KYUNGPOOK Math. J. 53(2013), 435-457
http://dx.doi.org/10.5666/KMJ.2013.53.3.435

On the Definition of Intuitionistic Fuzzy h-ideals of Hemi -rings

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ABSTRACT. Using the Lukasiewicz 3-valued implication operator, the notion of an (α, β)-intuitionistic fuzzy left (right) h-ideal of a hemiring is introduced, where $\alpha, \beta \in \{\in, q, \in \wedge q, \in \vee q\}$. We define intuitionistic fuzzy left (right) h-ideal with thresholds (s, t) of a hemiring R and investigate their various properties. We characterize intuitionistic fuzzy left (right) h-ideal with thresholds (s, t) and (α, β)-intuitionistic fuzzy left (right) h-ideal of a hemiring R by its level sets. We establish that an intuitionistic fuzzy set A of a hemiring R is a $(\in, \in)$ (or $(\in, \in \vee q)$ or $(\in \wedge q, \in)$)-intuitionistic fuzzy left (right) h-ideal of R if and only if A is an intuitionistic fuzzy left (right) h-ideal with thresholds $(0, 1)$ (or $(0, 0.5)$ or $(0.5, 1)$) of R respectively. It is also shown that A is a $(\in, \in)$ (or $(\in, \in \vee q)$ or $(\in \wedge q, \in)$)-intuitionistic fuzzy left (right) h-ideal if and only if for any $p \in (0, 1]$ (or $p \in (0, 0.5]$ or $p \in (0.5, 1]$), A_p is a fuzzy left (right) h-ideal. Finally, we prove that an intuitionistic fuzzy set A of a hemiring R is an intuitionistic fuzzy left (right) h-ideal with thresholds (s, t) of R if and only if for any $p \in (s, t]$, the cut set A_p is a fuzzy left (right) h-ideal of R.

1. Introduction

In abstract algebra, algebraic structures like semirings, play an important role in mathematics and numerous applications of this fundamental structures are seen in many disciplines such as combinatorics, functional analysis, graph theory, theoretical computer sciences, automata theory, information sciences, quantum physics, control engineering, discrete event dynamical systems and so on. From an algebraic point of view, hemirings (see [14, 16]) (semirings with zero and commutative addition) are an important generalization of rings. Ideals of semirings play a vital role

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* Corresponding Author.
Received June 3, 2011; accepted August 1, 2012.
2010 Mathematics Subject Classification: 08A72,16Y60,03B52.
Key words and phrases: Intuitionistic fuzzy set, Fuzzy h-ideal, Intuitionistic fuzzy ideal, Intuitionistic fuzzy h-ideal, Lukasiewicz implication operator.

in structure theory and are useful for many purposes.

Fuzzy set was introduced by Zadeh [24] in 1965, and since then the researchers have been carrying out research in various concepts of abstract algebra in fuzzy setting. Fuzzy subgroups of a group was introduced by Rosenfeld [21] in 1971. Consequently, many generalizations of this fundamental concept have been done. As an important generalization of Rosenfeld's fuzzy group, Bhakat and Das in [4, 5, 6], defined a new kind of fuzzy subgroups of a group using the notion of belongs to ($\in$) and quasi-coincident of a fuzzy point to a fuzzy set of the group. Based on that, Dudek et al., [11] introduced different types of (α, β)-fuzzy ideals of a hemiring. Davvaz et al. in [9, 10], generalized the concept to H_v-submodules and redefined fuzzy H_v-submodules by applying many valued implication operators. As a generalization of a fuzzy set, intuitionistic fuzzy set was introduced by Atanassov [1], also see [2, 3]. Since then various concepts of fuzzy setting has been generalized to intuitionistic fuzzy set. Different types of (α, β)-intuitionistic fuzzy subgroups A of a group using the notions of grades of a fuzzy point belongs to A or quasi-coincident with A or belongs to and quasi-coincident ($\in \wedge q$) or belongs to or quasi-coincident ($\in \vee q$) has been introduced in [22].

In this article, using the notions of grades of a fuzzy point an (α, β)-intuitionistic fuzzy h-ideals is defined by applying the Lukasiewicz 3-valued implication operator. We define intuitionistic fuzzy h-ideals with thresholds (s, t) of a hemiring R . It is established that, for $\alpha \neq \in \wedge q$, the support of an (α, β)-intuitionistic fuzzy left (resp. right) h-ideal of a hemiring R is a left (resp. right) h-ideal R. We prove that the level set of an intuitionistic fuzzy left (resp. right) h-ideal with thresholds (s, t) of a hemiring R is a left (resp. right) h-ideal of R. We obtain necessary and sufficient conditions between (α, β)-intuitionistic fuzzy left (resp. right) h-ideal and intuitionistic fuzzy left (resp. right) h-ideal with thresholds (s, t). It is established that an intuitionistic fuzzy set A of a hemiring R is a $(\in, \in)$ (or $(\in, \in \vee q$) or $(\in \wedge q, \in)$)-intuitionistic fuzzy left (resp. right) h-ideal of R if and only if A is an intuitionistic fuzzy left (resp. right) h-ideal with thresholds $(0, 1)$ (or $(0, 0.5)$ or $(0.5, 1)$) of R respectively. We establish that A is a $(\in, \in)$ (or $(\in, \in \vee q$) or $(\in \wedge q, \in)$)-intuitionistic fuzzy left (resp. right) h-ideal of a hemiring R if and only if for any $p \in (0, 1]$ (or $p \in (0, 0.5]$ or $p \in (0.5, 1]$), A_p is a fuzzy left (resp. right) h-ideal of R respectively. Finally, we show that an intuitionistic fuzzy set of a hemiring is an intuitionistic fuzzy left (resp. right) h-ideal with thresholds (s, t) of the ring if and only if for any $p \in (s, t]$, the cut set A_p is a fuzzy left (resp. right) h-ideal of R.

2. Preliminaries

A semiring is an algebraic system $(R, +, .)$ consisting of a nonempty set R together with two binary operations on R called addition and multiplication (denoted in the usual manner) such that $(R, +)$ and $(R, .)$ are semigroups and for all $x, y, z \in R$, the following distributive laws hold:

$x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.

An element $0 \in R$ such that $0x = x0 = 0$ and $0 + x = x + 0 = x$ for all $x \in R$ is known as zero. A semiring with zero and a commutative semigroup $(R, +)$ is called a hemiring.

A nonempty subset A of R is said to be a left (resp. right) ideal if it is closed with respect to the addition and satisfies $RA \subseteq A$ (resp. $AR \subseteq A$). A left (resp. right) ideal A is called a left (resp. right) h-ideal if for any $x, z \in R$ and $a, b \in A$, $x + a + z = b + z$ implies $x \in A$.

A Fuzzy set is defined as follows:

Definition 2.1([24]). By a fuzzy set of a non-empty set X, we mean any mapping μ from X to $[0, 1]$. By $[0, 1]^X$ we will denote the set of all fuzzy subsets of X.

For each fuzzy set μ in X and any $\alpha \in [0, 1]$, we define two sets,

$U(\mu,\alpha)=\{x \in X|\mu(x) \geq \alpha\}$ and $L(\mu, \alpha) = \{x \in X|\mu(x) \leq \alpha\}$,

which are called an upper level cut and a lower level cut of μ respectively. The complement of μ, denoted by μ^c, is the fuzzy set on X defined by $\mu^c(x) = 1 - \mu(x)$.

Definition 2.2([4]). Let $x \in X$ and $t \in (0, 1]$, then a fuzzy subset $\mu \in [0, 1]^X$ is called a fuzzy point if

$$\mu(y) = \begin{cases} t, & \text{if } y = x; \\ 0, & \text{for } y \neq x. \end{cases}$$

and it is denoted by x_t.

Definition 2.3([4]). Let μ be a fuzzy subset of X and x_a be a fuzzy point. Then

(1) If $\mu(x) \geq a$, then we say x_a belongs to μ, and is denoted $x_a \in \mu$.

(2) If $\mu(x) + a > 1$, then we say x_a is quasi-coincident with μ, and is denoted $x_a q\mu$.

(3) $x_a \in \wedge q\mu \Leftrightarrow x_a \in \mu$ and $x_a q\mu$.

(4) $x_a \in \vee q\mu \Leftrightarrow x_a \in \mu$ or $x_a q\mu$.

The symbol $\overline{\alpha}$ means that α does not hold.

Let $\mu, \sigma \in [0, 1]^X$, then the intersection and union of μ and σ are given by the fuzzy sets $\mu \cap \sigma$ and $\mu \cup \sigma$ and are defined as follows:

(1) $(\mu \cap \sigma)(x)= \mu(x) \wedge \sigma(x)$;

(2)$(\mu \cup \sigma)(x)= \mu(x) \vee \sigma(x)$,

where $\mu(x) \wedge \sigma(x) = min\{\mu(x), \sigma(x)\}$ and $\mu(x) \vee \sigma(x) = max\{\mu(x), \sigma(x)\}$.

Definition 2.4([17]). A fuzzy set μ of a hemiring R is called a fuzzy left (resp. right) ideal, if for all $x, y \in R$ the following two conditions hold:

(1) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$;

(2) $\mu(yx) \geq \mu(x)$ (resp. $\mu(xy) \geq \mu(x)$).

Definition 2.5([17]). A fuzzy set μ of a hemiring R is called a fuzzy left (resp. right) h-ideal, if μ is a fuzzy left (resp. right) ideal and for all $a, b, x, z \in R$ the following condition hold:

$x + a + z = b + z \longrightarrow \mu(x) \geq \mu(a) \wedge \mu(b)$.

An Intuitionistic fuzzy set (abbreviated as IFS) introduced by Atanassov in [1] is defined as follows:

Definition 2.6. An intuitionistic fuzzy set in a non-empty set X, is an object of the form
$A = \{(x, \mu_A(x), \nu_A(x)) | x \in X\}$, where μ_A and ν_A, fuzzy sets in X, denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. By IFS(X) we denote the set of all IFSs of X.

For our convenience we shall use the notation $A(x) \geq B(x)$, when $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for all $x \in X$.

Definition 2.7([12, 13])**.** An IFS $A = (\mu_A, \nu_A)$ on a hemiring R is called an intuitionistic fuzzy left h-ideal (IF left h-ideal for short) if
(1) $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$;
(2) $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$;
(3) $\mu_A(yx) \geq \mu_A(x)$;
(4) $\nu_A(yx) \leq \nu_A(x)$;
(5) $x + a + z = b + z \longrightarrow \mu_A(x) \geq \mu_A(a) \wedge \mu_A(b)$;
(6) $x + a + z = b + z \longrightarrow \nu_A(x) \leq \nu_A(a) \vee \nu_A(b)$, hold for all $a, b, x, y, z \in R$.

An IFS $A = (\mu_A, \nu_A)$ satisfying the first four conditions is called an intuitionistic fuzzy left ideal. The family of all intuitionistic fuzzy left h-ideals of a hemiring R will be denoted by IFI(R).

Definition 2.8([23]). Let $A = (\mu_A, \nu_A)$ be an IFSs of X, and $a \in [0, 1]$. Then
(1)
$$A_a(x) = \begin{cases} 1, & \text{if } \mu_A(x) \geq a; \\ \frac{1}{2}, & \text{if } \mu_A(x) < a \leq 1 - \nu_A(x); \\ 0, & \text{for } a > 1 - \nu_A(x). \end{cases}$$
and
$$A_{\underline{a}}(x) = \begin{cases} 1, & \text{if } \mu_A(x) > a; \\ \frac{1}{2}, & \text{if } \mu_A(x) \leq a < 1 - \nu_A(x); \\ 0, & \text{for } a \geq 1 - \nu_A(x). \end{cases}$$
are called the a-upper cut set and a- strong upper cut set of A, respectively.
(2)
$$A^a(x) = \begin{cases} 1, & \text{if } \nu_A(x) \geq a; \\ \frac{1}{2}, & \text{if } \nu_A(x) < a \leq 1 - \mu_A(x); \\ 0, & \text{for } a > 1 - \mu_A(x). \end{cases}$$
and
$$A^{\underline{a}}(x) = \begin{cases} 1, & \text{if } \nu_A(x) > a; \\ \frac{1}{2}, & \text{if } \nu_A(x) \leq a < 1 - \mu_A(x); \\ 0, & \text{for } a \geq 1 - \mu_A(x). \end{cases}$$

are called the a-lower cut set and a- strong lower cut set of A, respectively.
(3)

$$A_{[a]}(x) = \begin{cases} 1, & \text{if } \mu_A(x) + a \geq 1; \\ \frac{1}{2}, & \text{if } \nu_A(x) \leq a < 1 - \mu_A(x); \\ 0, & \text{for } a < \nu_A(x). \end{cases}$$

and

$$A_{[\underline{a}]}(x) = \begin{cases} 1, & \text{if } \mu_A(x) + a > 1; \\ \frac{1}{2}, & \text{if } \nu_A(x) < a \leq 1 - \mu_A(x); \\ 0, & \text{for } a \leq \nu_A(x). \end{cases}$$

are called the a-upper Q-cut set and a- strong upper Q-cut set of A, respectively.
(4)

$$A^{[a]}(x) = \begin{cases} 1, & \text{if } \nu_A(x) + a \geq 1; \\ \frac{1}{2}, & \text{if } \mu_A(x) \leq a < 1 - \nu_A(x); \\ 0, & \text{for } a < \mu_A(x). \end{cases}$$

and

$$A^{[\underline{a}]}(x) = \begin{cases} 1, & \text{if } \nu_A(x) + a > 1; \\ \frac{1}{2}, & \text{if } \mu_A(x) < a \leq 1 - \nu_A(x); \\ 0, & \text{for } a \leq \mu_A(x). \end{cases}$$

are called the a-lower Q-cut set and a- strong lower Q-cut set of A, respectively.

Property 2.9. (1) $A_{[\underline{a}]}(x) = A_{\underline{1-a}}(x)$; (2) $A_{\underline{a}} \subset A_a$, (3) $a < b \Rightarrow A_{\underline{a}} \supset A_{\underline{b}}$.

Definition 2.10([22]). Let $A = (\mu_A, \nu_A)$ be an IFS of X, and $a \in [0,1], x \in X$. Then

(1) The grades of $x_a \in A$ and $x_a qA$ denoted by $[x_a \in A]$ and $[x_a qA]$ respectively are given by the following relations:

$$[x_a \in A] = A_a(x) \text{ and } [x_a qA] = A_{[\underline{a}]}(x).$$

(2) The grades of $x_a \in \wedge qA$ and $x_a \in \vee qA$ denoted by $[x_a \in \wedge qA]$ and $[x_a \in \vee qA]$ respectively are given by the following relations:

$$[x_a \in \wedge qA] = [x_a \in A] \wedge [x_a qA] = A_a(x) \wedge A_{[\underline{a}]}(x)$$

and

$$[x_a \in \vee qA] = [x_a \in A] \vee [x_a qA] = A_a(x) \vee A_{[\underline{a}]}(x).$$

(3) The grades of $x_a \overline{\in} A$ and $x_a \overline{q} A$ denoted by $[x_a \overline{\in} A]$ and $[x_a \overline{q} A]$ respectively are given by the following relations:

$$[x_a \overline{\in} A] = A^a(x) \text{ and } [x_a \overline{q} A] = A^{[\underline{a}]}(x).$$

(4) The grades of $x_a \overline{\in \wedge q} A$ and $x_a \overline{\in \vee q} A$ denoted by $[x_a \overline{\in \wedge q} A]$ and $[x_a \overline{\in \vee q} A]$ respectively are given by the following relations:

$$[x_a \overline{\in \wedge q} A] = [x_a \overline{\in} \vee \overline{q} A] = [x_a \overline{\in} A] \vee [x_a \overline{q} A] = A^a(x) \vee A^{\underline{[a]}}(x)$$

and

$$[x_a \overline{\in \vee q} A] = [x_a \overline{\in} \wedge \overline{q} A] = [x_a \overline{\in} A] \wedge [x_a \overline{q} A] = A^a(x) \wedge A^{\underline{[a]}}(x).$$

$\rightarrow$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

Table 1: The table of truth value of Lukasiewicz implication.

Property 2.11([22]). (1) $[x_a \overline{\in} A] = [x_a \in A^c],\ [x_a \overline{q} A] = [x_a q A^c]$.
(2) $[x_a \overline{\in} \wedge \overline{q} A] = [x_a \in \wedge q A^c],\ [x_a \overline{\in} \vee \overline{q} A] = [x_a \in \vee q A^c]$.
(3) $[x_a \in (\bigcap_{t\in T} A_t)] = \bigwedge_{t\in T}[x_a \in A],\ [x_a q(\bigcup_{t\in T} A_t)] = \bigvee_{t\in T}[x_a q A]$.
(4) $[x_a \overline{\in}(\bigcup_{t\in T} A_t)] = \bigwedge_{t\in T}[x_a \overline{\in} A],\ [x_a \overline{q}(\bigcap_{t\in T} A_t)] = \bigvee_{t\in T}[x_a \overline{q} A]$.

In the following sections we present our main results.

3. (α, β)-intuitionistic Fuzzy Ideals

Let R be a hemiring and $\alpha, \beta \in \{\in,\ q,\ \in \wedge q,\ \in \vee q\}$. Then for $a \in [0,1]$, $x \in R$, x_a is a fuzzy point and $[x_a \alpha A]$, $[x_a \beta A] \in \{0, 1/2, 1\}$.

Definition 3.1. Let R be a hemiring and $A = (\mu_A, \nu_A)$ be an IF set in R. If for any $\alpha, \beta \in \{\in,\ q,\ \in \wedge q,\ \in \vee q\}$, $s, t \in (0,1]$, the following conditions are satisfied
(1) for all $x, y \in R$, $([x_s \alpha A] \wedge [y_t \alpha A] \rightarrow [(x_s + y_t)\beta A]) = 1$;
(2) for all $x, y \in R$, $([x_s \alpha A] \rightarrow [(y_s x_s)\beta A]) = 1$,
(resp. $([x_s \alpha A] \rightarrow [(x_s y_s)\beta A]) = 1$);
then A is called an (α, β)- intuitionistic fuzzy left (resp. right) ideal of R, where $x_s + y_t = (x+y)_{s\wedge t}$, and $y_s x_s = (yx)_s$ (resp. $x_s y_s = (xy)_s$).

It is noted that, for $p,\ q \in \{0,\ 1/2,\ 1\}$, we have from the **Table1**, $(p \rightarrow q) = 1 \Leftrightarrow q \geq p$. Therefore, Definition 3.1. is equivalent to the following definition.

Definition 3.2. Let R be a hemiring and $A = (\mu_A, \nu_A)$ be an IF set in R. If for any $\alpha, \beta \in \{\in,\ q,\ \in \wedge q,\ \in \vee q\}$, $s, t \in (0,1]$, the following conditions are satisfied
(1) for all $x, y \in R$, $[(x_s + y_t)\beta A] \geq [x_s \alpha A] \wedge [y_t \alpha A]$;
(2) for all $x, y \in R$, $[y_s x_s \beta A] \geq [x_s \alpha A]$,
(resp. $[x_s y_s \beta A] \geq [x_s \alpha A]$);
then A is called an (α, β)- intuitionistic fuzzy left (resp. right) ideal of R, where $x_s + y_t = (x+y)_{s\wedge t}$, and $y_s x_s = (yx)_s$ (resp. $x_s y_s = (xy)_s$).

Theorem 3.3. *Let $A = (\mu_A, \nu_A)$ be a non-zero (i.e. $A \neq (0, 1)$) (α, β)-intuitionistic fuzzy left (resp. right) ideal of a hemiring R. If $\alpha \neq \in \wedge q$, then $A_{\underline{0}}$ is a fuzzy left (resp. right) ideal of R.*

Proof. To show (1) for all $x, y \in R$, $A_{\underline{0}}(x+y) \geq A_{\underline{0}}(x) \wedge A_{\underline{0}}(y)$,

(2) for all $x, y \in R$, $A_{\underline{0}}(yx) \geq A_{\underline{0}}(x)$, (respectively $A_{\underline{0}}(xy) \geq A_{\underline{0}}(x)$)

(1) First we show $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) = 1 \Rightarrow A_{\underline{0}}(x+y) = 1$.

Let $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) = 1$. Then $A_{\underline{0}}(x) = 1$, $A_{\underline{0}}(y) = 1$, and so $\mu_A(x) > 0$, $\mu_A(y) > 0$. Put $t = \mu_A(x) \wedge \mu_A(y)$, then $t > 0$. Therefore, we must have $s \in (0, 1)$ such that $0 < 1 - s < t = \mu_A(x) \wedge \mu_A(y)$. Also, we have $\mu_A(x) \geq t$, $\mu_A(y) \geq t$. Thus we have $[x_t \in A] = 1$, $[y_t \in A] = 1$, $[x_s qA] = 1$ and $[y_s qA] = 1$.

If $\alpha = \in$ or $\alpha = \in \vee q$, then $[x_t \alpha A] = 1$, $[y_t \alpha A] = 1$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 3.2., $1 \geq [(x_t + y_t)\beta A] \geq [x_t \alpha A] \wedge [y_t \alpha A] = 1$. Therefore,

$[(x+y)_t \beta A] = 1$,

$\Rightarrow$ either $A_t(x+y) = 1$ or $A_{[\underline{t}]}(x+y) = 1$,

$\Rightarrow$ either $\mu_A(x+y) \geq t > 0$ or $\mu_A(x+y) > 1 - t \geq 0$,

$\Rightarrow \mu_A(x+y) > 0 \Rightarrow A_{\underline{0}}(x+y) = 1$.

If $\alpha = q$, then $[x_s \alpha A] = 1$ and $[y_s \alpha A] = 1$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 3.2., $1 \geq [(x_s + y_s)\beta A] \geq [x_s \alpha A] \wedge [y_s \alpha A] = 1$. Therefore,

$[(x+y)_s \beta A] = 1$,

$\Rightarrow$ either $A_s(x+y) = 1$ or $A_{[\underline{s}]}(x+y) = 1$,

$\Rightarrow$ either $\mu_A(x+y) \geq s > 0$ or $\mu_A(x+y) > 1 - s \geq 0$,

$\Rightarrow \mu_A(x+y) > 0 \Rightarrow A_{\underline{0}}(x+y) = 1$.

Next we claim $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) = 1/2 \Rightarrow A_{\underline{0}}(x+y) \geq 1/2$. Let $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) = 1/2$. Then $A_{\underline{0}}(x) \geq 1/2$ and $A_{\underline{0}}(y) \geq 1/2$, and so $\nu_A(x) < 1$ and $\nu_A(y) < 1$. Thus $\nu_A(x) \vee \nu_A(y) < 1$. So, there exists $s, t \in (0, 1)$ such that $\nu_A(x) \vee \nu_A(y) < 1 - t < s < 1$. Then $0 < t < 1 - \nu_A(x) \vee \nu_A(y) = (1 - \nu_A(x)) \wedge (1 - \nu_A(y))$ implies $1 - \nu_A(x) > t$ and $1 - \nu_A(y) > t$. Thus $[x_t \in A] \geq 1/2$ and $[y_t \in A] \geq 1/2$. Also, $\nu_A(x) \vee \nu_A(y) < s < 1$ implies, $\nu_A(x) < s$ and $\nu_A(y) < s$. Thus $[x_s qA] \geq 1/2$ and $[y_s qA] \geq 1/2$.

If $\alpha = \in$ or $\alpha = \in \vee q$, then $[x_t \alpha A] \geq 1/2$, $[y_t \alpha A] \geq 1/2$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 3.2., $[(x_t + y_t)\beta A] \geq [x_t \alpha A] \wedge [y_t \alpha A] \geq 1/2$. Therefore,

$[(x+y)_t \beta A] \geq 1/2$,

$\Rightarrow A_t(x+y) \geq 1/2$ or $A_{[\underline{t}]}(x+y) \geq 1/2$,

$\Rightarrow \nu_A(x+y) \leq 1 - t < 1 - 0$ or $\nu_A(x+y) < t < 1 - 0$,

$\Rightarrow \nu_A(x+y) < 1 - 0 \Rightarrow A_{\underline{0}}(x+y) \geq 1/2$.

If $\alpha = q$, then $[x_s \alpha A] \geq 1/2$ and $[y_s \alpha A] \geq 1/2$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 3.2., $[(x_s + y_s)\beta A] \geq [x_s \alpha A] \wedge [y_s \alpha A] \geq 1/2$. Therefore,

$[(x+y)_s \beta A] \geq 1/2$,

$\Rightarrow$ either $A_s(x+y) \geq 1/2$ or $A_{[\underline{s}]}(x+y) \geq 1/2$,

$\Rightarrow$ either $\nu_A(x+y) \leq 1 - s < 1$ or $\nu_A(x+y) < s < 1$,

$\Rightarrow \nu_A(x+y) < 1 \Rightarrow A_{\underline{0}}(x+y) \geq 1/2$.

Hence we have $A_{\underline{0}}(x+y) \geq A_{\underline{0}}(x) \wedge A_{\underline{0}}(y)$, for all $x, y \in R$.

(2) First we prove, $A_{\underline{0}}(x) = 1 \Rightarrow A_{\underline{0}}(yx) = 1$. Let $A_{\underline{0}}(x) = 1$. Then we have $\mu_A(x) > 0$. Put $t = \mu_A(x)$, then $t > 0$. Therefore, we must have $s \in (0,1)$ such that $0 < 1 - s < t = \mu_A(x)$. Thus $[x_t \in A] = 1$ and $[x_s qA] = 1$.

If $\alpha = \in$ or $\alpha = \in \vee q$, then $[x_t \alpha A] = 1$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 3.2., $1 \geq [y_t x_t \beta A] \geq [x_t \alpha A] = 1$. Therefore,

$[(yx)_t \beta A] = 1$,

$\Rightarrow$ either $A_t(yx) = 1$ or $A_{[\underline{t}]}(yx) = 1$,

$\Rightarrow$ either $\mu_A(yx) \geq t > 0$ or $\mu_A(yx) > 1 - t \geq 0$,

$\Rightarrow \mu_A(yx) > 0 \Rightarrow A_{\underline{0}}(yx) = 1$.

If $\alpha = q$, then $[x_s qA] = 1$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 3.2., $1 \geq [(y_s x_s)\beta A] \geq [x_s \alpha A] = 1$. Therefore,

$[(yx)_s \beta A] = 1$,

$\Rightarrow$ either $A_s(yx) = 1$ or $A_{[\underline{s}]}(yx) = 1$,

$\Rightarrow$ either $\mu_A(yx) \geq s > 0$ or $\mu_A(yx) > 1 - s \geq 0$,

$\Rightarrow \mu_A(yx) > 0 \Rightarrow A_{\underline{0}}(yx) = 1$.

Next we show, $A_{\underline{0}}(x) = 1/2 \Rightarrow A_{\underline{0}}(yx) \geq 1/2$. Let $A_{\underline{0}}(x) = 1/2$. Then $\nu_A(x) < 1$. So, there exists $s, t \in (0,1)$ such that $\nu_A(x) < 1 - t < s < 1$. Then $0 < t < 1 - \nu_A(x)$, and so $A_t(x) \geq 1/2$. Thus $[x_t \in A] \geq 1/2$ and $A_{[\underline{s}]}(x) \geq 1/2 \Rightarrow [x_s qA] \geq 1/2$.

If $\alpha = \in$ or $\alpha = \in \vee q$, then $[x_t \alpha A] \geq 1/2$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 3.2., $[y_t x_t \beta A] \geq [x_t \alpha A] \geq 1/2$. Therefore,

$[(yx)_t \beta A] \geq 1/2$,

$\Rightarrow$ either $A_t(yx) \geq 1/2$ or $A_{[\underline{t}]}(yx) \geq 1/2$,

$\Rightarrow$ either $\nu_A(yx) \leq 1 - t < 1 - 0$ or $\nu_A(yx) < t < 1 - 0$,

$\Rightarrow \nu_A(yx) < 1 - 0 \Rightarrow A_{\underline{0}}(yx) \geq 1/2$.

If $\alpha = q$, then $[x_s \alpha A] \geq 1/2$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, we have from Definition 3.2., $[y_s x_s \beta A] \geq [x_s \alpha A] \geq 1/2$. Therefore,

$[(yx)_s \beta A] \geq 1/2$,

$\Rightarrow$ either $A_s(yx) \geq 1/2$ or $A_{[\underline{s}]}(yx) \geq 1/2$,

$\Rightarrow$ either $\nu_A(yx) \leq 1 - s < 1$ or $\nu_A(yx) < s < 1$,

$\Rightarrow \nu_A(yx) < 1 \Rightarrow A_{\underline{0}}(yx) \geq 1/2$.

Thus we have $A_{\underline{0}}(yx) \geq A_{\underline{0}}(x)$, for all $x, y \in R$. Similarly, if A is an (α, β) IF right ideal of R, then $A_{\underline{0}}(xy) \geq A_{\underline{0}}(x)$, for all $x, y \in R$. Hence $A_{\underline{0}}$ is a fuzzy left (resp. right) ideal of R. □

Definition 3.4. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X. Then by the support of A, we mean a crisp subset, A^* of X, and it is defined as follows:

$$A^* = \{x \in X | \mu_A(x) \vee (1 - \nu_A(x)) > 0\}$$

That is , $A^* = \{x \in X | A_{\underline{0}}(x) > 0\}$.

Theorem 3.5. *Let $A = (\mu_A, \nu_A)$ be a non-zero (α, β)-intuitionistic fuzzy left (resp. right) ideal of a hemiring R. If $\alpha \neq \in \wedge q$, then the support A^* is a left (resp.*

right) ideal of R.

Proof. Let $x, y \in A^*$ and $r \in R$. Then $A_{\underline{0}}(x) > 0$ and $A_{\underline{0}}(y) > 0$. From Theorem 3.3., we have $A_{\underline{0}}(x+y) \geq A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) > 0$. Thus $x + y \in A^*$. Also, $A_{\underline{0}}(rx) \geq A_{\underline{0}}(x) > 0$, because $A_{\underline{0}}(x) > 0$, and so $rx \in A^*$. Similarly, $xr \in A^*$ if A is an (α, β)-intuitionistic fuzzy right ideal of a hemiring R. Hence A^* is a left (right) ideal of R. □

4. (α, β)-intuitionistic Fuzzy h-ideals

Definition 4.1. Let R be a hemiring and $A = (\mu_A, \nu_A)$ be an IF set in R. If for any $\alpha, \beta \in \{\in, q, \in \wedge q, \in \vee q\}$, $s, t \in (0, 1]$, A is an (α, β)- intuitionistic fuzzy left (resp. right) ideal of R and for all $x, z, a, b \in R$, $x + a + z = b + z$ implies $([a_s \alpha A] \wedge [b_t \alpha A] \to [x_{s \wedge t} \beta A]) = 1$, then A is called an (α, β)- intuitionistic fuzzy left (resp. right) h-ideal of R.

In view of Table1, the Definition 4.1. is equivalent to the following definition.

Definition 4.2. Let R be a hemiring ring and $A = (\mu_A, \nu_A)$ be an IF set in R. If for any $\alpha, \beta \in \{\in, q, \in \wedge q, \in \vee q\}$, $s, t \in (0, 1]$, A is an (α, β)- intuitionistic fuzzy left (resp. right) ideal of R and for all $x, z, a, b \in R$, $x + a + z = b + z$ implies $[x_{s \wedge t} \beta A] \geq [a_s \alpha A] \wedge [b_t \alpha A]$, then A is called an (α, β)- intuitionistic fuzzy left (resp. right) h-ideal of R.

It is noted that a $(\in, \in)$-IF h-ideal is also a $(\in, \in \vee q)$-IF h-ideal.

Theorem 4.3. *Let $A = (\mu_A, \nu_A)$ be a non-zero (α, β)-intuitionistic fuzzy left (resp. right) $h-$ideal of a hemiring R. If $\alpha \neq \in \wedge q$, then $A_{\underline{0}}$ is a fuzzy left (resp. right) h-ideal of R.*

Proof. In view of Theorem 3.3, it is sufficient to show that for all $x, z, a, b \in R$, $x + a + z = b + z$ implies $A_{\underline{0}}(x) \geq A_{\underline{0}}(a) \wedge A_{\underline{0}}(b)$.

Let $x, z, a, b \in R$ be such that $x + a + z = b + z$. First we show $A_{\underline{0}}(a) \wedge A_{\underline{0}}(b) = 1 \Rightarrow A_{\underline{0}}(x) = 1$.

Let $A_{\underline{0}}(a) \wedge A_{\underline{0}}(b) = 1$. Then $A_{\underline{0}}(a) = 1$, $A_{\underline{0}}(b) = 1$, and so $\mu_A(a) > 0$, $\mu_A(b) > 0$. Put $t = \mu_A(a) \wedge \mu_A(b)$, then $t > 0$. Therefore, we must have $s \in (0, 1)$ such that $0 < 1 - s < t = \mu_A(a) \wedge \mu_A(b)$. Also, we have $\mu_A(a) \geq t$, $\mu_A(b) \geq t$. Thus we have $[a_t \in A] = 1$, $[b_t \in A] = 1$, $[a_s qA] = 1$ and $[b_s qA] = 1$.

If $\alpha = \in$ or $\alpha = \in \vee q$, then $[a_t \alpha A] = 1$, $[b_t \alpha A] = 1$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 4.1., $1 \geq [x_t \beta A] \geq [a_t \alpha A] \wedge [b_t \alpha A] = 1$. Therefore,

$[x_t \beta A] = 1$,

$\Rightarrow$ either $A_t(x) = 1$ or $A_{[\underline{t}]}(x) = 1$,

$\Rightarrow$ either $\mu_A(x) \geq t > 0$ or $\mu_A(x) > 1 - t \geq 0$,

$\Rightarrow \mu_A(x) > 0 \Rightarrow A_{\underline{0}}(x) = 1$.

If $\alpha = q$, then $[a_s \alpha A] = 1$ and $[b_s \alpha A] = 1$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 4.1., $1 \geq [x_s \beta A] \geq [a_s \alpha A] \wedge [b_s \alpha A] = 1$. Therefore,

$[x_s \beta A] = 1$,

$\Rightarrow$ either $A_s(x) = 1$ or $A_{[\underline{s}]}(x) = 1$,
$\Rightarrow$ either $\mu_A(x) \geq s > 0$ or $\mu_A(x) > 1 - s \geq 0$,
$\Rightarrow \mu_A(x) > 0 \Rightarrow A_{\underline{0}}(x) = 1$.

Next we claim $A_{\underline{0}}(a) \wedge A_{\underline{0}}(b) = 1/2 \Rightarrow A_{\underline{0}}(x) \geq 1/2$. Let $A_{\underline{0}}(a) \wedge A_{\underline{0}}(b) = 1/2$. Then $A_{\underline{0}}(a) \geq 1/2$ and $A_{\underline{0}}(b) \geq 1/2$, and so $\nu_A(a) < 1$ and $\nu_A(b) < 1$. Thus $\nu_A(a) \vee \nu_A(b) < 1$. So, there exists $s, t \in (0, 1)$ such that $\nu_A(a) \vee \nu_A(b) < 1-t < s < 1$. Then $0 < t < 1-\nu_A(a) \vee \nu_A(b) = (1-\nu_A(a)) \wedge (1-\nu_A(b))$ implies $1-\nu_A(a) > t$ and $1-\nu_A(b) > t$. Thus $[a_t \in A] \geq 1/2$ and $[b_t \in A] \geq 1/2$. Also, $\nu_A(a) \vee \nu_A(b) < s < 1$ implies, $\nu_A(a) < s$ and $\nu_A(b) < s$. Thus $[a_s q A] \geq 1/2$ and $[b_s q A] \geq 1/2$.

If $\alpha = \in$ or $\alpha = \in \vee q$, then $[a_t \alpha A] \geq 1/2$, $[b_t \alpha A] \geq 1/2$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 4.1., $[x_t \beta A] \geq [a_t \alpha A] \wedge [b_t \alpha A] \geq 1/2$. Therefore,
$[x_t \beta A] \geq 1/2$,
$\Rightarrow A_t(x) \geq 1/2$ or $A_{[\underline{t}]}(x) \geq 1/2$,
$\Rightarrow \nu_A(x) \leq 1 - t < 1 - 0$ or $\nu_A(x) < t < 1 - 0$,
$\Rightarrow \nu_A(x) < 1 - 0 \Rightarrow A_{\underline{0}}(x) \geq 1/2$.

If $\alpha = q$, then $[a_s \alpha A] \geq 1/2$ and $[b_s \alpha A] \geq 1/2$, and so for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from Definition 4.1., $[x_s \beta A] \geq [a_s \alpha A] \wedge [b_s \alpha A] \geq 1/2$. Therefore,
$[x_s \beta A] \geq 1/2$,
$\Rightarrow$ either $A_s(x) \geq 1/2$ or $A_{[\underline{s}]}(x) \geq 1/2$,
$\Rightarrow$ either $\nu_A(x) \leq 1 - s < 1$ or $\nu_A(x) < s < 1$,
$\Rightarrow \nu_A(x) < 1 \Rightarrow A_{\underline{0}}(x) \geq 1/2$.

Thus we have for all $x, z, a, b \in R$, $x+a+z = b+b$ implies $A_{\underline{0}}(x) \geq A_{\underline{0}}(a) \wedge A_{\underline{0}}(b)$. Hence $A_{\underline{0}}$ is a fuzzy left (resp. right) h-ideal of R. □

Theorem 4.4. *Let $A = (\mu_A, \nu_A)$ be a non-zero (α, β)-intuitionistic fuzzy left (resp. right) h-ideal of R. If $\alpha \neq \in \wedge q$, then the support A^* is a left (resp. right) h-ideal of R.*

Proof. In view of Theorem 3.5., it is sufficient to show for all $x, z \in R$, $a, b \in A^*$, $x + a + z = b + z$ implies $x \in A^*$. Now $a, b \in A^*$ implies $A_{\underline{0}}(a) > 0$ and $A_{\underline{0}}(b) > 0$. Since $x + a + z = b + z$, so by Theorem 4.3. we have $A_{\underline{0}}(x) \geq A_{\underline{0}}(a) \wedge A_{\underline{0}}(b) > 0$. Hence $x \in A^*$. □

5. Intuitionistic Fuzzy h-ideals with Thresholds

Definition 5.1. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set of a hemiring R. Then A is said to be an intuitionistic fuzzy left (resp. right) h-ideal with thresholds (s, t) of R, if it satisfies the following properties:

(1) for all $x, y \in R$, $\mu_A(x + y) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t$;
(2) for all $x, y \in R$, $\nu_A(x + y) \wedge (1 - s) \leq \nu_A(x) \vee \nu_A(y) \vee (1 - t)$;
(3) for all $x, y \in R$, $\mu_A(yx) \vee s \geq \mu_A(x) \wedge t$,
(resp. $\mu_A(xy) \vee s \geq \mu_A(x) \wedge t$);
(4) for all $x, y \in R$, $\nu_A(yx) \wedge (1 - s) \leq \nu_A(x) \vee (1 - t)$,
(resp. $\nu_A(xy) \wedge (1 - s) \leq \nu_A(x) \vee (1 - t)$);
(5) for all $x, z, a, b \in R$, $x + a + z = b + z$ implies $\mu_A(x) \vee s \geq \mu_A(a) \wedge \mu_A(b) \wedge t$;

(6) for all $x, z, a, b \in R$, $x + a + z = b + z$ implies $\nu_A(x) \wedge (1 - s) \leq \nu_A(a) \vee \nu_A(b) \vee (1 - t)$,
where $s, t \in [0, 1]$.

An IFS $A = (\mu_A, \nu_A)$ of R satisfying the first four conditions is called an intuitionistic fuzzy left (resp. right) ideal with thresholds (s, t) of R.

Theorem 5.2. *An IF set $A = (\mu_A, \nu_A)$ in a hemiring R, is an intuitionistic fuzzy left (resp. right) h-ideal with thresholds $(0, 1)$ of R if and only if A is an intuitionistic fuzzy left (resp. right) h-ideal of R.*

Example 5.3. Consider the hemiring $R = \{0, 1, 2, 3\}$ with addition and multiplication operations defined as follows:

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

and

·	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

Let $A = (\mu_A, \nu_A)$ be an IFS in R defined by $\mu_A(0) = 0.4$, $\mu_(x) = 0.2$ and $\mu_A(0) = 0.2$, $\nu_A(x) = 0.7$ for $x \neq 0$. Then A is an IF h-ideal of R (See [12]). It can be easily verified that A is an $(\in, \in)$, $(\in, \in \vee q)$-IF h-ideal of R. Moreover, A is an IF h-ideal of R with thresholds $(0, 1)$.

Definition 5.4. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X and $\alpha \in [0, 1]$. Then by a α-level set of A, we mean a crisp subset, $A_{\overline{\alpha}}$ of X, and it is defined as follows:

$$A_{\overline{\alpha}} = \{x \in X | [x_\alpha \in A] > 0\}$$

Theorem 5.5. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy left (resp. right) h-ideal with thresholds (s, t) of R. If for any $p \in (s, t]$, $A_{\overline{p}}$ is a non-empty subset of R, then $A_{\overline{p}}$ is a left (resp. right) h-ideal of R.*

Proof. Let $x, y \in A_{\overline{p}} = \{x \in R | [x_p \in A] > 0\}$. Then $[x_p \in A] > 0$ and $[y_p \in A] > 0$, which implies $p \leq 1 - \nu_A(x)$ and $p \leq 1 - \nu_A(y)$. Now $\nu_A(x + y) \wedge (1 - s) \leq (\nu_A(x) \vee \nu_A(y)) \vee (1 - t)$, implies $(1 - \nu_A(x + y)) \vee s \geq (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \wedge t \geq p \wedge p \wedge t = p$. Thus $1 - \nu_A(x + y) \geq p$, and so $[(x + y)_p \in A] \geq 1/2 > 0$. Therefore, $x + y \in A_{\overline{p}}$. Let $r \in R$. Now $\nu_A(rx) \wedge (1 - s) \leq \nu_A(x) \vee (1 - t)$, implies

$(1-\nu_A(rx)) \vee s \geq (1-\nu_A(x)) \wedge t \geq p \wedge t = p$. Thus $1-\nu_A(rx) \geq p$, and so $[(rx)_p \in A] \geq 1/2 > 0$. Therefore, $rx \in A_{\overline{p}}$. Similarly, if A is an intuitionistic fuzzy right h-ideal with thresholds (s,t) of R, then we have $xr \in A_{\overline{p}}$. Finally, let $a, b \in A_{\overline{p}}$, $x, z \in R$ be such that $x+a+z = b+z$. Then $[a_p \in A] > 0$ and $[b_p \in A] > 0$, which implies $p \leq 1-\nu_A(a)$ and $p \leq 1-\nu_A(b)$. Now $\nu_A(x) \wedge (1-s) \leq (\nu_A(a) \vee \nu_A(b)) \vee (1-t)$, implies $(1-\nu_A(x)) \vee s \geq (1-\nu_A(a)) \wedge (1-\nu_A(b)) \wedge t \geq p \wedge p \wedge t = p$. Thus $1-\nu_A(x) \geq p$, and so $[(x)_p \in A] \geq 1/2 > 0$. Therefore, $x \in A_{\overline{p}}$. Hence $A_{\overline{p}}$ is a left (resp. right) h-ideal of R. □

Theorem 5.6. *An IFS $A = (\mu_A, \nu_A)$ of a hemiring R is a $(\in, \in)$-intuitionistic fuzzy left (resp. right) h-ideal of R if and only if A is an intuitionistic fuzzy left (resp. right) h-ideal of R with thresholds $(0, 1)$.*

Proof. Suppose, $A = (\mu_A, \nu_A)$ is a $(\in, \in)$-intuitionistic fuzzy left (resp. right) h-ideal of R. To show A is an intuitionistic fuzzy left (resp. right) h-ideal of R with thresholds $(0, 1)$. i.e. to show

(1) for all $x, y \in R$, $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$;
(2) for all $x, y \in R$, $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$;
(3) for all $x, y \in R$, $\mu_A(yx) \geq \mu_A(x)$, (resp. $\mu_A(xy) \geq \mu_A(x)$);
(4) for all $x, y \in R$, $\nu_A(yx) \leq \nu_A(x)$, (resp. $\nu_A(xy) \leq \nu_A(x)$);
(5) for all $x, z, a, b \in R$, $x+a+z = b+z$ implies $\mu_A(x) \geq \mu_A(a) \wedge \mu_A(b)$;
(6) for all $x, z, a, b \in R$, $x+a+z = b+z$ implies $\nu_A(x) \leq \nu_A(a) \vee \nu_A(b)$.

(1) Let $t = \mu_A(x) \wedge \mu_A(y)$, then $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$, which implies $A_t(x) = 1$ and $A_t(y) = 1$, and so $[x_t \in A] = 1$ and $[y_t \in A] = 1$. Now $1 \geq [(x_t+y_t) \in A] \geq [x_t \in A] \wedge [y_t \in A] = 1 \Rightarrow (x_t+y_t) \in A] = 1 \Rightarrow \mu_A(x+y) \geq t = \mu_A(x) \wedge \mu_A(y)$.

(2) If $\nu_A(x+y) = 0$, then it is obvious. Let $s = \nu_A(x+y) > 0$ and let $t \in [0,1]$ be such that $t > 1-s = 1-\nu_A(x+y)$, then we have $0 = [(x_t+y_t) \in A] \geq [x_t \in A] \wedge [y_t \in A] \Rightarrow [x_t \in A] \wedge [y_t \in A] = 0 \Rightarrow [x_t \in A] = 0$ or $[y_t \in A] = 0$ i.e., either $t > 1-\nu_A(x)$ or $t > 1-\nu_A(y) \Rightarrow$ either $\nu_A(x) > 1-t$ or $\nu_A(y) > 1-t \Rightarrow \nu_A(x) \vee \nu_A(y) > 1-t$. Therefore, $\nu_A(x) \vee \nu_A(y) \geq \vee\{1-t | t > 1-s\} = \vee\{1-t | s > 1-t\} = s = \nu_A(x+y)$. Thus $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$.

(3) Let $t = \mu_A(x)$. Then $A_t(x) = 1$, and so $[x_t \in A] = 1$. Now $1 \geq [(y_t x_t) \in A] \geq [x_t \in A] = 1 \Rightarrow [(yx)_t \in A] = 1 \Rightarrow \mu_A(yx) \geq t = \mu_A(x)$. Similarly, if A is a $(\in, \in)$-intuitionistic fuzzy right h-ideal of R, then we have $\mu_A(xy) \geq \mu_A(x)$.

(4) If $\nu_A(yx) = 0$, then it is obvious. Let $s = \nu_A(yx) > 0$ and let $t \in [0,1]$ be such that $t > 1-s = 1-\nu_A(yx)$, then we have $0 = [(yx)_t \in A] \geq [x_t \in A] \Rightarrow [x_t \in A] = 0 \Rightarrow t > 1-\nu_A(x) \Rightarrow \nu_A(x) > 1-t$. Therefore, $\nu_A(x) \geq \vee\{1-t | t > 1-s\} = \vee\{1-t | s > 1-t\} = s = \nu_A(yx)$. Thus $\nu_A(yx) \leq \nu_A(x)$. Similarly, if A is a $(\in, \in)$-intuitionistic fuzzy right h-ideal of R, then we have $\nu_A(xy) \leq \nu_A(x)$.

(5) Let $t = \mu_A(a) \wedge \mu_A(b)$, then $\mu_A(a) \geq t$ and $\mu_A(b) \geq t$, which implies $A_t(a) = 1$ and $A_t(b) = 1$, and so $[a_t \in A] = 1$ and $[b_t \in A] = 1$. Now $1 \geq [x_t \in A] \geq [a_t \in A] \wedge [b_t \in A] = 1 \Rightarrow [x_t \in A] = 1 \Rightarrow \mu_A(x) \geq t = \mu_A(a) \wedge \mu_A(b)$.

(6) If $\nu_A(x) = 0$, then it is obvious. Let $s = \nu_A(x) > 0$ and let $t \in [0,1]$ be such that $t > 1-s = 1-\nu_A(x)$, then we have $0 = [x_t \in A] \geq [a_t \in A] \wedge [b_t \in A] \Rightarrow$

$[a_t \in A] \wedge [b_t \in A] = 0 \Rightarrow [a_t \in A] = 0$ or $[b_t \in A] = 0$ i.e., either $t > 1 - \nu_A(a)$ or $t > 1 - \nu_A(b) \Rightarrow$ either $\nu_A(a) > 1 - t$ or $\nu_A(b) > 1 - t \Rightarrow \nu_A(a) \vee \nu_A(b) > 1 - t$. Therefore, $\nu_A(a) \vee \nu_A(b) \geq \vee\{1 - t | t > 1 - s\} = \vee\{1 - t | s > 1 - t\} = s = \nu_A(x)$. Thus $\nu_A(x) \leq \nu_A(a) \vee \nu_A(b)$.

Conversely, we assume A is an intuitionistic fuzzy left h-ideal of R with thresholds $(0, 1)$. We need to show $A = (\mu_A, \nu_A)$ is a $(\in, \in)$-intuitionistic fuzzy left h-ideal of R. Let $x, y \in R$ and $s, t \in (0, 1]$. Let $r = [x_s \in A] \wedge [y_t \in A]$.

Case I. $r = 1$. Then $[x_s \in A] = 1$ and $[y_t \in A] = 1 \Rightarrow \mu_A(x) \geq s$ and $\mu_A(y) \geq t \Rightarrow \mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t \Rightarrow [(x_s + y_t) \in A] = 1 \geq 1 = [x_s \in A] \wedge [y_t \in A]$.

Case II. $r = 1/2$. Then $[x_s \in A] \geq 1/2$ and $[y_t \in A] \geq 1/2 \Rightarrow 1 - \nu_A(x) \geq s$ and $1 - \nu_A(y) \geq t \Rightarrow 1 - \nu_A(x + y) \geq 1 - \nu_A(x) \vee \nu_A(y) = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t \Rightarrow [(x_s + y_t) \in A] \geq 1/2 = [x_s \in A] \wedge [y_t \in A]$. Hence $[(x_s + y_t) \in A] \geq [x_s \in A] \wedge [y_t \in A]$.

Similarly, we have for all $x, z, a, b \in R$, $x + a + z = b + z$, it follows $[x_{s\wedge t} \in A] \geq [a_s \in A] \wedge [b_t \in A]$. Let $r = [x_s \in A]$.

Case I. $r = 1$. Then $\mu_A(x) \geq s \Rightarrow \mu_A(yx) \geq \mu_A(x) \geq s \Rightarrow [y_s x_s \in A] = 1 \geq 1 = [x_s \in A]$.

Case II. $r = 1/2$. Then $1 - \nu_A(x) \geq s \Rightarrow 1 - \nu_A(yx) \geq 1 - \nu_A(x) \geq s \Rightarrow [y_s x_s \in A] \geq 1/2 = [x_s \in A]$. Hence A is a $(\in, \in)$-intuitionistic fuzzy left h-ideal of R.

Similarly, if A is an intuitionistic fuzzy right h-ideal of R with thresholds $(0, 1)$, then A is a $(\in, \in)$-intuitionistic fuzzy right h-ideal of R. □

As a consequence of Theorem 5.5. and Theorem 5.6., we have the following

Theorem 5.7. *Let $A = (\mu_A, \nu_A)$ be a $(\in, \in)$-intuitionistic fuzzy left (resp. right) h-ideal of a hemiring R. If for any $p \in (0, 1]$, $A_{\overline{p}}$ is a non-empty subset of R, then $A_{\overline{p}}$ is a left (resp. right) h-ideal of R.*

Theorem 5.8. *An IFS $A = (\mu_A, \nu_A)$ of a hemiring R is a $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) h-ideal of R if and only if A is an intuitionistic fuzzy left (resp. right) h-ideal of R with thresholds $(0, 0.5)$.*

Proof. Suppose, $A = (\mu_A, \nu_A)$ is a $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) h-ideal of R. To show A is an intuitionistic fuzzy left (resp. right) h-ideal of R with thresholds $(0, 0.5)$.

Let $x, y \in R$. (1) Let $t = \mu_A(x) \wedge \mu_A(y) \wedge 0.5$, then we have $\mu_A(x) \geq t$, $\mu_A(y) \geq t$, and so $[x_t \in A] = 1$, $[y_t \in A] = 1$. Therefore, from definition 4.2. we have,

$1 \geq [(x_t + y_t) \in \vee q A] \geq [x_t \in A] \wedge [y_t \in A] = 1$,
$\Rightarrow [(x_t + y_t) \in \vee q A] = 1$,
$\Rightarrow [(x_t + y_t) \in A] \vee [(x_t + y_t) q A] = 1$,
$\Rightarrow [(x_t + y_t) \in A] = 1$ or $[(x_t + y_t) q A] = 1$,
$\Rightarrow \mu_A(x + y) \geq t$ or $\mu_A(x + y) + t > 1$,
$\Rightarrow \mu_A(x + y) \geq t$ or $\mu_A(x + y) > 1 - t \geq 0.5 \geq t$,
$\Rightarrow \mu_A(x + y) \geq t = (\mu_A(x) \wedge \mu_A(y)) \wedge 0.5$.

(2) Let $\nu_A(x) \vee \nu_A(y) \vee 0.5 = 1 - s$ then $\nu_A(x) \leq 1 - s$ and $\nu_A(y) \leq 1 - s \Rightarrow$

$s \leq 1-\nu_A(x)$ and $s \leq 1-\nu_A(y) \Rightarrow [x_s \in A] \geq 1/2$ and $[y_s \in A] \geq 1/2$. Therefore, from of definition 4.2. we have,

$1 \geq [(x_t+y_t) \in \vee qA] \geq [x_t \in A] \wedge [y_t \in A] \geq 1/2$,

$\Rightarrow [(x_t+y_t) \in A] \vee [(x_t+y_t)qA] \geq 1/2$,

$\Rightarrow [(x_t+y_t) \in A] \geq 1/2$ or $[(x_t+y_t)qA] \geq 1/2$,

$\Rightarrow$ either $s \leq 1-\nu_A(x+y)$ or $\nu_A(x+y) < s \leq 1-s$, [since $1-s \geq 0.5$ so, $s \leq 0.5$]

$\Rightarrow \nu_A(x+y) \leq 1-s = \nu_A(x) \vee \nu_A(y) \vee 0.5$.

(3) Let $t=\mu_A(x) \wedge 0.5$. This implies, $\mu_A(x) \geq t$, and so $[x_t \in A]=1$. Therefore, from definition 4.2. we have,

$1 \geq [y_t x_t) \in \vee qA] \geq [x_t \in A]=1$,

$\Rightarrow [y_t x_t \in \vee qA]=1$,

$\Rightarrow [y_t x_t \in A]=1$ or $[y_t x_t qA]=1$,

$\Rightarrow \mu_A(yx) \geq t$ or $\mu_A(yx)+t>1$,

$\Rightarrow \mu_A(yx) \geq t$ or $\mu_A(yx) > 1-t \geq 0.5 \geq t$.

Thus $\mu_A(yx) \geq t = \mu_A(x) \wedge 0.5$. Similarly, if A is a $(\in, \in \vee q)$-intuitionistic fuzzy right h-ideal of R, then we have $\mu_A(xy) \geq \mu_A(x) \wedge 0.5$.

(4) Let $\nu_A(x) \vee 0.5 = 1-s$, then we have $(1-\nu_A(x)) \wedge 0.5 = s$, and so $[x_s \in A] \geq 1/2$. Thus $[y_s x_s \in \vee qA] \geq [x_s \in A] \geq 1/2$, [By definition 4.2.]. Therefore, we have $[y_s x_s \in A] \geq 1/2$ or $[y_s x_s qA] \geq 1/2$, which implies $s \leq 1-\nu_A(yx)$ or $\nu_A(yx) < s \leq 1-s$, [Since $1-s \geq 0.5$, so $s \leq 0.5$]. Thus $\nu_A(yx) \leq 1-s$ or $\nu_A(yx) \leq 1-s$. Hence $\nu_A(yx) \leq 1-s = \nu_A(x) \vee 0.5$. Similarly, if A is a $(\in, \in \vee q)$-intuitionistic fuzzy right h-ideal of R, then we have $\nu_A(xy) \leq \nu_A(x) \vee 0.5$.

Let $x, z, a, b \in R$ be such that $x+a+z=b+z$.

(5) Let $t=\mu_A(a) \wedge \mu_A(b) \wedge 0.5$. Then we have $[a_t \in A]=1$, $[b_t \in A]=1$. Therefore, from definition 4.2. we have,

$1 \geq [x_t \in \vee qA] \geq [a_t \in A] \wedge [b_t \in A]=1$,

$\Rightarrow [x_t \in \vee qA]=1$,

$\Rightarrow [x_t \in A]=1$ or $[x_t qA]=1$,

$\Rightarrow \mu_A(x) \geq t$ or $\mu_A(x) > 1-t \geq 0.5 \geq t$,

$\Rightarrow \mu_A(x) \geq t = \mu_A(a) \wedge \mu_A(b) \wedge 0.5$.

(6) Let $\nu_A(a) \vee \nu_A(b) \vee 0.5 = 1-s$, then $\nu_A(a) \leq 1-s$ and $\nu_A(b) \leq 1-s$. Thus we have $[a_s \in A] \geq 1/2$ and $[b_s \in A] \geq 1/2$. Therefore, from definition 4.2. we have,

$1 \geq [x_t \in \vee qA] \geq [a_t \in A] \wedge [b_t \in A] \geq 1/2$,

$\Rightarrow [x_t \in A] \geq 1/2$ or $[x_t qA] \geq 1/2$,

$\Rightarrow$ either $s \leq 1-\nu_A(x)$ or $\nu_A(x) < s \leq 1-s$, [since $1-s \geq 0.5$ so, $s \leq 0.5$]

$\Rightarrow \nu_A(x) \leq 1-s = \nu_A(a) \vee \nu_A(b) \vee 0.5$.

Conversely, we assume A is an intuitionistic fuzzy left h-ideal of R with thresholds $(0, 0.5)$. We claim A is a $(\in, \in \vee q)$-intuitionistic fuzzy left h-ideal of R. Let $s, t \in [0,1]$.

(1) Let $x, y \in R$ and let $a = [x_s \in A] \wedge [y_t \in A]$.

Case I. $a=1$. Then $[x_s \in A]=1$ and $[y_t \in A]=1$, which implies $\mu_A(x) \geq s$ and $\mu_A(y) \geq t$.

If $[(x_s + y_t) \in \vee qA] \leq 1/2$, then $\mu_A(x+y) < s \wedge t$ and $\mu_A(x+y) \leq 1 - s \wedge t$. Thus $0.5 > \mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5$. So, $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t$, a contradiction to $\mu_A(x+y) < s \wedge t$. Thus we must have $[(x_s + y_t) \in \vee qA] = 1$.

Case II. $a = 1/2$. Then $[x_s \in A] \geq 1/2$ and $[y_t \in A] \geq 1/2$, which implies $1 - \nu_A(x) \geq s$ and $1 - \nu_A(y) \geq t$. Now

$$1 - \nu_A(x) \vee \nu_A(y) = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$$

If $[(x_s + y_t) \in \vee qA] = 0$, then $(1 - \nu_A(x+y)) < s \wedge t$ and $\nu_A(x+y) \geq s \wedge t$. Now from $0.5 < \nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y) \vee 0.5$, we get $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$, and so we have $1 - \nu_A(x+y) \geq 1 - \nu_A(x) \vee \nu_A(y) = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$, which contradicts $(1 - \nu_A(x+y)) < s \wedge t$. Therefore, we must have $[(x_s + y_t) \in \vee qA] \geq 1/2 = [x_s \in A] \wedge [y_t \in A]$. Hence $[(x_s + y_t) \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A]$.

Next we prove $[y_s x_s \in \vee qA] \geq [x_s \in A]$. Let $b = [x_s \in A]$.

Case I. $b = 1$. Then $\mu_A(x) \geq s$. If $[y_s x_s \in \vee qA] \leq 1/2$, then $[y_s x_s \in A] \leq 1/2$ and $[y_s x_s qA] \leq 1/2$, which implies $\mu_A(yx) < s$ and $s \leq 1 - \mu_A(yx)$, and hence $\mu_A(yx) < s$ and $\mu_A(yx) \leq 1 - s$. Now, $0.5 > \mu_A(yx) \geq \mu_A(x) \wedge 0.5$ implies $\mu_A(yx) \geq \mu_A(x) \geq s$, a contradiction to $\mu_A(yx) < s$. Therefore, we must have $[y_s x_s \in \vee qA] = 1$.

Case II. $b = 1/2$. Then we have $s \leq 1 - \nu_A(x)$. If $[y_s x_s \in \vee qA] = 0$, then $[y_s x_s \in A] = 0$ and $[y_s x_s qA] = 0$, which implies $s > 1 - \nu_A(yx)$ and $s \leq \nu_A(yx)$, and these implies $\nu_A(yx) > 1 - s$ and $s \leq \nu_A(yx)$. Therefore, we have $0.5 < \nu_A(yx) \leq \nu_A(x) \vee 0.5 \Rightarrow \nu_A(yx) \leq \nu_A(x)$. Now, $1 - \nu_A(yx) \geq 1 - \nu_A(x) \geq s$, a contradiction to $s > 1 - \nu_A(yx)$. Therefore, we have $[y_s x_s \in \vee qA] \geq 1/2 = [x_s \in A]$. Hence $[y_s x_s \in \vee qA] \geq [x_s \in A]$.

(3) Let $x, z, a, b \in R$ be such that $x + a + z = b + z$. We claim $[x_{s\wedge t} \in \vee qA] \geq [a_s \in A] \wedge [b_t \in A]$. Let $c = [a_s \in A] \wedge [b_t \in A]$.

Case I. $c = 1$. Then $[a_s \in A] = 1$ and $[b_t \in A] = 1$, which implies $\mu_A(a) \geq s$ and $\mu_A(b) \geq t$.

If $[x_{s\wedge t} \in \vee qA] \leq 1/2$, then $\mu_A(x) < s \wedge t$ and $\mu_A(x) \leq 1 - s \wedge t$. Thus $0.5 > \mu_A(x) \geq \mu_A(a) \wedge \mu_A(b) \wedge 0.5$. So, $\mu_A(x) \geq \mu_A(a) \wedge \mu_A(b) \geq s \wedge t$, a contradiction to $\mu_A(x) < s \wedge t$. Thus we must have $[x_{s\wedge t} \in \vee qA] = 1$.

Case II. $c = 1/2$. Then $[a_s \in A] \geq 1/2$ and $[b_t \in A] \geq 1/2$, which implies $1 - \nu_A(a) \geq s$ and $1 - \nu_A(b) \geq t$. Now

$$1 - \nu_A(a) \vee \nu_A(b) = (1 - \nu_A(a)) \wedge (1 - \nu_A(b)) \geq s \wedge t$$

If $[x_{s\wedge t} \in \vee qA] = 0$, then $1 - \nu_A(x) < s \wedge t$ and $\nu_A(x) \geq s \wedge t$. Now from $0.5 < \nu_A(x) \leq \nu_A(a) \vee \nu_A(b) \vee 0.5$, we get $\nu_A(x) \leq \nu_A(a) \vee \nu_A(b)$, and so we have $1 - \nu_A(x) \geq 1 - \nu_A(a) \vee \nu_A(b) = (1 - \nu_A(a)) \wedge (1 - \nu_A(b)) \geq s \wedge t$, which contradicts to $1 - \nu_A(x) < s \wedge t$. Therefore, we must have $[x_{s\wedge t} \in \vee qA] \geq 1/2 = [a_s \in A] \wedge [b_t \in A]$. Hence $[x_{s\wedge t} \in \vee qA] \geq a_s \in A] \wedge [b_t \in A]$.

Hence A is a $(\in, \in \vee q)$-intuitionistic fuzzy left h-ideal of R. Similarly, if A is an intuitionistic fuzzy right h-ideal with thresholds $(0, 0.5)$ of R, then A is a $(\in, \in \vee q)$-intuitionistic fuzzy right h-ideal of R. □

As a consequence of Theorem 5.5. and Theorem 5.8., we have the following

Theorem 5.9. *Let $A = (\mu_A, \nu_A)$ be a $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) h-ideal of a hemiring R. If for any $p \in (0, 0.5]$, $A_{\overline{p}}$ is a non-empty subset of R, then $A_{\overline{p}}$ is a left (resp. right) h-ideal of R.*

Theorem 5.10. *An IFS $A = (\mu_A, \nu_A)$ of a hemiring R is a $(\in \wedge q, \in)$-intuitionistic fuzzy left (resp. right) h-ideal of R if and only if A is an intuitionistic fuzzy left (resp. right) h-ideal of R with thresholds $(0.5, 1)$.*

Proof. Suppose, $A = (\mu_A, \nu_A)$ is a $(\in \wedge q, \in)$-intuitionistic fuzzy left h-ideal of R. To show A is an intuitionistic fuzzy left h-ideal of R with thresholds $(0.5, 1)$. Let $x, y \in R$.

(1) Let $t = \mu_A(x) \wedge \mu_A(y)$. Now if $\mu_A(x+y) \vee 0.5 < t = \mu_A(x) \wedge \mu_A(y)$, then $\mu_A(x) \geq t > 0.5$ and $\mu_A(y) \geq t > 0.5$,
$\Rightarrow [x_t \in A] = 1$, $[x_t qA] = 1$, $[y_t \in A] = 1$, $[y_t qA] = 1$,
$\Rightarrow [x_t \in \wedge qA] = 1$, $[y_t \in \wedge qA] = 1$,
$\Rightarrow [x_t \in \wedge qA] \wedge [y_t \in \wedge qA] = 1$,

Therefore, $[(x_t + y_t) \in A] \geq [x_t \in \wedge qA] \wedge [y_t \in \wedge qA] = 1$, which gives $[(x_t + y_t) \in A] = 1 \Rightarrow \mu_A(x+y) \geq t$, a contradiction to our assumption $\mu_A(x+y) \leq \mu_A(x+y) \vee 0.5 < t$. Therefore, we have $\mu_A(x+y) \vee 0.5 \geq t = \mu_A(x) \wedge \mu_A(y)$.

(2) let $t = 1 - s = \nu_A(x) \vee \nu_A(y)$, then $1 - s \geq \nu_A(x), 1 - s \geq \nu_A(y)$. If $\nu_A(x+y) \wedge 0.5 > t$, then we have $s \leq 1 - \nu_A(x)$, $s \leq 1 - \nu_A(y)$, $\nu_A(x+y) > t$ and $s > 0.5 > t$, and so $[x_s \in A] \geq 1/2$, $[y_s \in A] \geq 1/2$, $\nu_A(x+y) > t$ and $s > 0.5 > t$. Also, $\nu_A(x) \leq t < s$ and $\nu_A(y) \leq t < s$ implies $[x_s qA] \geq 1/2$, $[y_s qA] \geq 1/2$. Therefore, from $[(x_s + y_s) \in A] \geq [x_s \in \wedge qA] \wedge [y_s \in \wedge qA] \geq 1/2$ we have $[(x_s + y_s) \in A] \geq 1/2$. This implies, $s \leq 1 - \nu_A(x+y)$, which contradict to $\nu_A(x+y) > t$. Hence $\nu_A(x+y) \wedge 0.5 \leq t = \nu_A(x) \vee \nu_A(y)$.

(3) let $t = \mu_A(x)$. If $\mu_A(yx) \vee 0.5 < t$, then $\mu_A(x) = t > 0.5$, and this implies $[x_t \in \wedge qA] = 1$. Now from $[y_t x_t \in A] \geq [x_t \in \wedge qA] = 1$, we get $[y_t x_t \in A] = 1$, and so $\mu_A(yx) \geq t$, which contradicts our assumption $\mu_A(yx) < t$. Therefore, we must have $\mu_A(yx) \vee 0.5 \geq t = \mu_A(x)$.

(4) Let $t = 1 - s = \nu_A(x)$. Then we have $[x_s \in A] \geq 1/2$. If $\nu_A(yx) \wedge 0.5 > t$, then $\nu_A(yx) > t$ and $t < 0.5 < s$. Therefore, $\nu_A(x) = 1 - s = t < s$, this implies $[x_s qA] \geq 1/2$. Thus we have $[x_s \in A] \geq 1/2$ and $[x_s qA] \geq 1/2$, and these imply $[x_s \in \wedge qA] \geq 1/2$. Now from $[y_s x_s \in A] \geq [x_s \in \wedge qA] \geq 1/2$, we have $[y_s x_s \in A] \geq 1/2 \Rightarrow s \leq 1 - \nu_A(yx)$. Therefore, $\nu_A(yx) \leq 1 - s = t$, which contradicts $\nu_A(yx) > t$. Hence $\nu_A(yx) \wedge 0.5 \leq t = \nu_A(x)$.

Let $x, z, a, b \in R$ be such that $x + a + z = b + z$.

(5) Let $t = \mu_A(a) \wedge \mu_A(b)$. Now if $\mu_A(x) \vee 0.5 < t = \mu_A(a) \wedge \mu_A(b)$, then $\mu_A(a) \geq t > 0.5$ and $\mu_A(b) \geq t > 0.5$,
$\Rightarrow [a_t \in A] = 1$, $[a_t qA] = 1$, $[b_t \in A] = 1$, $[b_t qA] = 1$,
$\Rightarrow [a_t \in \wedge qA] = 1$, $[b_t \in \wedge qA] = 1$,
$\Rightarrow [a_t \in \wedge qA] \wedge [b_t \in \wedge qA] = 1$,

Therefore, $[x_t \in A] \geq [a_t \in \wedge qA] \wedge [b_t \in \wedge qA] = 1$, which gives $[x_t \in A] = 1 \Rightarrow$

$\mu_A(x) \geq t$, a contradiction to our assumption $\mu_A(x) \leq \mu_A(x) \vee 0.5 < t$. Therefore, we have $\mu_A(x) \vee 0.5 \geq t = \mu_A(a) \wedge \mu_A(b)$.

(6) let $t = 1 - s = \nu_A(a) \vee \nu_A(b)$, then $1 - s \geq \nu_A(a), 1 - s \geq \nu_A(b)$. If $\nu_A(x) \wedge 0.5 > t$, then we have $s \leq 1 - \nu_A(a)$, $s \leq 1 - \nu_A(b)$, $\nu_A(x) > t$ and $s > 0.5 > t$, and so $[a_s \in A] \geq 1/2$, $[b_s \in A] \geq 1/2$, $\nu_A(x) > t$ and $s > 0.5 > t$. Also, $\nu_A(a) \leq t < s$ and $\nu_A(b) \leq t < s$ implies $[a_s qA] \geq 1/2$, $[b_s qA] \geq 1/2$. Therefore, from $[x_s \in A] \geq [a_s \in \wedge qA] \wedge [b_s \in \wedge qA] \geq 1/2$ we have $[x_s \in A] \geq 1/2$. This implies, $s \leq 1 - \nu_A(x)$, which contradicts $\nu_A(x) > t$. Hence $\nu_A(x) \wedge 0.5 \leq t = \nu_A(a) \vee \nu_A(b)$.

Similarly, if A is a $(\in \wedge q, \in)$-intuitionistic fuzzy right h-ideal of R, then A is an intuitionistic fuzzy right h-ideal of R with thresholds $(0.5, 1)$.

Conversely, we assume A is an intuitionistic fuzzy left h-ideal with thresholds $(0.5, 1)$ of R. Let $x, y \in R$ and $s, t \in [0, 1]$, let $a = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$. Now,

Case I. $a = 1$. Then $\mu_A(x) \geq s, \mu_A(x) + s > 1, \mu_A(y) \geq t, \mu_A(y) + t > 1$. This implies $\mu_A(x) \geq 0.5$ and $\mu_A(y) \geq 0.5$. Now, $\mu_A(x+y) \vee 0.5 \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t$, implies $\mu_A(x+y) \geq s \wedge t$, from which we get $[(x_s + y_t) \in A] = 1$.

Case II. $a = 1/2$. Then $s \leq 1 - \nu_A(x)$, $\nu_A(x) < s$, $t \leq 1 - \nu_A(y)$, $\nu_A(y) < t$,

$\Rightarrow 1 - \nu_A(x) \geq s > \nu_A(x)$, $1 - \nu_A(y) \geq t > \nu_A(y)$,

$\Rightarrow \nu_A(x) < 0.5$, $\nu_A(x) < 0.5$.

Therefore, $\nu_A(x+y) \wedge 0.5 \leq \nu_A(x) \vee \nu_A(y) \Rightarrow \nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$, which implies, $1 - \nu_A(x+y) \geq (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$. Thus $[(x_s + y_t) \in A] \geq 1/2$. Hence $[(x_s + y_t) \in A] \geq [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Next, let $b = [x_s \in \wedge qA]$.

Case I. $b = 1$. Then $\mu_A(x) \geq s, \mu_A(x) + s > 1$. This implies, $\mu_A(x) \geq 0.5$. Now, $\mu_A(yx) \geq \mu_A(x) \geq s$, from which we get $[y_s x_s \in A] = 1$.

Case II. $a = 1/2$. Then $s \leq 1 - \nu_A(x)$, $\nu_A(x) < s$, and these imply $1 - \nu_A(x) \geq s > \nu_A(x)$. Therefore, we have $\nu_A(x) < 0.5$.

Therefore, $\nu_A(yx) \wedge 0.5 \leq \nu_A(x) \Rightarrow \nu_A(yx) \leq \nu_A(x)$, which implies $1 - \nu_A(yx) \geq 1 - \nu_A(x) \geq s$. Thus $[y_s x_s \in A] \geq 1/2$. Hence $[y_s x_s \in A] \geq [x_s \in \wedge qA]$.

Lastly, let $x, z, a, b \in R$ be such that $x + a + z = b + z$. let $c = [a_s \in \wedge qA] \wedge [b_t \in \wedge qA]$.

Case I. $c = 1$. Then $\mu_A(a) \geq s, \mu_A(a) + s > 1, \mu_A(b) \geq t, \mu_A(b) + t > 1$. This implies $\mu_A(a) \geq 0.5$ and $\mu_A(b) \geq 0.5$. Now, $\mu_A(x) \vee 0.5 \geq \mu_A(a) \wedge \mu_A(b) \geq s \wedge t$, implies $\mu_A(x) \geq s \wedge t$, from which we get $[x_{s \wedge t} \in A] = 1$.

Case II. $c = 1/2$. Then

$s \leq 1 - \nu_A(a)$, $\nu_A(a) < s$, $t \leq 1 - \nu_A(b)$, $\nu_A(b) < t$,

$\Rightarrow 1 - \nu_A(a) \geq s > \nu_A(a)$, $1 - \nu_A(b) \geq t > \nu_A(b)$,

$\Rightarrow \nu_A(a) < 0.5$, $\nu_A(b) < 0.5$.

Therefore, $\nu_A(x) \wedge 0.5 \leq \nu_A(a) \vee \nu_A(b) \Rightarrow \nu_A(x) \leq \nu_A(a) \vee \nu_A(b)$, which implies, $1 - \nu_A(x) \geq (1 - \nu_A(a)) \wedge (1 - \nu_A(b)) \geq s \wedge t$. Thus $[x_{s \wedge t} \in A] \geq 1/2$. Hence $[x_{s \wedge t} \in A] \geq [a_s \in \wedge qA] \wedge [b_t \in \wedge qA]$.

Therefore, A is a $(\in \wedge q, \in)$-intuitionistic fuzzy left h-ideal of R. Similarly, if A is an intuitionistic fuzzy right h-ideal with thresholds $(0.5, 1)$ of R, then A is a $(\in \wedge q, \in)$-intuitionistic fuzzy right h-ideal of R. □

As a consequence of Theorem 5.5 and Theorem 5.10, we have the following

Theorem 5.11. *Let $A = (\mu_A, \nu_A)$ be a $(\in \wedge q, \in)$-intuitionistic fuzzy left (resp. right) h-ideal of a hemiring R. If for any $p \in (0.5, 1]$, $A_{\overline{p}}$ is a non-empty subset of R, then $A_{\overline{p}}$ is a left (resp. right) h-ideal of R.*

Theorem 5.12. *An intuitionistic fuzzy set, $A = (\mu_A, \nu_A)$ of a hemiring R is a $(\in, \in)$-intuitionistic fuzzy left (resp. right) h-ideal of R if and only if for any $p \in [0, 1]$, A_p is a fuzzy left (resp. right) h-ideal of R.*

Proof. Suppose, A is a $(\in, \in)$-intuitionistic fuzzy left h-ideal of R. Let $x, y \in R$ and $p \in [0, 1]$, then

$A_p(x + y) = [(x + y)_p \in A] = [(x_p + y_p) \in A] \geq [x_p \in A] \wedge [y_p \in A] = A_p(x) \wedge A_p(y)$,

$A_p(yx) = [y_p x_p \in A] \geq [x_p \in A] = A_p(x)$,

Let $x, z, a, b \in R$ be such that $x + a + z = b + z$. Then

$A_p(x) = [x_p \in A] \geq [a_p \in A] \wedge [b_p \in A] = A_p(a) \wedge A_p(b)$.

Hence A_p is a fuzzy left h-ideal of R. Similarly, if A is a $(\in, \in)$-intuitionistic fuzzy right h-ideal of R, then A_p is a fuzzy left h-ideal of R.

Conversely, we assume for any $p \in [0, 1]$, A_p is a fuzzy left h-ideal of R. Let $x, y \in R$ and $s, t \in [0, 1]$. We will prove

(1) $[(x_s + y_t) \in A] \geq [x_s \in A] \wedge [y_t \in A]$,

(2) $[y_s x_s \in A] \geq [x_s \in A]$, and

(3) for all $x, z, a, b \in R$, $x + a + z = b + z$, implies $[x_{s\wedge t} \in A] \geq [a_s \in A] \wedge [b_t \in A]$.

(2) Let $c = [x_s \in A]$. Now,

Case I. $c = 1$. Then $A_s(x) = 1$, so from $A_s(yx) \geq A_s(x) = 1$, we have $A_s(yx) = 1$, which implies $\mu_A(yx) \geq s$. Thus $[y_s x_s \in A] = 1$.

Case II. $c = 1/2$. Then $A_s(x) = 1/2$, so from $A_s(yx) \geq A_s(x) = 1/2$, we have $A_s(yx) \geq 1/2$, which implies $s \leq 1 - \nu_A(yx)$. Thus $[y_s x_s \in A] \geq 1/2$. Hence $[y_s x_s \in A] \geq [x_s \in A]$.

(3) Let $x, z, a, b \in R$ be such that $x + a + z = b + z$ and $s, t \in [0, 1]$.
Let $c = [a_s \in A] \wedge [b_t \in A]$. Now,

Case I. $c = 1$. Then $A_s(a) = 1$ and $A_t(b) = 1$, so from $A_{s\wedge t}(x) \geq A_{s\wedge t}(a) \wedge A_{s\wedge t}(b) \geq A_s(a) \wedge A_t(b) = 1$, we have $A_{s\wedge t}(x) = 1$, which implies $\mu_A(x) \geq s \wedge t$. Thus $[x_{s\wedge t} \in A] = 1$.

Case II. $c = 1/2$. Then $A_s(a) = 1/2$ and $A_t(b) = 1/2$, so from $A_{s\wedge t}(x) \geq A_{s\wedge t}(a) \wedge A_{s\wedge t}(b) \geq A_s(a) \wedge A_t(b) = 1/2$, we have $A_{s\wedge t}(x) \geq 1/2$, which implies $s \wedge t \leq 1 - \nu_A(x)$. Thus $[x_{s\wedge t} \in A] \geq 1/2$. Hence $[x_{s\wedge t} \in A] \geq [a_s \in A] \wedge [b_t \in A]$.

In a similar manner we can prove (1). Hence A is a $(\in, \in)$-intuitionistic fuzzy left h-ideal of R. Similarly, if for any $p \in [0, 1]$, A_p is a fuzzy right h-ideal of R, then we have A is a $(\in, \in)$-intuitionistic fuzzy right h-ideal of R. □

Theorem 5.13. *An intuitionistic fuzzy set, $A = (\mu_A, \nu_A)$ of a hemiring R is a $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) h-ideal of R if and only if for any $p \in [0, 0.5]$, A_p is a fuzzy (resp. right) h-ideal of R.*

Proof. Suppose, A is a $(\in, \in \vee q)$-intuitionistic fuzzy right h-ideal of R, then for any $p \in (0, 0.5]$ and $x, y \in R$, we have

$$[x_p y_p \in \vee q] \geq [x_p \in A] \Rightarrow A_p(xy) \vee A_{[\underline{p}]}(xy) \geq A_p(x)$$

Since $0 < p \leq 0.5$, therefore we have $p \leq 0.5 \leq 1 - p$. Then

$$A_{[\underline{p}]}(xy) = A_{\underline{1-p}}(xy) \leq A_{\underline{p}}(xy) \leq A_p(xy)$$

Therefore, $A_p(x) \leq A_p(xy) \vee A_{[\underline{p}]}(xy) \leq A_p(xy) \vee A_p(xy) = A_p(xy)$, and so $A_p(xy) \geq A_p(x)$. Let $x, z, a, b \in R$ be such that $x + a + z = b + z$. Then we have

$$[x_p \in \vee q] \geq [a_p \in A] \wedge [b_p \in A] \Rightarrow A_p(x) \vee A_{[\underline{p}]}(x) \geq A_p(a) \wedge A_p(b)$$

Since $0 < p \leq 0.5$, therefore we have $p \leq 0.5 \leq 1 - p$. Then

$$A_{[\underline{p}]}(x) = A_{\underline{1-p}}(x) \leq A_{\underline{p}}(x) \leq A_p(x)$$

Therefore, $A_p(a) \wedge A_p(b) \leq A_p(x) \vee A_{[\underline{p}]}(x) \leq A_p(x) \vee A_p(x) = A_p(x)$, and so $A_p(x) \geq A_p(a) \wedge A_p(b)$. In a similar manner we can prove that for all $x, y \in R$, $A_p(x+y) \geq A_p(x) \wedge A_p(y)$. Therefore, for any $p \in [0, 0.5]$, A_p is a fuzzy right h-ideal of R. Similarly, if A is a $(\in, \in \vee q)$-intuitionistic fuzzy left h-ideal of R, then for any $p \in (0, 0.5]$, A_p is a fuzzy left h-ideal of R.

Conversely, we assume for any $p \in [0, 0.5]$, A_p is a fuzzy right h-ideal of R. let $s, t \in [0, 1]$ and $x, z, a, b \in R$ be such that $x + a + z = b + z$.

(1) If $s \wedge t \leq 0.5$, then let $a = [a_s \in A] \wedge [b_t \in A]$.

Case I. $a = 1$. Then $A_s(a) = 1$ and $A_t(b) = 1$, and so $A_{s \wedge t}(x) \geq A_{s \wedge t}(a) \wedge A_{s \wedge t}(b) \geq A_s(a) \wedge A_t(b) = 1$. Therefore, we have $A_{s \wedge t}(x) = 1 \Rightarrow [x_{s \wedge t} \in A] = 1$. Thus $[x_{s \wedge t} \in \vee q A] = [x_{s \wedge t} \in A] \vee [x_{s \wedge t} q A] = 1$.

Case II. $a = 1/2$. Then $A_s(a) \geq 1/2$ and $A_t(b) \geq 1/2$, and so $A_{s \wedge t}(x) \geq A_{s \wedge t}(a) \wedge A_{s \wedge t}(b) \geq A_s(a) \wedge A_t(b) \geq 1/2$. Therefore, we have $A_{s \wedge t}(x) \geq 1/2 \Rightarrow [x_{s \wedge t} \in A] \geq 1/2$. Thus $[x_{s \wedge t} \in \vee q A] = [x_{s \wedge t} \in A] \vee [x_{s \wedge t} q A] \geq 1/2$. Therefore, $[x_{s \wedge t} \in \vee q A] \geq [a_s \in A] \wedge [b_t \in A]$.

If $s \wedge t > 0.5$, then let $p \in (0, 1)$ such that $1 - s \wedge t < p < 0.5 < s \wedge t$. Now $A_{[\underline{s \wedge t}]}(x) = A_{\underline{1-s \wedge t}}(x) \geq A_{s \wedge t}(x)$, and $A_{[\underline{s \wedge t}]}(x) = A_{\underline{1-s \wedge t}}(x) \geq A_p(x)$.

Therefore, $[x_{s \wedge t} \in \vee q A] = [x_{s \wedge t} \in A] \vee [x_{s \wedge t} q A] = A_{s \wedge t}(x) \vee A_{[\underline{s \wedge t}]}(x) = A_{[\underline{s \wedge t}]}(x) \geq A_p(x) \geq A_p(a) \wedge A_p(b) \geq A_s(a) \wedge A_t(b) = [a_s \in A] \wedge [b_t \in A]$, and hence $[x_{s \wedge t} \in \vee q A] \geq [a_s \in A] \wedge [b_t \in A]$.

In a similar manner we can prove that for all $x, y \in R$, $[(x_s + y_t) \in \vee q A] \geq [x_s \in A] \wedge [b_t \in A]$.

(3) If $s \leq 0.5$, then let $c = [x_s \in A]$.

Case I. $c = 1$. Then $A_s(x) = 1$, therefore from $A_s(xy) \geq A_s(x) = 1$, we have $A_s(xy) = 1$. This implies $\mu_A(xy) \geq s$. Therefore,$[x_s y_s \in A] = 1$. Thus $[x_s y_s \in \vee q A] = [x_s y_s \in A] \vee [x_s y_s q A] = 1$.

Case II. $c = 1/2$. Then $A_s(x) = 1/2$, therefore from $A_s(xy) \geq A_s(x) = 1/2$, we have $s \le 1-\nu_A(xy)$. Therefore, $[x_sy_s \in A] \ge 1/2$, and so $[x_sy_s \in \vee qA] = [x_sy_s \in A] \vee [x_sy_sqA] \ge 1/2$. Therefore, $[x_sy_s \in \vee qA] \ge [x_s \in A]$.

If $s > 0.5$, then let $p \in (0,1)$ be such that $1-s < p < 0.5 < s$. Now $A_{[\underline{s}]}(xy) = A_{\underline{1-s}}(xy) \ge A_s(xy)$, and $A_{[\underline{s}]}(xy) = A_{\underline{1-s}}(xy) \ge A_p(xy)$.

Therefore, $[x_sy_s \in \vee qA] = [x_sy_s \in A] \vee [x_sy_sqA] = A_s(xy) \vee A_{[\underline{s}]}(xy) = A_{[\underline{s}]}(xy) \ge A_p(xy) \ge A_p(x) \ge A_s(x) = [x_s \in A]$, and hence $[x_sy_s \in \vee qA] \ge [x_s \in A]$. Hence A is a $(\in, \in \vee q)$-intuitionistic fuzzy right h-ideal of R.

Similarly, if for any $p \in [0, 0.5]$, A_p is a fuzzy left h-ideal of R, then A is a $(\in, \in \vee q)$-intuitionistic fuzzy left h-ideal of R. □

Theorem 5.14. *An intuitionistic fuzzy set, $A = (\mu_A, \nu_A)$ of a hemiring R is a $(\in \wedge q, \in)$-intuitionistic fuzzy left (resp. right) h-ideal of R if and only if for any $p \in (0.5, 1]$, A_p is a fuzzy left (resp. right) h-ideal of R.*

Proof. Suppose, A is a $(\in \wedge q, \in)$-intuitionistic fuzzy right h-ideal of R. Let $p \in (0.5, 1]$ and $x, y \in R$, then $A_{[\underline{p}]}(x) \ge A_p(x)$. Thus for all $x, y \in R$, we have

$A_p(x+y) = [(x_p+y_p) \in A] \ge [x_p \in \wedge qA] \wedge [y_p \in \wedge qA] = A_p(x) \wedge A_{[\underline{p}]}(x) \wedge A_p(y) \wedge A_{[\underline{p}]}(y) = A_p(x) \wedge A_p(y)$. Therefore, $A_p(x+y) \ge A_p(x) \wedge A_p(y)$.

$A_p(xy) = [x_py_p \in A] \ge [x_p \in \wedge qA] = A_p(x) \wedge A_{[\underline{p}]}(x) = A_p(x)$. Therefore, $A_p(xy) \ge A_p(x)$.

Let $x, z, a, b \in R$ be such that $x+a+z = b+z$. Then $A_p(x) = [x_p \in A] \ge [a_p \in \wedge qA] \wedge [b_p \in \wedge qA] = A_p(a) \wedge A_{[\underline{p}]}(a) \wedge A_p(b) \wedge A_{[\underline{p}]}(b) = A_p(a) \wedge A_p(b)$. Therefore, $A_p(x) \ge A_p(a) \wedge A_p(b)$. Hence A_p is a fuzzy right h-ideal of R.

Similarly, if A is a $(\in \wedge q, \in)$-intuitionistic fuzzy left h-ideal of R, then for any $p \in (0.5, 1]$, A_p is a fuzzy left h-ideal of R.

Conversely, we assume for any $p \in (0.5, 1]$, A_p is a fuzzy right h-ideal of R. Let $x, y \in R, s, t \in (0, 1]$.

(1) Let $c = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Case I. $c = 1$. Then $\mu_A(x) \ge s$, $\mu_A(x) > 1-s$, $\mu_A(y) \ge t$, $\mu_A(y) > 1-t$. Therefore, $\mu_A(x) > 0.5$, $\mu_A(y) > 0.5$. Let $p = \mu_A(x) \wedge \mu_A(y)$, then $p > 0.5$ and $\mu_A(x) \ge p$, $\mu_A(y) \ge p$, and so $A_p(x) = 1$, $A_p(y) = 1$. Thus $A_p(x+y) \ge A_p(x) \wedge A_p(y) = 1$ implies $A_p(x+y) = 1$, and so $\mu_A(x+y) \ge p = \mu_A(x) \wedge \mu_A(y) \ge s \wedge t$. Therefore, $[(x_s+y_t) \in A] = 1$.

Case II. $c = 1/2$. Then $1-\nu_A(x) \ge s$, $s > \nu_A(x)$ and $1-\nu_A(y) \ge t$, $t > \nu_A(y)$, which imply $\nu_A(x) < 0.5, \nu_A(y) < 0.5$. Thus $1-\nu_A(x) > 0.5$, $1-\nu_A(y) > 0.5$. Let $p = (1-\nu_A(x)) \wedge (1-\nu_A(y))$, then $p > 0.5$. Therefore, $A_p(x+y) \ge A_p(x) \wedge A_p(y) \ge 1/2 \wedge 1/2 = 1/2$, [Since $1-\nu_A(x) \ge p$, $1-\nu_A(y) \ge p$]. This implies $1-\nu_A(x+y) \ge p = (1-\nu_A(x)) \wedge (1-\nu_A(y)) \ge s \wedge t$. Therefore, $[(x_s+y_t) \in A] \ge 1/2 = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

(2) Let $c = [x_s \in \wedge qA]$.

Case I. $c = 1$. Then we have $\mu_A(x) \ge s$, $\mu_A(x) > 1-s$. Therefore, $\mu_A(x) > 0.5$. Let $p = \mu_A(x)$, then $p > 0.5$ and $A_p(x) = 1$. Thus $A_p(xy) \ge A_p(x) = 1$, implies $A_p(xy) = 1$, and so $\mu_A(xy) \ge p = \mu_A(x) = s$. Therefore, $[x_sy_s \in A] = 1$.

Case II. $c = 1/2$. Then we have $1-\nu_A(x) \ge s$, $s > \nu_A(x)$, from which we get $\nu_A(x) < 0.5$. Thus $1 - \nu_A(x) > 0.5$. Let $p = 1 - \nu_A(x)$, then $p > 0.5$. Therefore, $A_p(xy) \geq A_p(x) = 1/2$, [Since $1 - \nu_A(x) = p$]. This implies $1 - \nu_A(xy) \geq p = 1 - \nu_A(x) = s$. Therefore, $[x_s y_s \in A] \geq 1/2 = [x_s \in \wedge qA]$.

(3) Let $x, z, a, b \in R$ be such that $x + a + z = b + z$. Let $c = [a_s \in \wedge qA] \wedge [b_t \in \wedge qA]$.

Case I. $c = 1$. Then $\mu_A(a) \geq s$, $\mu_A(a) > 1 - s$, $\mu_A(b) \geq t$, $\mu_A(b) > 1 - t$. Therefore, $\mu_A(a) > 0.5$, $\mu_A(b) > 0.5$. Let $p = \mu_A(a) \wedge \mu_A(b)$, then $p > 0.5$ and $\mu_A(a) \geq p$, $\mu_A(b) \geq p$, and so $A_p(a) = 1$, $A_p(b) = 1$. Thus $A_p(x) \geq A_p(a) \wedge A_p(b) = 1$ implies $A_p(x) = 1$, and so $\mu_A(x) \geq p = \mu_A(a) \wedge \mu_A(b) \geq s \wedge t$. Therefore, $[x_{s\wedge t} \in A] = 1$.

Case II. $c = 1/2$. Then $1 - \nu_A(a) \geq s$, $s > \nu_A(a)$ and $1 - \nu_A(b) \geq t$, $t > \nu_A(b)$, which implies $\nu_A(a) < 0.5, \nu_A(b) < 0.5$. Thus $1 - \nu_A(a) > 0.5, 1 - \nu_A(b) > 0.5$. Let $p = (1 - \nu_A(a)) \wedge (1 - \nu_A(b))$, then $p > 0.5$. Therefore, $A_p(x) \geq A_p(a) \wedge A_p(b) \geq 1/2 \wedge 1/2 = 1/2$, [Since $1 - \nu_A(a) \geq p, 1 - \nu_A(b) \geq p$]. This implies $1 - \nu_A(x) \geq p = (1 - \nu_A(a)) \wedge (1 - \nu_A(b)) \geq s \wedge t$. Therefore, $[x_{s\wedge t} \in A] \geq 1/2 = [a_s \in \wedge qA] \wedge [b_t \in \wedge qA]$.

Hence A is a $(\in \wedge q, \in)$-intuitionistic fuzzy right h-ideal of R. Similarly, if for any $p \in (0.5, 1]$, A_p is a fuzzy left h-ideal of R, then A is a $(\in \wedge q, \in)$-intuitionistic fuzzy left h-ideal of R. □

Theorem 5. 15. *An intuitionistic fuzzy set, $A = (\mu_A, \nu_A)$ of a hemiring R is an intuitionistic fuzzy left (resp. right) h-ideal with thresholds (s, t) of R if and only if for any $p \in (s, t]$, A_p is a fuzzy left (resp. right) h-ideal of R.*

Proof. Suppose, A is an intuitionistic fuzzy right h-ideal with thresholds (s, t) of R. Let $p \in (s, t], x, y \in R$. Let $c = A_p(x)$.

Case I. $c = 1$. This implies $\mu_A(x) \geq p > s$. Now, $\mu_A(xy) \vee s \geq \mu_A(x) \wedge t \geq p \wedge t = p$. Therefore $\mu_A(xy) \geq p$, which implies $A_p(xy) = 1$.

Case II. $c = 1/2$. This implies $1 - \nu_A(x) \geq p$. Thus $\nu_A(x) \leq 1 - p < 1 - s$. Now $\nu_A(xy) \wedge (1 - s) \leq \nu_A(x) \vee (1 - t) \leq (1 - p) \vee (1 - t) = 1 - p$, [Since $t \geq p$]. Therefore, $1 - \nu_A(xy) \geq p$, and so $A_p(xy) \geq 1/2 = A_p(x)$. Hence $A_p(xy) \geq A_p(x)$.

Similarly, we have $A_p(x + y) \geq A_p(x) \wedge A_p(y)$ for all $x, z, a, b \in R$ with $x + a + z = b + z$,it follows $A_p(x) \geq A_p(a) \wedge A_p(b)$. Therefore A_p is a fuzzy right h-ideal of R. Similarly, if A is an intuitionistic fuzzy left h-ideal with thresholds (s, t) of R, then A_p is a fuzzy left h-ideal of R.

Conversely, we assume for any $p \in (s, t]$, A_p is a fuzzy right h-ideal of R.
Let $x, z, a, b \in R$ be such that $x + a + z = b + z$. First we show $\mu_A(x) \vee s \geq \mu_A(a) \wedge \mu_A(b) \wedge t$. If $\mu_A(x) \vee s < p = \mu_A(a) \wedge \mu_A(b) \wedge t$, then $p \in (s, t]$ and $\mu_A(a) \geq p, \mu_A(b) \geq p$. Thus from $A_p(x) \geq A_p(a) \wedge A_p(b) = 1$, we have $A_p(x) = 1$, and so $\mu_A(x) \geq p$, which contradicts $\mu_A(x) < p$. Therefore, we have $\mu_A(x) \vee s \geq \mu_A(a) \wedge \mu_A(b) \wedge t$.

Similarly, we have $\mu_A(x + y) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t$ for all $x, y \in R$.

Next we show $\mu_A(xy) \vee s \geq \mu_A(x) \wedge t$ for all $x, y \in R$. If $\mu_A(xy) \vee s < p = \mu_A(x) \wedge t$, then $p \in (s, t]$ and $\mu_A(x) \geq p$. Thus from $A_p(xy) \geq A_p(x) = 1$, we have $A_p(xy) = 1$, and so $\mu_A(xy) \geq p$, which contradicts $\mu_A(xy) < p$. Therefore, $\mu_A(xy) \vee s \geq \mu_A(x) \wedge t$.

Let $x, z, a, b \in R$ be such that $x + a + z = b + z$. To show $\nu_A(x) \wedge (1 - s) \leq \nu_A(a) \vee \nu_A(b) \vee (1 - t)$. If $\nu_A(x) \wedge (1 - s) > r = \nu_A(a) \vee \nu_A(b) \vee (1 - t)$, then $(1 - \nu_A(x)) \vee s < p = 1 - r = (1 - \nu_A(a)) \wedge (1 - \nu_A(b)) \wedge t$, and so $p \in (s, t]$ and $(1 - \nu_A(a)) \geq p$, $(1 - \nu_A(b)) \geq p$. Thus from $A_p(x) \geq A_p(a) \wedge A_p(b) \geq 1/2$, we have $A_p(x) \geq 1/2$, and so $1 - \nu_A(x) \geq p = 1 - r$. Therefore, $\nu_A(x) \leq r$, which contradicts $\nu_A(x) > r$. Hence $\nu_A(x) \wedge (1 - s) \leq \nu_A(a) \vee \nu_A(b) \vee (1 - t)$.

Similarly, we have $\nu_A(x + y) \wedge (1 - s) \leq \nu_A(x) \vee \nu_A(y) \vee (1 - t)$, for all $x, y \in R$.

Lastly, we show $\nu_A(xy) \wedge (1 - s) \leq \nu_A(x) \vee (1 - t)$. If $\nu_A(xy) \wedge (1 - s) > r = \nu_A(x) \vee (1 - t)$, then $(1 - \nu_A(xy)) \vee s < p = 1 - r = (1 - \nu_A(x)) \wedge t$, and so $p \in (s, t]$ and $(1 - \nu_A(x)) \geq p$. Thus from $A_p(xy) \geq A_p(x) \geq 1/2$, we have $A_p(xy) \geq 1/2$, and so $1 - \nu_A(xy) \geq p = 1 - r$. Therefore, $\nu_A(xy) \leq r$, which contradicts $\nu_A(xy) > r$. Thus $\nu_A(xy) \wedge (1 - s) \leq \nu_A(x) \vee (1 - t)$.

Hence $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy right h-ideal with thresholds (s, t) of R. Similarly, if for any $p \in (s, t]$, A_p is a fuzzy left h-ideal of R, then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy left h-ideal with thresholds (s, t) of R. □

6. Conclusion

In this article, we have defined a new kind of fuzzy ideal of hemiring namely, (α, β)-intuitionistic fuzzy left (right) h-ideal of a hemiring R, where $\alpha, \beta \in \{\in, q, \in \wedge q, \in \vee q\}$. We have also defined intuitionistic fuzzy left (right) h-ideal with thresholds (s, t) of a hemiring R. Among the 16 number of (α, β) intuitionistic fuzzy left (right) h-ideals, $(\in, \in)$, $(\in, \in \vee q)$ and $(\in \wedge q, \in)$ are significant. We have investigated various properties of (α, β)-intuitionistic fuzzy left (right) h-ideal and established necessary and sufficient conditions with intuitionistic fuzzy left (right) h-ideal with thresholds (s, t). In our opinion this is an opening for investigations of different types of (α, β)-intuitionistic fuzzy left (right) h-ideal.

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