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Weighted Carlson Mean of Positive Definite Matrices

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Abstract. Taking the weighted geometric mean [11] on the cone of positive definite matrix, we propose an iterative mean algorithm involving weighted arithmetic and geometric means of $n-$positive definite matrices which is a weighted version of Carlson mean presented by Lee and Lim [13]. We show that each sequence of the weigthed Carlson iterative mean algorithm has a common limit and the common limit of satisfies weighted multidimensional versions of all properties like permutation symmetry, concavity, monotonicity, homogeneity, congruence invariancy, duality, mean inequalities.

1. Introduction

For positive real numbers a and b, the sequences $\{a_n\}$ and $\{b_n\}$ defined by

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \frac{1}{2}(a_n + b_n), \quad b_{n+1} = \sqrt{a_{n+1}b_n}$$

converge to a common limit. This is called Borchardt's algorithm [5]. In a generalization of Borchardt's algorithm with a suitable incomplete elliptic integral representation and with "permutation symmetry," Carlson [6] has found a 3-dimensional iterative mean algorithm involving arithmetic and geometric means of positive reals. For positive reals $a_0 = a, b_0 = b$ and $c_0 = c$, the three sequences defined by

$$\begin{aligned} a_{n+1} &= \Big(\frac{a_n + b_n}{2}\frac{a_n + c_n}{2}\Big)^{\frac{1}{2}}, \\ b_{n+1} &= \Big(\frac{b_n + c_n}{2}\frac{a_n + b_n}{2}\Big)^{\frac{1}{2}}, \\ c_{n+1} &= \Big(\frac{a_n + c_n}{2}\frac{b_n + c_n}{2}\Big)^{\frac{1}{2}} \end{aligned}$$

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approach to a common limit. This satisfies the permutation symmetry and includes Borchardt's algorithm (when $b = c$) and the common limit has a symmetric integral representation. A key observation in the Carlson algorithm is that it is a composition of two symmetrization procedures: $(a_{n+1}, b_{n+1}, c_{n+1}) = \delta(a_n, b_n, c_n)$ where $\delta = \gamma \circ \beta$ and

$$\beta(a, b, c) = \frac{1}{2}(b + c, a + c, a + b), \quad \gamma(a, b, c) = (\sqrt{bc}, \sqrt{ac}, \sqrt{ab}).$$

The geometric mean of two positive definite matrices A and B is defined by $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ and is regarded as a unique positive definite solution of the Riccati equation $XA^{-1}X = B$ [12]. In the Riemannian manifold of positive definite matrices equipped with the Riemannian metric (it coincides with the Hessian metric of the logarithmic barrier functional) $ds = ||A^{-1/2}dA\, A^{-1/2}||_2 = \big(\mathrm{tr}(A^{-1}dA)^2\big)^{1/2}$ ([10, 2, 3, 12]), $A\#B$ is the unique metric midpoint of A and B for the Riemannian metric distance satisfying the symmetry $A\#B = B\#A$.
q It has been a long-standing problem to extend the two-variable geometric mean of positive definite matrices to n-variables, $n \geq 3$, and a variety of attempts may be found in the literature. Two recent approaches have been given by Ando-Li-Mathias [1] and Bini-Meini-Poloni [4] via "symmetrization methods" and induction. From these successful extensions of geometric means of n-positive definite matrices, Lee and Lim [13] have generalized Carlson's algorithm from the case that variables vary over 3-dimensinal positive real numbers to the case that variables vary over multi-variable positive definite matrices preserving permutation symmetry.
In [11], the authors have constucted a weighted geometric mean $\mathfrak{G}_n(\omega; A_1, \ldots, A_n)$ where ω varies over $n-$dimentional positive probability vectors via weighted version of Bini-Meini-Poloni symmetrization procedure and induction, satisfying the weighted version of ten properties of ALM geometric mean.
The main purpose of this paper is to propose weighted Carlson means based on a weighted version of the generalized permutation symmetrization procedure in [13]. For a positive probability vector $\omega = (w_1, \ldots, w_n)$ and an n-tuple of positive definite matrices $A = (A_1, \ldots, A_n)$, our symmetrization method is given by

$$(A_1^{(0)}, \ldots, A_n^{(0)}) = (A_1, \ldots, A_n),$$
$$(A_1^{(r+1)}, \ldots, A_n^{(r+1)}) = \delta_\omega(A_1^{(r)}, \ldots, A_n^{(r)}),$$

where $\delta_\omega = \gamma_\omega \circ \beta_\omega$, β_ω the weighted Ando-Li-Mathias symmetrization procedure of arithmetic mean and γ_ω is the weighted Bini-Meini-Poloni symmetrization procedure of geometric mean. We show that the sequences $\{A_i^{(r)}\}_{r=0}^{\infty}$, $i = 1, \ldots, n$, converge to a common limit, yielding a weighted Carlson mean $\mathfrak{C}_n(\omega; A)$. In Section 4, we present all properties of the weighted Carlson mean of positive definite matrices.

2. The Convex Cone of Positive Definite Matrices

Let $\mathrm{M}(m)$ be the space of $m \times m$ complex matrices equipped with the operator norm $||\cdot||$, $\mathrm{H}(m)$ the space of $m \times m$ complex Hermitian matrices, and $\Omega = \Omega(m)$ the convex cone of positive definite Hermitian matrices. The general linear group $\mathrm{GL}(m, C)$ acts on $\Omega(m)$ transitively via congruence transformations $\Gamma_M(X) = MXM^*$. For $X, Y \in \mathrm{H}(m)$, we write that $X \leq Y$ if $Y - X$ is positive semidefinite, and $X < Y$ if $Y - X$ is positive definite (positive semidefinite and invertible). Each positive semidefinite matrix A has a unique positive semidefinite square root, denoted by $A^{1/2}$. For $A \in \mathrm{H}(m)$, $\lambda_j(A)$ are the eigenvalues of A in non-increasing order: $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_m(A)$. We note that the operator norm of a Hermitian matrix A coincides with its spectral norm $||A|| = \max\{|\lambda_j(A)| : 1 \leq j \leq m\}$. The Thompson metric given by

$$d(A, B) = \max\{\log M(B/A), \log M(A/B)\}, \tag{2.1}$$

where $M(B/A) = \inf\{\alpha > 0 : B \leq \alpha A\} = \lambda_1(A^{-1/2}BA^{-1/2})$ is a complete metric on the open convex cone $\Omega = \Omega(m)$. For $0 < A \leq B$, we denote $[A, B] = \{X > 0 : A \leq X \leq B\}$.

Lemma 2.1. *Let $0 < A \leq B$. If $X, Y \in [A, B]$ then $d(X, Y) \leq d(A, B)$.*

Lemma 2.2. *The Thompson metric on $\Omega(m)$ satisfies*

$$d\left(\sum_{i=1}^{n} A_i, \sum_{i=1}^{n} B_i\right) \leq \max\{d(A_i, B_i)\}_{i=1}^{n}$$

for any $A_i, B_i \in \Omega(m), 1 \leq i \leq n$.

The following additive contraction theorem will play a key role for our purpose.

Proposition 2.3.[[14]] *Let A be a $l \times l$ positive semidefinite matrix. Then*

$$d(A + X, A + Y) \leq \frac{\alpha}{\alpha + \beta} d(X, Y), \quad X, Y \in \Omega(m)$$

where $\alpha = \max\{\lambda_1(X), \lambda_1(Y)\}$ and $\beta = \lambda_l(A)$.

3. Weighted Geometric Means of Positive Definite Matrices

The curve $t \mapsto A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ is a minimal geodesic line between A and B and its geodesic middle $A\#B := A\#_{1/2}B$ is known as the geometric mean of A and B.

Theorem 3.1.[[2]] *For $A, B, C \in \Omega$ and $M \in \mathrm{GL}(m, C)$,*

(1) $d(A, B) = d(A^{-1}, B^{-1}) = d(MAM^*, MBM^*)$;

(2) $d(A\#B, A) = d(A\#B, B) = \frac{1}{2}d(A, B)$;

(3) *for all* $s, t \in [0, 1]$, $d(A\#_tB, A\#_sB) = |s - t|d(A, B)$ *and*

$$d(A\#_tB, C\#_tD) \leq (1 - t)d(A, C) + td(B, D). \tag{3.1}$$

The non-positive curvature property (3.1) for the Thompson metric is appeared in [8] even for positive definite operators on a Hilbert space.
The following properties for the weighted geometric mean $A\#_tB$ are well-known.

Lemma 3.2. *Let $A, B, C, D \in \Omega$ and let $t \in [0, 1]$. Then*

(i) $A\#_tB = A^{1-t}B^t$ *if* $AB = BA$;

(ii) $(aA)\#_t(bB) = a^{1-t}b^t(A\#_tB)$ *for* $a, b > 0$;

(iii) (Löwner-Heinz inequality) $A\#_tB \leq C\#_tD$ *if* $A \leq C$ *and* $B \leq D$;

(iv) $M(A\#_tB)M^* = (MAM^*)\#_t(MBM^*)$ *for non-singular* M;

(v) $A\#_tB = B\#_{1-t}A, (A\#_tB)^{-1} = A^{-1}\#_tB^{-1}$;

(vi) $(\lambda A+(1-\lambda)B)\#_t(\lambda C+(1-\lambda)D) \geq \lambda(A\#_tC)+(1-\lambda)(B\#_tD)$ *for* $\lambda \in [0, 1]$;

(vii) $\det(A\#_tB) = \det(A)^{1-t}\det(B)^t$;

(viii) $((1 - t)A^{-1} + tB^{-1})^{-1} \leq A\#_tB \leq (1 - t)A + tB$.

$A\#_tB$ as a two-variable weighted mean, denoted by $\mathfrak{G}_2(1 - t, t; A, B)$, the authors of [11] have constucted for each $n > 2$ a weighted geometric mean $\mathfrak{G}_n(\omega; A_1, \ldots, A_n)$, where $\omega = (w_1, \ldots, w_n)$ varies over n-dimentional positive probability vectors via weighted version of Bini-Meini-Poloni symmetrization procedure [4]. Let (w_1, w_2, w_3) be a positive probability vector and let A_1, A_2, A_3 be positive definite matrices. Starting with $(A_1^{(0)}, A_2^{(0)}, A_3^{(0)}) = (A_1, A_2, A_3)$ define

$$\begin{aligned}
\left(A_1^{(1)}, A_2^{(1)}, A_3^{(1)}\right) = \Big(&A_1\#_{1-w_1}(A_2\#_{\frac{w_3}{1-w_1}}A_3),\\
&A_2\#_{1-w_2}(A_1\#_{\frac{w_3}{1-w_2}}A_3),\\
&A_3\#_{1-w_3}(A_1\#_{\frac{w_2}{1-w_3}}A_2)\Big),\\
&\vdots\\
\left(A_1^{(r)}, A_2^{(r)}, A_3^{(r)}\right) = \Big(&A_1^{(r-1)}\#_{1-w_1}(A_2^{(r-1)}\#_{\frac{w_3}{1-w_1}}A_3^{(r-1)}),\\
&A_2^{(r-1)}\#_{1-w_2}(A_1^{(r-1)}\#_{\frac{w_3}{1-w_2}}A_3^{(r-1)}),\\
&A_3^{(r-1)}\#_{1-w_3}(A_1^{(r-1)}\#_{\frac{w_2}{1-w_3}}A_2^{(r-1)})\Big).
\end{aligned}$$

It is shown that the sequences $\{A_i^{(r)}\}_{r=0}^{\infty}, i = 1, 2, 3$, converge to a common limit, yielding geometric means of 3-positive definite matrices $\mathfrak{G}_3(w_1, w_2, w_3; A_1, A_2, A_3)$. Inductively, for $n-$dimensional positive probability vector $\omega = (w_1, w_2, \ldots, w_n)$, the weighted symmetrization procedure of n-positive definite matrices is defined by

$$\gamma_\omega(A) \quad = \quad (A_1 \#_{1-w_1} \mathfrak{G}_{n-1}(\hat{\omega}_{\neq 1}; A_{k\neq 1}), \ldots, A_n \#_{1-w_n} \mathfrak{G}_{n-1}(\hat{\omega}_{\neq n}; A_{k\neq n}))$$

where $\hat{\omega}_{\neq i} = \frac{1}{1-w_i}(w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)$ is $(n-1)$-dimensional positive probability vector and $A_{k\neq i} = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ of $A = (A_1, \ldots, A_n)$. Then each component of the iteration $\gamma_\omega^r(A) = (A_1^{(r)}, \ldots, A_n^{(r)})$ approaches to a common limit, yielding n-dimensional weighted geometric mean $\mathfrak{G}_n(\omega; A)$.
The weighted geometric means of $n-$positive definite matrices satisfy the following properties.

Theorem 3.3. *Let* $A = (A_1, A_2, \ldots, A_n), B = (B_1, B_2, \ldots, B_n) \in \Omega^n$ *and let* $\omega = (w_1, w_2, \ldots, w_n)$ *be a positive probability vector.*

(P1) $\mathfrak{G}_n(\omega; A_1, \ldots A_n) = A_1^{w_1} \cdots A_n^{w_n}$ *for commuting* A_i*'s.*

(P2) (*Joint homogeneity*)

$$\mathfrak{G}_n(\omega; a_1A_1, \ldots, a_nA_n) = a_1^{w_1} \cdots a_n^{w_n} \mathfrak{G}_n(\omega; A_1, \ldots, A_n).$$

(P3) (*Permutation invariance*) *For any permutation* σ*,*

$$\mathfrak{G}_n(\omega_\sigma; A_{\sigma(1)}, \ldots, A_{\sigma(n)}) = \mathfrak{G}_n(\omega; A_1, \ldots, A_n).$$

(P4) (*Monotonicity*) *If* $B_i \leq A_i$ *for all* $1 \leq i \leq n$*, then*

$$\mathfrak{G}_n(\omega; B_1, \ldots, B_n) \leq \mathfrak{G}_n(\omega; A_1, \ldots, A_n).$$

(P5) (*Continuity*) *The map* $\mathfrak{G}_n(\omega; \cdot)$ *is continuous.*

(P6) (*Congruence invariance*) *For any invertible matrix* M*,*

$$\mathfrak{G}_n(\omega; MA_1M^*, \ldots, MA_nM^*) = M\mathfrak{G}_n(\omega; A_1, \ldots, A_n)M^*.$$

(P7) (*Joint Concavity*) *For* $0 \leq \lambda \leq 1$*,*

$$\begin{aligned} &\mathfrak{G}_n(\omega; \lambda A_1 + (1-\lambda)B_1, \ldots, \lambda A_n + (1-\lambda)B_n) \\ &\qquad \geq \lambda \mathfrak{G}_n(\omega; A_1, \ldots, A_n) + (1-\lambda)\mathfrak{G}_n(\omega; B_1, \ldots, B_n). \end{aligned}$$

(P8) (*Self-duality*) $\mathfrak{G}_n(\omega; A_1^{-1}, \ldots, A_n^{-1}) = \mathfrak{G}_n(\omega; A_1, \ldots, A_n)^{-1}$.

(P9) (*Determinantal identity*) $\det(\mathfrak{G}_n(\omega; A_1, \ldots, A_n)) = \prod_{i=1}^n (\det A_i)^{\omega_i}$.

(P10) (*Arithmetic-geometric-harmonic mean inequality*)

$$\left(\sum_{i=1}^{n} w_i A_i^{-1}\right)^{-1} \leq \mathfrak{G}_n(\omega; A_1 \ldots, A_n) \leq \sum_{i=1}^{n} w_i A_i.$$

We call a mean of n-variables satisfying these properties a weighted geometric mean. We note that for $\omega = (1/n, \ldots, 1/n)$ the geometric means obtained by Ando-Li-Mathias [1] and by Bini-Meini-Poloni [4] satisfy (P1)-(P10).

Remark 3.4. *In [11], the authors obtained a stronger version of* $(P5)$ *for the Thompson metric;*

(P11) $d(\mathfrak{G}_n(\omega; A_1, \ldots, A_n), \mathfrak{G}_n(\omega; B_1, \ldots, B_n)) \leq \sum_{i=1}^{n} w_i d(A_i, B_i)$.

Proposition 3.5. *Let* $\omega = (w_1, \ldots, w_n)$ *and* $\nu = (v_1, \ldots, v_n)$ *be positive probability vectors. Then*

$$d(\mathfrak{G}_n(\omega; A_1, \ldots, A_n), \mathfrak{G}_n(\nu; B_1, \ldots, B_n)) \leq \sum_{i,j=1}^{n} w_i v_j d(A_i, B_j). \tag{3.2}$$

Proof. By (P1) and (P11), we have

$$\begin{aligned}
&d(\mathfrak{G}_n(\omega; A_1, \ldots, A_n), \mathfrak{G}_n(\nu; B_1, \ldots, B_n)) \\
&\quad= d(\mathfrak{G}_n(\omega; A_1, \ldots, A_n), \mathfrak{G}_n(\omega; \mathfrak{G}_n(\nu; B_1, \ldots, B_n), \ldots, \mathfrak{G}_n(\nu; B_1, \ldots, B_n)) \\
&\quad\leq \sum_{i=1}^{n} w_i d(A_i, \mathfrak{G}_n(\nu; B_1, \ldots, B_n)) \\
&\quad= \sum_{i=1}^{n} w_i d(\mathfrak{G}_n(\nu; A_i, \ldots, A_i), \mathfrak{G}_n(\nu; B_1, \ldots, B_n)) \\
&\quad\leq \sum_{i=1}^{n} \sum_{j=1}^{n} w_i v_j d(A_i, B_j)).
\end{aligned}$$

□

4. Higher Order Weighted Carlson's Algorithm

Let $\Delta_n = \{(w_1, \ldots, w_n) \in (0,1)^n : \sum_{i=1}^{n} w_i = 1\}$ be the set of $n(n \geq 2)$-dimensional positive probability vectors. For $\omega = (w_1, \ldots, w_n) \in \Delta_n$, we denote

$$\begin{aligned}
\omega_{\neq j} &= (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n) \\
\omega_{\neq i,j} &= (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n) \quad (i < j) \\
\hat{\omega}_{\neq j} &= \frac{1}{1 - w_j} \omega_{\neq j} \in \Delta_{n-1}, \ (n \geq 3)
\end{aligned}$$

and $w_{jk} = \begin{cases} \frac{w_k}{1-w_j}, & \text{if } j \neq k \\ 0, & \text{if } j = k. \end{cases}$ Then we have $\hat{\omega}_{\neq j} = (\omega_{j1}, \dots, \omega_{j(j-1)}, \omega_{j(j+1)}, \dots, \omega_{jn})$.
For $A = (A_1, \dots, A_{n+1}) \in \Omega^{n+1}$ and $\omega = (w_1, \dots, w_{n+1}) \in \Delta_{n+1}$, we consider the symmetrization procedure of arithmetic and harmonic means

$$\beta_\omega(A) = \left(\sum_{k=1}^{n+1} w_{1k} A_k, \dots, \sum_{k=1}^{n+1} w_{(n+1)k} A_k \right), \tag{4.1}$$

$$\beta_\omega^*(A) = \left((\sum_{k=1}^{n+1} w_{1k} A_k^{-1})^{-1}, \dots, (\sum_{k=1}^{n+1} w_{(n+1)k} A_k^{-1})^{-1} \right). \tag{4.2}$$

It is not difficult to see that there exist positive definite matrices X^* and Y^* such that

$$\lim_{r\to\infty} \beta_\omega^r(A) = (X^*, \dots, X^*) \text{ and } \lim_{r\to\infty} (\beta_\omega^*)^r(A) = (Y^*, \dots, Y^*).$$

Indeed, the map β has the linear representation

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n+1} \end{bmatrix} \mapsto \begin{bmatrix} 0 & w_{12} & w_{13} & \cdots & w_{1(n+1)} \\ w_{21} & 0 & w_{23} & \cdots & w_{2(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(n+1)1} & w_{(n+1)2} & w_{(n+1)3} & \cdots & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n+1} \end{bmatrix}$$

where $w_{ij} = \frac{w_j}{1-w_i}$ and

$$\begin{bmatrix} 0 & w_{12} & w_{13} & \cdots & w_{1(n+1)} \\ w_{21} & 0 & w_{23} & \cdots & w_{2(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(n+1)1} & w_{(n+1)2} & w_{(n+1)3} & \cdots & 0 \end{bmatrix}^r \to \begin{bmatrix} z_1 & z_2 & z_3 & \cdots & z_{n+1} \\ z_1 & z_2 & z_3 & \cdots & z_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1 & z_2 & z_3 & \cdots & z_{n+1} \end{bmatrix}.$$

A stochastic matrix is said to be *regular* if some power has all positive entries. We note that the obtained $(n+1) \times (n+1)$ matrix is a regular stochastic matrix with an eigenvector $z = (z_1, z_2, , \dots, z_{n+1})^T$. It is well-known (cf. Chapter 8, [9]) that every regular stochastic matrix M has a unique probability vector z with all positive components such that $Mz = z$, and the sequence $\{M^k\}$ converges to a matrix S whose columns are the fixed column vector z.
The case β_ω^* follows from $\beta_\omega(A^{-1})^{-1} = \beta_\omega^*(A)$.
Define a self-map on Ω^{n+1};

$$\gamma_\omega(A) = \left(A_1 \#_{1-w_1} \mathfrak{G}_n(\hat{\omega}_{\neq 1}; A_{k\neq 1}), \dots, A_{n+1} \#_{1-w_{n+1}} \mathfrak{G}_n(A_{k\neq n+1})\right) \tag{4.3}$$

where $A = (A_1, \dots, A_{n+1}) \in \Omega^{n+1}$ and

$$A_{k\neq i} := (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{n+1}) \in \Omega^n.$$

We consider the compositions δ_ω and μ_ω on Ω^{n+1} defined as

$$\delta_\omega := \gamma_\omega \circ \beta_\omega \quad \text{ and } \quad \mu_\omega := \gamma_\omega \circ \beta^*_\omega.$$

Theorem 4.1. *For each $A \in \Omega^{n+1}$, the sequences $\{\delta^r_\omega\}_{r\in N}$ and $\{\mu^r_\omega\}_{r\in N}$ are power convergent, i.e., there exist $X^*, X_* \in \Omega$ such that*

$$\lim_{r\to\infty} \delta^r_\omega(A) = (X^*, X^*, \dots, X^*),$$
$$\lim_{r\to\infty} \mu^r_\omega(A) = (X_*, X_*, \dots, X_*).$$

Proof. Let $A \in \Omega^{n+1}$. Setting $\gamma = \gamma_\omega, \delta = \delta_\omega$ and

$$\delta^r(A) = (A^{(r)}_1, A^{(r)}_2, \dots, A^{(r)}_{n+1}), \tag{4.4}$$

$$(\beta \circ \delta^r)(A) = (X^{(r)}_1, X^{(r)}_2, \dots, X^{(r)}_{n+1}), \tag{4.5}$$

we have $A^{(0)}_i = A_i, X^{(0)}_i = \sum_{k=1}^{n+1} w_{ik} A_k$ and

$$A^{(r+1)}_i = X^{(r)}_i \#_{1-w_i} \mathfrak{G}_n(\hat{\omega}_{\neq i}; (X^{(r)}_k)_{k\neq i}), \quad X^{(r)}_i = \sum_{k=1}^{n+1} w_{ik} A^{(r)}_k.$$

By Theorem 3.1 (i) and Lemma 2.2,

$$\begin{aligned}
d(X^{(r)}_i, X^{(r)}_j) =& d\left(\sum_{k=1}^{n+1} w_{ik} A^{(r)}_k, \sum_{l=1}^{n+1} w_{jl} A^{(r)}_l\right)\\
=& d\left(\sum_{k=1}^{n+1} w_{ik} A^{(r)}_k, \sum_{k=1}^{n+1} w_{ik} \left(\sum_{l=1}^{n+1} w_{jl} A^{(r)}_l\right)\right)\\
\le& \max_k d\left(w_{ik} A^{(r)}_k, w_{ik} \sum_{l=1}^{n+1} w_{jl} A^{(r)}_l\right)\\
=& \max_k d\left(A^{(r)}_k, \sum_{l=1}^{n+1} w_{jl} A^{(r)}_l\right)\\
=& \max_k d\left(\sum_{l=1}^{n+1} w_{jl} A^{(r)}_k, \sum_{l=1}^{n+1} w_{jl} A^{(r)}_l\right)\\
\le& \max_{k,l} d\left(w_{jl} A^{(r)}_k, w_{jl} A^{(r)}_l\right)\\
=& \Delta(A^{(r)}_1, \dots, A^{(r)}_{n+1})
\end{aligned}$$

and

$$\Delta((X_1^{(r)},\dots,X_{n+1}^{(r)})) \le \Delta(A_1^{(r)},\dots,A_{n+1}^{(r)}) \tag{4.6}$$

where $\Delta(A_1,\dots,A_{n+1}) = \max_{1\le i,j\le n+1}\{d(A_i,A_j)\}$, the diameter of $\{A_i\}_{i=1}^{n+1}$.
By Theorem 3.1 (3),

$$\begin{aligned}
d(A_i^{(r+1)},A_j^{(r+1)}) =& d(X_i^{(r)}\#_{1-w_i}\mathfrak{G}_n(\hat{\omega}_{\neq i};(X_k^{(r)})_{k\neq i}),X_j^{(r)}\#_{1-w_j}\mathfrak{G}_n(\hat{\omega}_{\neq j};(X_l^{(r)})_{l\neq j}))\\
\le& w_iw_j\ d(X_i^{(r)},X_j^{(r)})\\
&+w_i(1-w_j)\ d(X_i^{(r)},\mathfrak{G}_n(\hat{\omega}_{\neq j};(X_l^{(r)})_{l\neq j}))\\
&+(1-w_i)w_j\ d(\mathfrak{G}_n(\hat{\omega}_{\neq i};(X_k^{(r)})_{k\neq i}),X_j^{(r)})\\
&+(1-w_i)(1-w_j)\ d(\mathfrak{G}_n(\hat{\omega}_{\neq i};(X_k^{(r)})_{k\neq i}),\mathfrak{G}_n(\hat{\omega}_{\neq j};(X_l^{(r)})_{l\neq j}).
\end{aligned}$$

By Proposition 3.5, we have following inequalities

$$\begin{aligned}
&d(\mathfrak{G}_n(\hat{\omega}_{\neq i};(X_k^{(r)})_{k\neq i}),\mathfrak{G}_n(\hat{\omega}_{\neq j};(X_l^{(r)})_{l\neq j}))\\
&\quad\le \sum_{k\neq i}\sum_{l\neq j}\frac{w_k}{1-w_i}\frac{w_l}{1-w_j}d(X_k^{(r)},X_l^{(r)})\\
&\quad= \sum_{k\neq i,l}\sum_{l\neq j}\frac{w_k}{1-w_i}\frac{w_l}{1-w_j}d(X_k^{(r)},X_l^{(r)})+\sum_{l\neq i,j}\frac{w_l}{1-w_i}\frac{w_l}{1-w_j}d(X_l^{(r)},X_l^{(r)})\\
&\quad\le \sum_{k\neq i,l}\sum_{l\neq j}\frac{w_k}{1-w_i}\frac{w_l}{1-w_j}\Delta(X_1^{(r)},\dots,X_{n+1}^{(r)})\\
&\quad= \left(1-\sum_{l\neq i,j}\frac{w_l}{1-w_i}\frac{w_l}{1-w_j}\right)\Delta(X_1^{(r)},\dots,X_{n+1}^{(r)}),
\end{aligned}$$

$$\begin{aligned}
d(X_i^{(r)},\mathfrak{G}_n(\hat{\omega}_{\neq j};(X_l^{(r)})_{l\neq j})) &\le \sum_{l\neq j}\frac{w_l}{1-w_j}d(X_i^{(r)},X_l^{(r)})\\
&= \sum_{l\neq i,j}\frac{w_l}{1-w_j}d(X_i^{(r)},X_l^{(r)})+\frac{w_i}{1-w_j}d(X_i^{(r)},X_i^{(r)})\\
&\le \sum_{l\neq i,j}\frac{w_l}{1-w_j}\Delta(X_1^{(r)},\dots,X_{n+1}^{(r)})\\
&= \left(1-\frac{w_i}{1-w_j}\right)\Delta(X_1^{(r)},\dots,X_{n+1}^{(r)})
\end{aligned}$$

and

$$d(\mathfrak{G}_n(\hat{\omega}_{\neq i};(X_k^{(r)})_{k\neq i}),X_j^{(r)}) \le \left(1-\frac{w_j}{1-w_i}\right)\Delta(X_1^{(r)},\dots,X_{n+1}^{(r)}).$$

Therefore

$$
\begin{aligned}
& d(A_i^{(r+1)}, A_j^{(r+1)}) \\
& \quad \leq \left(w_i w_j + w_i(1-w_j)\left(1 - \frac{w_i}{1-w_j}\right) + (1-w_i)w_j\left(1-\frac{w_j}{1-w_i}\right) \right. \\
& \qquad \left. +(1-w_i)(1-w_j)\left(1 - \sum_{l\neq i,j} \frac{w_l}{1-w_i}\frac{w_l}{1-w_j}\right)\right) \Delta(X_1^{(r)}, \ldots, X_{n+1}^{(r)}) \\
& \quad = \left(1 - \sum_{k=1}^{n+1} w_k^2\right) \Delta(X_1^{(r)}, \ldots, X_{n+1}^{(r)}) \\
& \quad \leq \left(1 - \sum_{k=1}^{n+1} w_k^2\right) \Delta(A_1^{(r)}, \ldots, A_{n+1}^{(r)}).
\end{aligned}
$$

the last inequality follows by (4.9). Inductively, we have

$$
d(A_i^{(r)}, A_j^{(r)}) \leq \left(1 - \sum_{k=1}^{n+1} w_k^2\right) \Delta(A_1^{(r-1)}, \ldots, A_{n+1}^{(r-1)}) \tag{4.7}
$$

$$
\leq \cdots \leq \left(1 - \sum_{k=1}^{n+1} w_k^2\right)^r \Delta(A_1^{(0)}, \ldots, A_{n+1}^{(0)}). \tag{4.8}
$$

By Lemma 2.2,

$$
\begin{aligned}
d(A_i^{(r)}, X_j^{(r)}) &= d\left(A_i^{(r)}, \sum_{k\neq j} w_{jk} A_k^{(r)}\right) = d\left(\sum_{k\neq j} w_{jk} A_i^{(r)}, \sum_{k\neq j} w_{jk} A_k^{(r)}\right) \\
&\leq \max_{k\neq j}\left\{ d\left(A_i^{(r)}, A_k^{(r)}\right)\right\} \overset{(4.7)}{\leq} \left(1 - \sum_{k=1}^{n+1} w_k^2\right)^r \Delta(A)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
d(A_i^{(r)}, A_i^{(r+1)}) &= d(A_i^{(r)}, X_i^{(r)} \#_{1-w_i} \mathfrak{G}_n(\hat{\omega}_{\neq i}; (X_k^{(r)})_{k\neq i})) \\
&\leq w_i d(A_i^{(r)}, X_i^{(r)}) + (1-w_i)\ d(A_i^{(r)}, g((X_k^{(r)})_{k\neq i})) \\
&\leq w_i d(A_i^{(r)}, X_i^{(r)}) + \sum_{k\neq i} w_k d(A_i^{(r)}, X_k^{(r)}) \\
&\leq w_i \left(1 - \sum_{k=1}^{n+1} w_k^2\right)^r \Delta(A) + \sum_{k\neq i} w_k \left(1 - \sum_{k=1}^{n+1} w_k^2\right)^r \Delta(A) \\
&= \left(1 - \sum_{k=1}^{n+1} w_k^2\right)^r \Delta(A).
\end{aligned}
$$

This together with (4.7) shows that the sequences $A_i^{(r)}, 1 \leq i \leq n+1$, have a common limit.
The proof for μ_ω is similar to that of δ_ω by using the invariancy of the Thompson metric and the weighted geometric mean under the inversion (Lemma 3.2). □

Definition 4.2. We denote $\mathfrak{C}(\omega; A)$ (resp. $\mathfrak{C}^*(\omega; A)$) by the common limit of the iteration $\delta_\omega = \gamma_\omega \circ \beta_\omega$ (resp. μ_ω) at A.

Remark 4.3. *We consider the iterative mean algorithm $(\beta \circ \gamma)(A)$. From $(\beta \circ \gamma)^{r+1} = \beta \circ (\gamma \circ \beta)^r \circ \gamma = \beta \circ \delta^r \circ \gamma$, we have $(\beta \circ \gamma)^{r+1}(A) = \beta(\delta^r(\gamma(A)))$. Passing to the limit as $r \to \infty$ yields*

$$\begin{aligned}\lim_{r\to\infty}(\beta \circ \gamma)^{r+1}(A) &= \beta(\lim_{r\to\infty}\delta^r(\gamma(A))) = \beta(\mathfrak{C}(\omega; \gamma(A)), \ldots, \mathfrak{C}(\omega; \gamma(A))) \\ &= (\mathfrak{C}(\omega; \gamma(A)), \ldots, \mathfrak{C}(\omega; \gamma(A)))\end{aligned}$$

where the last equality follows from the fact that $\beta(A, A, \ldots, A) = (A, A, \ldots, A)$ for all $A > 0$.

One may have interest in finding some properties of $\mathfrak{C}(\omega; A)$ and $\mathfrak{C}^*(\omega; A)$. The following results actually show that most of common properties of the arithmetic mean and the weighted geometric mean $\mathfrak{G}_n$ are preserved by $\mathfrak{C}$.

Theorem 4.4. *The map $\mathfrak{C} : \Delta_{n+1} \times \Omega^{n+1} \to \Omega$ satisfies the following properties; for $A = (A_1, A_2, \ldots, A_{n+1}), B = (B_1, B_2, \ldots, B_{n+1}) \in \Omega^{n+1}$, a permutation σ on $n+1$-letters, an invertible matrix M, and for $0 \leq \lambda \leq 1$,*

(C1) (*Idempotency*) $\mathfrak{C}(\omega; A, A, \ldots, A) = A$;

(C2) (*Homogeneity*) *for any $s > 0$, $\mathfrak{C}(\omega; sA) = s\ \mathfrak{C}(\omega; A)$;*

(C3) (*Permutation symmetry*) $= \mathfrak{C}(\omega_\sigma; A_{\sigma(1)}, \ldots, A_{\sigma(n+1)})$;

(C4) (*Monotonicity*) *If $B_i \leq A_i$ for all i, then $\mathfrak{C}(\omega; B) \leq \mathfrak{C}(\omega; A)$;*

(C5) (*Continuity*) *$\mathfrak{C}(\omega; \cdot)$ is continuous;*

(C6) (*Congruence Invariancy*) $\mathfrak{C}(\omega; MA_1M^*, \ldots, MA_nM^*) = M\mathfrak{C}(\omega; A)M^*$;

(C7) (*Joint Concavity*) $\mathfrak{C}(\omega; \lambda A + (1-\lambda)B) \geq \lambda\mathfrak{C}(\omega; A) + (1-\lambda)\mathfrak{C}(\omega; B)$;

(C8) (*Duality*) $(\mathfrak{C}(\omega; A^{-1}))^{-1} = \mathfrak{C}^*(\omega; A)$;

For regular stochastic matrices

$$R_1 = \begin{bmatrix} w_1 & w_2 & w_3 & \cdots & w_{n+1} \\ w_1 & w_2 & w_3 & \cdots & w_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_1 & w_2 & w_3 & \cdots & w_{n+1} \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0 & \frac{w_2}{1-w_1} & \frac{w_3}{1-w_1} & \cdots & \frac{w_{n+1}}{1-w_1} \\ \frac{w_1}{1-w_2} & 0 & \frac{w_3}{1-w_2} & \cdots & \frac{w_{n+1}}{1-w_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{w_1}{1-w_{n+1}} & \frac{w_2}{1-w_{n+1}} & \frac{w_3}{1-w_{n+1}} & \cdots & 0 \end{bmatrix}.$$

and an eigenvector $z = (z_1, \ldots, z_{n+1})^T$ *of the regular stochastic matrix* $(R_1 R_2)^T$ *corresponding to the eigenvalue 1, we have*

(C9) (*Determinantal inequality*) $\det \mathfrak{C}(\omega; A) \le \prod_{i=1}^{n+1} (\det A_i)^{z_i}$;

(C10) (*ACH mean inequalities*)

$$\left(\sum_{k=1}^{n+1} z_k A_k^{-1} \right)^{-1} \le \mathfrak{C}^*(\omega; \mathbb{A}) \le \mathfrak{C}(\omega; \mathbb{A}) \le \sum_{k=1}^{n+1} z_k A_k$$

Proof. Let $A = (A_1, \ldots, A_{n+1}), B = (B_1, \ldots, B_{n+1}) \in \Omega^{n+1}$. Set $\gamma = \gamma_\omega, \delta = \delta_\omega$ and

$$\begin{aligned} \delta_\omega^r(A) =& (A_1^{(r)}, \ldots, A_{n+1}^{(r)}), \quad \delta_\omega^r(B) = (B_1^{(r)}, \ldots, B_{n+1}^{(r)}) \\ (\beta_\omega \circ \delta_\omega^r)(A) =& (X_1^{(r)}, \ldots, X_{n+1}^{(r)}), \quad (\beta_\omega \circ \delta_\omega^r)(B) = (Y_1^{(r)}, \ldots, Y_{n+1}^{(r)}). \end{aligned}$$

We consider the partial order on Ω^{n+1}; $B \le A$ if and only if $B_i \le A_i, \quad i = 1, \ldots, n+1$. One may see that β_ω and γ_ω are monotone functions by the monotonicity of $\mathfrak{G}_n$ and two-variable weighted geometric means ((P4) and Lemma 3.2). In particular, $\delta_\omega = \gamma_\omega \circ \beta_\omega$ is monotone.

(C1) It follows from $\beta_\omega(A, \ldots, A) = \gamma_\omega(A, \ldots, A) = (A, \ldots, A)$.

(C2) It follows from $\beta_\omega(sA) = s\beta_\omega(A)$ and $\gamma_\omega(sA) = s\gamma_\omega(A)$.

(C3) Let σ be a permutation on $(n+1)$-letters. Put $\omega_\sigma = (w_{\sigma(1)}, w_{\sigma(2)}, \ldots, w_{\sigma(n+1)})$. We consider the sequences $\{B_i^{(r)}\}_{r=0}^\infty$ which determine the ω_σ−weighted Carlson mean $\mathfrak{G}_n(\omega_\sigma; A_\sigma)$. By definition, $B_i^{(0)} = B_i = A_{\sigma(i)} = A_{\sigma(i)}^{(0)}$ for all i. Suppose that $B_i^{(r)} = A_{\sigma(i)}^{(r)}$ for all i. Then $Y_i^{(r)} = \sum_{k \ne i} w_{\sigma(k)} B_k^{(r)} = \sum_{k \ne i} w_{\sigma(k)} A_{\sigma(k)}^{(r)} = \sum_{k \ne \sigma(i)} w_k A_k^{(r)} = X_{\sigma(i)}^{(r)}$ for all i. By the permutation invariancy of $\mathfrak{G}_n$,

$$\begin{aligned} B_i^{(r+1)} &= Y_i^{(r)} \#_{1-w_{\sigma(i)}} \mathfrak{G}_n(\widehat{(\omega_\sigma)}_{\ne i}; (Y_k^{(r)})_{k \ne i}) \\ &= X_{\sigma(i)}^{(r)} \#_{1-w_{\sigma(i)}} \mathfrak{G}_n(\widehat{(\omega_\sigma)}_{\ne i}; X_{\sigma(1)}^{(r)}, \ldots, X_{\sigma(i-1)}^{(r)}, X_{\sigma(i+1)}^{(r)}, \ldots X_{\sigma(n+1)}^{(r)}) \\ &= X_{\sigma(i)}^{(r)} \#_{1-w_{\sigma(i)}} \mathfrak{G}_n(\hat{\omega}_{\ne \sigma(i)}; X_1^{(r)}, \ldots, X_{\sigma(i)-1}^{(r)}, X_{\sigma(i)+1}^{(r)}, \ldots X_{n+1}^{(r)}) \\ &= X_{\sigma(i)}^{(r)} \#_{1-w_{\sigma(i)}} \mathfrak{G}_n(\hat{\omega}_{\ne \sigma(i)}; (X_k^{(r)})_{k \ne \sigma(i)}) = A_{\sigma(i)}^{(r+1)}. \end{aligned}$$

By definition of $\mathfrak{C}(\omega; A)$ and $\mathfrak{C}(\omega_\sigma; A_\sigma)$, we have

$$\mathfrak{C}(\omega; A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(n+1)}) = \lim_{r \to \infty} B_i^{(r)} = \lim_{r \to \infty} A_{\sigma(i)}^{(r)} = \mathfrak{C}(\omega; A).$$

(C4) Let $B_i > 0$ with $B_i \leq A_i$ for all $i = 1, \ldots, n+1$. Then $\beta_\omega(B) \leq \beta_\omega(A)$ and $\gamma_\omega(B) \leq \gamma_\omega(A)$ and hence $\delta_\omega(B) = (\gamma_\omega \circ \beta_\omega)(B) = \gamma_\omega(\beta_\omega(B)) \leq \gamma_\omega(\beta_\omega(A)) = (\gamma_\omega \circ \beta_\omega)(A) = \delta_\omega(A)$. Since $\leq$ is closed in Ω and $\delta^r_\omega(B) \leq \delta^r_\omega(A)$, we conclude that $\mathfrak{C}(\omega; B) \leq \mathfrak{C}(\omega; A)$.
(C5) By Lemma 2.2,

$$\begin{aligned} d(X_i^{(r)}, Y_i^{(r)}) =& d\left(\sum_{k\neq i} \frac{w_k}{1-w_i} A_k^{(r)}, \sum_{k\neq i} \frac{w_k}{1-w_i} B_k^{(r)}\right) \\ &\leq \max_{k\neq i} d\left(\frac{w_k}{1-w_i} A_k^{(r)}, \frac{w_k}{1-w_i} B_k^{(r)}\right) = \max_{k\neq i} d(A_k^{(r)}, B_k^{(r)}) \\ &\leq \max_j d(A_j^{(r)}, B_j^{(r)}) \end{aligned}$$

for all $1 \leq i \leq n+1$. By (3.1) and by $(P11)$,

$$\begin{aligned} d(A_i^{(r+1)}, & B_i^{(r+1)}) \\ &= d\left(X_i^{(r)} \#_{1-w_i} \mathfrak{G}_n(\hat{\omega}_{\neq i}; (X_k^{(r)})_{k\neq i}), Y_i^{(r)} \#_{1-w_i} \mathfrak{G}_n(\hat{\omega}_{\neq i}; (Y_k^{(r)})_{k\neq i})\right) \\ &\leq w_i d(X_i^{(r)}, Y_i^{(r)}) + (1-w_i) d\left(\mathfrak{G}_n(\hat{\omega}_{\neq i}; (X_k^{(r)})_{k\neq i}), \mathfrak{G}_n(\hat{\omega}_{\neq i}; (Y_k^{(r)})_{k\neq i})\right) \\ &\leq w_i d(X_i^{(r)}, Y_i^{(r)}) + (1-w_i) \sum_{k\neq i} \frac{w_k}{1-w_i} d(X_k^{(r)}, Y_k^{(r)}) \\ &= \sum_{k=1}^{n+1} w_k d(X_k^{(r)}, Y_k^{(r)}) \\ &\leq \max_j d(A_j^{(r)}, B_j^{(r)}) \end{aligned}$$

and therefore

$$d(A_i^{(r)}, B_i^{(r)}) \leq \max_j d(A_j^{(r-1)}, B_j^{(r-1)}) \leq \cdots \leq \max_j d(A_j^{(0)}, B_j^{(0)}) = \max_j d(A_j, B_j).$$

Passing to the limit as $r \to \infty$ yields $d(\mathfrak{C}(\omega; A), \mathfrak{C}(\omega; B)) \leq \max_j d(A_j, B_j)$.
(C6) It follows from $\beta_\omega(MAM^*) = M\beta_\omega(A)M^*$ and $\gamma_\omega(MAM^*) = M\gamma_\omega(A)M^*$ where $MAM^* = (MA_1M^*, \ldots, MA_{n+1}M^*)$.
(C7) Let $\{C_i^{(r)}\}_{r=0}^\infty$ (resp. $\{Z_i^{(r)}\}_{r=0}^\infty$) be defined in the same fashion as $\{A_i^{(r)}\}_{r=0}^\infty$ (resp. $\{X_i^{(r)}\}_{r=0}^\infty$), but starting from $C_i = (1-\lambda)A_i + \lambda B_i$.
We shall show that $C_i^{(r)} \geq (1-\lambda)A_i^{(r)} + \lambda B_i^{(r)}$ for all r and i. From $A_i^{(0)} = A_i, B_i^{(0)} = B_i, C_i^{(0)} = C_i$, it holds true for $r = 0$. Suppose that $C_i^{(r)} \geq (1-\lambda)A_i^{(r)} + \lambda B_i^{(r)}$ for

all $i = 1, 2, \ldots, n+1$. Then

$$\begin{aligned}
Z_i^{(r)} =& \sum_{k\neq i} w_k C_k^{(r)} \\
\geq& \sum_{k\neq i} w_k((1-\lambda)A_k^{(r)} + \lambda B_k^{(r)}) \\
=&(1-\lambda)\sum_{k\neq i} w_k A_k^{(r)} + \lambda \sum_{k\neq i} w_k B_k^{(r)} \\
=&(1-\lambda)X_i^{(r)} + \lambda Y_i^{(r)}
\end{aligned}$$

for all $i = 1, 2, \ldots, n+1$. By the monotonicity and concavity of $\mathfrak{G}_n$ ((P4),(P7)) and two-variable weighted geometric means (Lemma 3.2),

$$\begin{aligned}
C_i^{(r+1)} =& Z_i^{(r)} \#_{1-w_i} \mathfrak{G}_n(\hat{\omega}_{\neq i}; (Z_k^{(r)})_{k\neq i}) \\
\geq& \left((1-\lambda)X_i^{(r)} + \lambda Y_i^{(r)}\right) \#_{1-w_i} \mathfrak{G}_n(\hat{\omega}_{\neq i}; ((1-\lambda)X_k^{(r)} + \lambda Y_k^{(r)})_{k\neq i}) \\
\geq& \left((1-\lambda)X_i^{(r)} + \lambda Y_i^{(r)}\right) \#_{1-w_i} \\
& \qquad \left((1-\lambda)\mathfrak{G}_n(\hat{\omega}_{\neq i}; (X_k^{(r)})_{k\neq i}) + \lambda \mathfrak{G}_n(\hat{\omega}_{\neq i}; (Y_k^{(r)})_{k\neq i})\right) \\
\geq& (1-\lambda)\left(X_i^{(r)} \#_{1-w_i} \mathfrak{G}_n(\hat{\omega}_{\neq i}; (X_k^{(r)})_{k\neq i})\right) \\
& \qquad + \lambda \left(Y_i^{(r)} \#_{1-w_1} \mathfrak{G}_n(\hat{\omega}_{\neq i}; (Y_k^{(r)})_{k\neq i})\right) \\
=& (1-\lambda)A_i^{(r+1)} + \lambda B_i^{(r+1)}.
\end{aligned}$$

Passing to the limit as $r \to \infty$ yields

$$\mathfrak{C}(\omega; (1-\lambda)A+\lambda B) = \lim_{r\to\infty} C_i^{(r)} \geq \lim_{r\to\infty}((1-\lambda)A_i^{(r+1)}+\lambda B_i^{(r+1)}) = (1-\lambda)\mathfrak{C}(\omega; A)+\lambda\mathfrak{C}(\omega; B).$$

(C8) It follows from $\beta_\omega(A^{-1})^{-1} = \beta_\omega^*(A)$ and $\gamma_\omega(A^{-1})^{-1} = \gamma_\omega(A)$ (Lemma 3.2 (v)) where $A^{-1} = (A_1^{-1}, \ldots, A_{n+1}^{-1})$.
(C9) By the determinant identity of $\mathfrak{G}_n$ (P9) and two-variable weighted geometric means (Lemma 3.2),

$$\begin{aligned}
\det A_i^{(r+1)} =& \det\left(X_i^{(r)} \#_{1-w_i} \mathfrak{G}_n(\hat{\omega}_{\neq i}; (X_k^{(r)})_{k\neq i})\right) \\
=& (\det X_i^{(r)})^{w_i} \left(\det \mathfrak{G}_n(\hat{\omega}_{\neq i}; (X_k^{(r)})_{k\neq i})\right)^{1-w_i} \\
=& (\det X_i^{(r)})^{w_i} \left(\prod_{k\neq i} (\det X_k^{(r)})^{\frac{w_k}{1-w_i}}\right)^{1-w_i} \\
=& \prod_{k=1}^{n+1} (\det X_k^{(r)})^{w_k}
\end{aligned}$$

and

$$\det X_i^{(r)} = \det\left(\sum_{k\neq i}\frac{w_k}{1-w_i}A_i^{(r)}\right) \geq \prod_{k\neq i}\det A_i^{\frac{w_k}{1-w_i}}$$

where the inequality follows by Corollary 7.6.9 of [9] for $n = 2$ and by an appropriate symmetrization method for $n > 2$.
Setting $\mathbf{a}^{(r)} = (a_1^{(r)}, \ldots, a_{n+1}^{(r)})$ and $\mathbf{x}^{(r)} = (x_1^{(r)}, \ldots, x_{n+1}^{(r)})$ where $a_i^{(r)} = \log\det A_i^{(r)}$ and $x_i^{(r)} = \log\det X_i^{(r)}$, respectively. Then we have

$$a_i^{(r+1)} = \sum_{k=1}^{n+1} w_k x_k^{(r)} \quad \text{and} \quad x_i^{(r)} \geq \sum_{k\neq i}\frac{w_k}{1-w_i}a_k$$

and these relations can be rewritten by using the regular stochastic matrices R_1 and R_2 as

$$\mathbf{a}^{(r+1)} = R_1\mathbf{x}^{(r)} \quad \text{and} \quad \mathbf{x}^{(r)} \geq R_2\mathbf{a}^{(r)}.$$

Since every row of R_1 is nonnegative probability vector,

$$\mathbf{a}^{(r+1)} \geq R_1R_2\mathbf{a}^{(r)}$$

and hence, inductively, we have

$$\mathbf{a}^{(r)} \geq (R_1R_2)\mathbf{a}^{(r-1)} \geq \cdots \geq (R_1R_2)^r\mathbf{a}^{(0)}.$$

From the facts that every component of $\mathbf{a}^{(r)}$ converges to $\log\det\mathfrak{C}(\omega; A)$ and the sequence $\{(R_1R_2)^r\}$ converges to a matrix S whose rows are the fixed row $z = (z_1, \ldots, z_{n+1})$, we obtain

$$\det\mathfrak{C}(\omega; A) \geq \prod_{i=1}^{n+1}(\det A_i)^{z_i}.$$

(C10) From the arithmetic-harmonic mean inequality, we have $\beta_\omega^*(\mathbb{A}) \leq \beta_\omega(\mathbb{A})$ and hence by the monotonicity of γ_ω, $\mu_\omega(A) = \gamma_\omega(\beta_\omega^*(A)) \leq \gamma_\omega(\beta_\omega(A)) = \delta_\omega(A)$. By monotonicity of δ_ω and by induction, $\mu_\omega^r(A) \leq \delta_\omega^r(A)$ for all r and therefore $\mathfrak{C}^*(\omega; \mathbb{A}) \leq \mathfrak{C}(\omega; \mathbb{A})$.
We consider a self-map on Ω^{n+1} defined by $\mu(\mathbb{A}) = \left(\sum_{k=1}^{n+1} w_kA_k, \ldots, \sum_{k=1}^{n+1} w_kA_k\right)$ It is not difficult to see that $\lim_{r\to\infty}(\mu\circ\beta)^r(\mathbb{A}) = (X, \ldots, X)$ and $X = \sum_{i=1}^{n+1} z_iA_i$ by using its linear representation

$$A \mapsto (R_1R_2)A$$

and the corresponding matrix (R_1R_2) is transpose of a regular stochastic matrix with an eigenvector $z = (z_1, z_2, \ldots, z_n)$ of the eigenvalue 1.

We will show by induction that $\delta^r(\mathbb{A}) \leq (\mu\circ\beta)^r(\mathbb{A})$ for all positive integers r, which implies that $\mathfrak{C}(\omega;\mathbb{A}) \leq X = \sum_{i=1}^{n+1} z_iA_i$ by passing to the limit as $r\to\infty$. By the arithmetic-geometric mean inequality (Lemma 3.2) and (P10),

$$\begin{aligned} A_i\#_{1-w_i}\mathfrak{G}_n(\hat{\omega}_{\neq i};(A_k)_{k\neq i}) \leq & w_iA_i + (1-w_i)\mathfrak{G}_n(\hat{\omega}_{\neq i};(A_k)_{k\neq i}) \\ \leq & w_iA_i + (1-w_i)\sum_{k\neq i}\frac{w_k}{1-w_i}A_k \\ = & \sum_{k=1}^{n+1} w_kA_k \end{aligned}$$

for all i and therefore $\gamma(\mathbb{A}) \leq \mu(\mathbb{A})$. Replacing $\mathbb{A}$ to $\beta(\mathbb{A})$ yields $\delta(\mathbb{A}) = (\gamma\circ\beta)(A) = \gamma(\beta(A)) \leq (\mu\circ\beta)(\mathbb{A})$. Suppose that $\delta^r(\mathbb{A}) \leq (\mu\circ\beta)^r(\mathbb{A})$. Then by the monotonicity of δ, $\delta^{r+1}(\mathbb{A}) = \delta(\delta^r(\mathbb{A})) \leq \delta((\mu\circ\beta)^r(\mathbb{A})) \leq (\mu\circ\beta)((\mu\circ\beta)^r(\mathbb{A})) = (\mu\circ\beta)^{r+1}(\mathbb{A})$. The inequality $\left(\sum_{k=1}^{n+1} z_kA_k^{-1}\right)^{-1} \leq \mathfrak{C}^*(\omega;\mathbb{A})$ follows by the preceding one and the duality (C8).

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