_{M(\cdot, \cdot), \rho}^B(y^*)\| \leq \frac{1}{\rho(\alpha - \beta)} \|x^* - y^*\|, \quad \forall x^*, y^* \in E^*.$$

3. A System of Generalized Variational Inclusions with B -monotone Mappings and Iterative Algorithm

In this section, we will introduce a new system of generalized variational inclusions with B -monotone and construct a new iterative algorithm for this system.

Let E be a reflexive Banach space with the dual space E^* , $A, B : E \rightarrow E^*$, $N : E \times E \times E \rightarrow E^*$, $f, g : E \rightarrow E$ be a single-valued mappings, $M_1, M_2 : E \times E \rightarrow 2^{E^*}$ be B -monotone mappings, $S, T, W : E \rightarrow CB(E)$ be set-valued mappings. We consider the following problem: for any given $a_1, a_2 \in E$ find $x, y \in E$, $s \in S(x)$, $\bar{s} \in S(y)$, $t \in T(x)$, $\bar{t} \in T(y)$, $w \in W(x)$ and $\bar{w} \in W(y)$ such that

$$(3.1) \quad \begin{aligned} \rho a_1 &\in B(x) - B(y) + \rho[A(x) + M_1(f(x), g(x)) - N(s, t, w)], \\ \lambda a_2 &\in B(y) - B(x) + \lambda[A(x) + M_2(f(y), g(y)) - N(\bar{s}, \bar{t}, \bar{w})], \end{aligned}$$

where $\rho, \lambda > 0$ are constants.

Problem (3.1) is called a system of generalized variational inclusions with B -monotone mappings.

Below are some special cases of problem (3.1):

(I) If $x = y$, $\rho = 1$ and $\lambda = 0$, then the problem (3.1) reduces to the variational inclusion problem with B -monotone operators (1.1) introduced and studied by Luo and Huang [9].

(II) If $x = y$, $\rho = 1$, $\lambda = 0$ and $g = I$ is an identity mapping on E , then the problem (3.1) reduces to the following problem: for any given $a_1 \in E^*$, find $x \in E$, $s \in S(x)$, $t \in T(x)$ and $w \in W(x)$ such that

$$(3.2) \quad a_1 \in A(x) + M(f(x), x) - N(s, t, w).$$

Problem (3.2) was introduced and studied by Ding and Feng [6].

(III) If $x = y$, $\rho = 1$, $\lambda = 0$, $a_1 = 0$, $N = 0$ and $M(f(x), g(x)) = M(f(x))$, then the problem (4.1) reduces to the following problem: find $x \in E$ such that

$$(3.3) \quad 0 \in A(x) + M(f(x)).$$

Problem (3.3) was introduced and studied by Xia and Huang [11].

(IV) If E is a Hilbert space, $x = y$, $\rho = 1$, $\lambda = 0$, $a_1 = 0$, $N = 0$ and $M(f(x), g(x)) = M(x)$, then the problem (4.1) reduces to the following problem: find $x \in E$ such that

$$(3.4) \quad 0 \in A(x) + M(x).$$

Problem (3.4) was introduced and studied by Feng and Huang [8].

Theorem 3.1. *Let $A : E \rightarrow E^*$, $N : E \times E \times E \rightarrow E^*$, $f, g : E \rightarrow E$ be single-valued mappings, $S, T, W : E \rightarrow CB(E)$ be multi-valued mappings. Let $B : E \rightarrow E^*$ be a strictly monotone mapping and $M_1, M_2 : E \times E \rightarrow 2^{E^*}$ be B -monotone mappings. Then $(x, y, s, \bar{s}, t, \bar{t}, w, \bar{w})$ with $x, y \in E$, $s \in S(x)$, $\bar{s} \in S(y)$, $t \in T(x)$, $\bar{t} \in T(y)$, $w \in W(x)$ and $\bar{w} \in W(y)$ is a solution of problem (3.1) if and only if*

$$\begin{aligned} x &= R_{M_1(\cdot, \cdot), \rho}^B[\rho a_1 + B(y) - \rho A(x) + \rho N(s, t, w)], \\ y &= R_{M_2(\cdot, \cdot), \lambda}^B[\lambda a_2 + B(x) - \lambda A(x) + \lambda N(\bar{s}, \bar{t}, \bar{w})]. \end{aligned}$$

Proof. The fact directly follows from Definition 2.5 and some arguments. □

Remark 3.1. (i) If $x = y$, $\rho = 1$ and $\lambda = 0$, then Theorem 3.1 reduces to Theorem 4.1 in [9].

(ii) If $x = y$, $\rho = 1$, $\lambda = 0$, $g = I$ is an identity mapping on E and M is a general H -monotone mapping in the first argument, then Theorem 3.1 reduces to Theorem 3.1 in [6].

(iii) If $x = y$, $\rho = 1$, $\lambda = 0$, $a_1 = 0$, $N = 0$, $M(f(x), g(x)) = M(f(x))$ and M is a general H -monotone mapping, then Theorem 3.1 reduces to Theorem 3.3 in [11].

For any given $x_0, y_0 \in E$, take $s_0 \in S(x_0)$, $\bar{s}_0 \in S(y_0)$, $t_0 \in T(x_0)$, $\bar{t}_0 \in T(y_0)$, $w_0 \in W(x_0)$ and $\bar{w}_0 \in W(y_0)$. It follows from Theorem 3.1 that there exist $x_1, y_1 \in E$ such that

$$\begin{aligned} x_1 &= R_{M_1(\cdot, \cdot), \rho}^B[\rho a_1 + B(y_0) - \rho A(x_0) + \rho N(s_0, t_0, w_0)], \\ y_1 &= R_{M_2(\cdot, \cdot), \lambda}^B[\lambda a_2 + B(x_0) - \lambda A(x_0) + \lambda N(\bar{s}_0, \bar{t}_0, \bar{w}_0)]. \end{aligned}$$

Since $s_0 \in S(x_0)$, $\bar{s} \in S(y_0)$, $t_0 \in T(x_0)$, $\bar{t}_0 \in T(y_0)$, $w_0 \in W(x_0)$ and $\bar{w}_0 \in W(y_0)$, by Nadler's theorem [10], there exist $s_1 \in S(x_1)$, $\bar{s}_1 \in S(y_1)$, $t_1 \in T(x_1)$, $\bar{t}_1 \in T(y_1)$, $w_1 \in W(x_1)$ and $\bar{w}_1 \in W(y_1)$ such that

$$\|s_1 - s_0\| \leq (1 + 1)D(S(x_1), S(x_0)),$$

$$\|\bar{s}_1 - \bar{s}_0\| \leq (1 + 1)D(S(y_1), S(y_0)),$$

$$\|t_1 - t_0\| \leq (1 + 1)D(T(x_1), T(x_0)),$$

$$\|\bar{t}_1 - \bar{t}_0\| \leq (1 + 1)D(T(y_1), T(y_0)),$$

$$\|w_1 - w_0\| \leq (1 + 1)D(W(x_1), W(x_0)),$$

$$\|\bar{w}_1 - \bar{w}_0\| \leq (1 + 1)D(W(y_1), W(y_0)).$$

Hence we can construct an iterative algorithm for solving problem (3.1) as follows:

Algorithm 3.1. For any given $x_0, y_0 \in E$, $s_0 \in S(x_0)$, $\bar{s}_0 \in S(y_0)$, $t_0 \in T(x_0)$, $\bar{t}_0 \in T(y_0)$, $w_0 \in W(x_0)$, $\bar{w}_0 \in W(y_0)$, we can obtain the sequence $\{x_n\}$, $\{y_n\}$, $\{s_n\}$, $\{\bar{s}_n\}$, $\{t_n\}$, $\{\bar{t}_n\}$, $\{w_n\}$, $\{\bar{w}_n\}$ such that

$$(3.3) \quad \begin{aligned} x_{n+1} &= R_{M_1(\cdot, \cdot), \rho}^B[\rho a_1 + B(y_n) - \rho A(x_n) + \rho N(s_n, t_n, w_n)], \\ y_{n+1} &= R_{M_2(\cdot, \cdot), \lambda}^B[\lambda a_2 + B(x_n) - \lambda A(x_n) + \lambda N(\bar{s}_n, \bar{t}_n, \bar{w}_n)], \end{aligned}$$

$$s_n \in S(x_n), \quad \|s_{n+1} - s_n\| \leq (1 + \frac{1}{n+1})D(S(x_{n+1}), S(x_n)),$$

$$\bar{s}_n \in S(y_n), \quad \|\bar{s}_{n+1} - \bar{s}_n\| \leq (1 + \frac{1}{n+1})D(S(y_{n+1}), S(y_n)),$$

$$(3.4) \quad t_n \in T(x_n), \quad \|t_{n+1} - t_n\| \leq (1 + \frac{1}{n+1})D(T(x_{n+1}), T(x_n)),$$

$$\bar{t}_n \in T(y_n), \quad \|\bar{t}_{n+1} - \bar{t}_n\| \leq (1 + \frac{1}{n+1})D(T(y_{n+1}), T(y_n)),$$

$$w_n \in W(x_n), \quad \|w_{n+1} - w_n\| \leq (1 + \frac{1}{n+1})D(W(x_{n+1}), W(x_n)),$$

$$\bar{w}_n \in W(y_n), \quad \|\bar{w}_{n+1} - \bar{w}_n\| \leq (1 + \frac{1}{n+1})D(W(y_{n+1}), W(y_n)).$$

4. Convergence of an Iterative Algorithm

In this section, we show the existence of solutions for problem (3.1) and the convergence of the iterative sequences generated by Algorithm 3.1.

Theorem 4.1. *Let E be a reflexive Banach space with the dual space E^* . Let $f, g : E \rightarrow E$ be single-valued mappings, $A : E \rightarrow E^*$ be a τ -Lipschitz continuous mapping, $B : E \rightarrow E^*$ be a strictly monotone and δ -Lipschitz continuous mapping*

and $M_1, M_2 : E \times E \rightarrow 2^{E^*}$ be B -monotone mappings. Let $S, T, W : E \rightarrow CB(E)$ be D -Lipschitz continuous with constants $k_S > 0, k_T > 0, k_W > 0$, respectively and $N : E \times E \times E \rightarrow E^*$ be ξ_1 -Lipschitz continuous in the first argument, ξ_2 -Lipschitz continuous in the second argument, ξ_3 -Lipschitz continuous in the third argument. If there exist constants $\rho, \lambda > 0$ such that

$$(4.1) \quad \frac{1}{\alpha - \beta} \{ \tau + \xi_1 k_S + \xi_2 k_T + \xi_3 k_W + \frac{1}{\lambda} (\delta + \lambda \tau) \} < 1,$$

$$(4.2) \quad \frac{1}{\alpha - \beta} \{ \frac{\delta}{\rho} + \xi_1 k_S + \xi_2 k_T + \xi_3 k_W \} < 1,$$

then problem (3.1) has a solution $(x, y, s, \bar{s}, t, \bar{t}, w, \bar{w})$ and sequences $\{x_n\}, \{y_n\}, \{s_n\}, \{\bar{s}_n\}, \{t_n\}, \{\bar{t}_n\}, \{w_n\}, \{\bar{w}_n\}$ converge to $x, y, s, \bar{s}, t, \bar{t}, w, \bar{w}$, respectively, where $\{x_n\}, \{y_n\}, \{s_n\}, \{\bar{s}_n\}, \{t_n\}, \{\bar{t}_n\}, \{w_n\}, \{\bar{w}_n\}$ are the sequences generated by Algorithm 3.1.

Proof. For $n = 1, 2, \dots$, let

$$\begin{aligned} \Omega_n &= \rho a_1 + B(y_n) - \rho A(x_n) + \rho N(s_n, t_n, w_n), \\ \Delta_n &= \lambda a_2 + B(x_n) - \lambda A(x_n) + \lambda N(\bar{s}_n, \bar{t}_n, \bar{w}_n). \end{aligned}$$

By (3.3) and Lemma 2.2, we have

$$(4.3) \quad \begin{aligned} \|x_{n+1} - x_n\| &= \|R_{M(\cdot, \cdot), \rho}^B(\Omega_n) - R_{M(\cdot, \cdot), \rho}^B(\Omega_{n-1})\| \\ &\leq \frac{1}{\rho(\alpha - \beta)} \|\Omega_n - \Omega_{n-1}\|, \end{aligned}$$

$$(4.4) \quad \begin{aligned} \|y_{n+1} - y_n\| &= \|R_{M(\cdot, \cdot), \lambda}^B(\Delta_n) - R_{M(\cdot, \cdot), \lambda}^B(\Delta_{n-1})\| \\ &\leq \frac{1}{\lambda(\alpha - \beta)} \|\Delta_n - \Delta_{n-1}\|. \end{aligned}$$

And

$$(4.5) \quad \begin{aligned} \|\Omega_n - \Omega_{n-1}\| &\leq \|B(y_n) - B(y_{n-1})\| + \rho \|A(x_n) - A(x_{n-1})\| \\ &\quad + \rho \|N(s_n, t_n, w_n) - N(s_{n-1}, t_{n-1}, w_{n-1})\|, \end{aligned}$$

$$(4.6) \quad \begin{aligned} \|\Delta_n - \Delta_{n-1}\| &\leq \|B(x_n) - B(x_{n-1})\| + \lambda \|A(x_n) - A(x_{n-1})\| \\ &\quad + \lambda \|N(\bar{s}_n, \bar{t}_n, \bar{w}_n) - N(\bar{s}_{n-1}, \bar{t}_{n-1}, \bar{w}_{n-1})\|. \end{aligned}$$

Using the Lipschitz continuities of A, B, N, S, T and W we obtain

$$(4.7) \quad \|B(x_n) - B(x_{n-1})\| \leq \delta \|x_n - x_{n-1}\|,$$

$$(4.8) \quad \|B(y_n) - B(y_{n-1})\| \leq \delta \|y_n - y_{n-1}\|,$$

$$(4.9) \quad \|A(x_n) - A(x_{n-1})\| \leq \tau \|x_n - x_{n-1}\|,$$

$$(4.10) \quad \begin{aligned} \|s_n - s_{n-1}\| &\leq \left(1 + \frac{1}{n}\right) D(S(x_n), S(x_{n-1})) \\ &\leq \left(1 + \frac{1}{n}\right) k_S \|x_n - x_{n-1}\|, \end{aligned}$$

$$(4.11) \quad \begin{aligned} \|\bar{s}_n - \bar{s}_{n-1}\| &\leq \left(1 + \frac{1}{n}\right) D(S(y_n), S(y_{n-1})) \\ &\leq \left(1 + \frac{1}{n}\right) k_S \|y_n - y_{n-1}\|, \end{aligned}$$

$$(4.12) \quad \begin{aligned} \|t_n - t_{n-1}\| &\leq \left(1 + \frac{1}{n}\right) D(T(x_n), T(x_{n-1})) \\ &\leq \left(1 + \frac{1}{n}\right) k_T \|x_n - x_{n-1}\|, \end{aligned}$$

$$(4.13) \quad \begin{aligned} \|\bar{t}_n - \bar{t}_{n-1}\| &\leq \left(1 + \frac{1}{n}\right) D(T(y_n), T(y_{n-1})) \\ &\leq \left(1 + \frac{1}{n}\right) k_T \|y_n - y_{n-1}\|, \end{aligned}$$

$$(4.14) \quad \begin{aligned} \|w_n - w_{n-1}\| &\leq \left(1 + \frac{1}{n}\right) D(W(x_n), W(x_{n-1})) \\ &\leq \left(1 + \frac{1}{n}\right) k_W \|x_n - x_{n-1}\|, \end{aligned}$$

$$(4.15) \quad \begin{aligned} \|\bar{w}_n - \bar{w}_{n-1}\| &\leq \left(1 + \frac{1}{n}\right) D(W(y_n), W(y_{n-1})) \\ &\leq \left(1 + \frac{1}{n}\right) k_W \|y_n - y_{n-1}\|, \end{aligned}$$

$$(4.17) \quad \begin{aligned} &\|N(s_n, t_n, w_n) - N(s_{n-1}, t_{n-1}, w_{n-1})\| \\ &\leq \|N(s_n, t_n, w_n) - N(s_{n-1}, t_n, w_n)\| \\ &\quad + \|N(s_{n-1}, t_n, w_n) - N(s_{n-1}, t_{n-1}, w_n)\| \\ &\quad + \|N(s_{n-1}, t_{n-1}, w_n) - N(s_{n-1}, t_{n-1}, w_{n-1})\| \\ &\leq \xi_1 \|s_n - s_{n-1}\| + \xi_2 \|t_n - t_{n-1}\| + \xi_3 \|w_n - w_{n-1}\| \\ &\leq \left(1 + \frac{1}{n}\right) (\xi_1 k_S + \xi_2 k_T + \xi_3 k_W) \|x_n - x_{n-1}\|, \end{aligned}$$

$$\begin{aligned}
& \|N(\bar{s}_n, \bar{t}_n, \bar{w}_n) - N(\bar{s}_{n-1}, \bar{t}_{n-1}, \bar{w}_{n-1})\| \\
& \leq \|N(\bar{s}_n, \bar{t}_n, \bar{w}_n) - N(\bar{s}_{n-1}, \bar{t}_n, \bar{w}_n)\| \\
& \quad + \|N(\bar{s}_{n-1}, \bar{t}_n, \bar{w}_n) - N(\bar{s}_{n-1}, \bar{t}_{n-1}, \bar{w}_n)\| \\
& \quad + \|N(\bar{s}_{n-1}, \bar{t}_{n-1}, \bar{w}_n) - N(\bar{s}_{n-1}, \bar{t}_{n-1}, \bar{w}_{n-1})\| \\
& \leq \xi_1 \|\bar{s}_n - \bar{s}_{n-1}\| + \xi_2 \|\bar{t}_n - \bar{t}_{n-1}\| + \xi_3 \|\bar{w}_n - \bar{w}_{n-1}\| \\
(4.18) \quad & \leq (1 + \frac{1}{n})(\xi_1 k_S + \xi_2 k_T + \xi_3 k_W) \|y_n - y_{n-1}\|.
\end{aligned}$$

It follows from (4.3)-(4.18) that

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
& \leq \frac{1}{\rho(\alpha - \beta)} [\|B(y_n) - B(y_{n-1})\| + \rho \|A(x_n) - A(x_{n-1})\| \\
& \quad + \rho \|N(s_n, t_n, w_n) - N(s_{n-1}, t_{n-1}, w_{n-1})\|] \\
& \leq \frac{1}{\rho(\alpha - \beta)} [\delta \|y_n - y_{n-1}\| + \rho \tau \|x_n - x_{n-1}\| \\
(4.19) \quad & \quad + \rho(1 + \frac{1}{n})(\xi_1 k_S + \xi_2 k_T + \xi_3 k_W) \|x_n - x_{n-1}\|],
\end{aligned}$$

$$\begin{aligned}
& \|y_{n+1} - y_n\| \\
& \leq \frac{1}{\lambda(\alpha - \beta)} [\|B(x_n) - B(x_{n-1})\| + \lambda \|A(x_n) - A(x_{n-1})\| \\
& \quad + \lambda \|N(\bar{s}_n, \bar{t}_n, \bar{w}_n) - N(\bar{s}_{n-1}, \bar{t}_{n-1}, \bar{w}_{n-1})\|] \\
& \leq \frac{1}{\lambda(\alpha - \beta)} [\delta \|x_n - x_{n-1}\| + \lambda \tau \|x_n - x_{n-1}\| \\
(4.20) \quad & \quad + \lambda(1 + \frac{1}{n})(\xi_1 k_S + \xi_2 k_T + \xi_3 k_W) \|y_n - y_{n-1}\|].
\end{aligned}$$

Thus

$$\begin{aligned}
& \|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\
& \leq \frac{1}{\alpha - \beta} [\tau + (1 + \frac{1}{n})(\xi_1 k_S + \xi_2 k_T + \xi_3 k_W) + \frac{1}{\lambda}(\delta + \lambda \tau)] \|x_n - x_{n-1}\| \\
& \quad + \frac{1}{\alpha - \beta} [\frac{\delta}{\rho} + (1 + \frac{1}{n})(\xi_1 k_S + \xi_2 k_T + \xi_3 k_W)] \|y_n - y_{n-1}\| \\
(4.21) \quad & \leq \theta_n [\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|],
\end{aligned}$$

where

$$\begin{aligned}
\theta_n = \max\{ & \frac{1}{\alpha - \beta} [\tau + (1 + \frac{1}{n})(\xi_1 k_S + \xi_2 k_T + \xi_3 k_W) + \frac{1}{\lambda}(\delta + \lambda \tau)], \\
& \frac{1}{\alpha - \beta} [\frac{\delta}{\rho} + (1 + \frac{1}{n})(\xi_1 k_S + \xi_2 k_T + \xi_3 k_W)] \}.
\end{aligned}$$

Let

$$\theta = \max\left\{\frac{1}{\alpha - \beta}[\tau + \xi_1 k_S + \xi_2 k_T + \xi_3 k_W + \frac{1}{\lambda}(\delta + \lambda\tau)], \frac{1}{\alpha - \beta}[\frac{\delta}{\rho} + \xi_1 k_S + \xi_2 k_T + \xi_3 k_W]\right\}.$$

Then $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$. By (4.1)-(4.2), we know that $0 < \theta < 1$ and hence there exists an $n_0 > 0$ and $\theta_0 \in (0, 1)$ such that $\theta_n \leq \theta_0$ for all $n \geq n_0$. Therefore by (4.21), we have

$$\begin{aligned} & \|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\ & \leq \theta_0[\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|] \\ & \leq \theta_0^{n-n_0}[\|x_{n_0+1} - x_{n_0}\| + \|y_{n_0+1} - y_{n_0}\|], \quad \forall n \geq n_0. \end{aligned}$$

Hence for any $m \geq n \geq n_0$, it follows that

$$\begin{aligned} (4.22) \quad \|x_m - x_n\| & \leq \sum_{i=n}^{m-1} [\|x_{i+1} - x_i\| + \|y_{i+1} - y_i\|] \\ & \leq \sum_{i=n}^{m-1} \theta_0^{i-n_0} [\|x_{n_0+1} - x_{n_0}\| + \|y_{n_0+1} - y_{n_0}\|]. \end{aligned}$$

Since $0 < \theta_0 < 1$, it follows from (4.22) that

$$\|x_m - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and hence $\{x_n\}$ is a Cauchy sequence in E . By the same argument, we also have that $\{y_n\}$ is a Cauchy sequence. Thus, there exist $x, y \in E$ such that $x_n \rightarrow x$, $y_n \rightarrow y$ as $n \rightarrow \infty$.

Now we prove that $s_n \rightarrow s \in S(x)$, $\bar{s}_n \rightarrow \bar{s} \in S(y)$, $t_n \rightarrow t \in T(x)$, $\bar{t}_n \rightarrow \bar{t} \in T(y)$, $w_n \rightarrow w \in W(x)$ and $\bar{w}_n \rightarrow \bar{w} \in W(y)$ as $n \rightarrow \infty$.

In fact, it follows from (4.10)-(4.15) that $\{s_n\}$, $\{\bar{s}_n\}$, $\{t_n\}$, $\{\bar{t}_n\}$, $\{w_n\}$, $\{\bar{w}_n\}$ are also Cauchy sequences. Therefore there exist $s, \bar{s}, t, \bar{t}, w, \bar{w} \in E$ such that $s_n \rightarrow s$, $\bar{s}_n \rightarrow \bar{s}$, $t_n \rightarrow t$, $\bar{t}_n \rightarrow \bar{t}$, $w_n \rightarrow w$ and $\bar{w}_n \rightarrow \bar{w}$ as $n \rightarrow \infty$. Further,

$$\begin{aligned} d(s, S(x)) & \leq \|s - s_n\| + d(s_n, S(x)) \\ & \leq \|s - s_n\| + D(S(x_n), S(x)) \\ & \leq \|s - s_n\| + k_S \|x_n - x\| \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

$$\begin{aligned} d(\bar{s}, S(y)) & \leq \|\bar{s} - \bar{s}_n\| + d(\bar{s}_n, S(y)) \\ & \leq \|\bar{s} - \bar{s}_n\| + D(S(y_n), S(y)) \\ & \leq \|\bar{s} - \bar{s}_n\| + k_S \|y_n - y\| \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} d(t, T(x)) &\leq \|t - t_n\| + k_T \|x_n - x\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

$$\begin{aligned} d(\bar{t}, T(y)) &\leq \|\bar{t} - \bar{t}_n\| + k_T \|y_n - y\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

$$\begin{aligned} d(w, W(x)) &\leq \|w - w_n\| + k_W \|x_n - x\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

$$\begin{aligned} d(\bar{w}, W(y)) &\leq \|\bar{w} - \bar{w}_n\| + k_W \|y_n - y\| \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since $S(x)$, $S(y)$, $T(x)$, $T(y)$, $W(x)$, $W(y)$ are closed, we have $s \in S(x)$, $\bar{s} \in S(y)$, $t \in T(x)$, $\bar{t} \in T(y)$, $w \in W(x)$, $\bar{w} \in W(y)$. By the continuities of f , A , B , N , $R_{M_1(\cdot, \cdot), \rho}^B$, $R_{M_2(\cdot, \cdot), \lambda}^B$ and Algorithm 3.1, we know that x , y , s , \bar{s} , t , \bar{t} , w , \bar{w} satisfy the following relations:

$$x = R_{M_1(\cdot, \cdot), \rho}^B[\rho a_1 + B(y) - \rho A(x) + \rho N(s, t, w)],$$

$$y = R_{M_2(\cdot, \cdot), \lambda}^B[\lambda a_2 + B(x) - \lambda A(x) + \lambda N(\bar{s}, \bar{t}, \bar{w})].$$

By Theorem 3.1, $(x, y, s, \bar{s}, t, \bar{t}, w, \bar{w})$ is a solution of problem (3.1). This completes the proof. \square

Remark 4.1. (i) If $x = y$, $\rho = 1$ and $\lambda = 0$, then Theorem 4.1 reduces to Theorem 4.2 in [9].

(ii) If $x = y$, $\rho = 1$, $\lambda = 0$, $g = I$ is an identity mapping on E and M_1 is a general H -monotone mapping in the first argument, then Theorem 4.1 reduces to Theorem 3.2 in [6].

(iii) If $x = y$, $\rho = 1$, $\lambda = 0$, $a_1 = 0$, $N = 0$, $M_1(f(x), g(x)) = M_1(f(x))$ and M_1 is a general H -monotone mapping, then Theorem 4.1 reduces to Theorem 3.4 in [11].

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