ON THE ADMISSIBILITY OF THE SPACE $L_0(A, X)$ OF VECTOR-VALUED MEASURABLE FUNCTIONS

Diana Caponetti, Grzegorz Lewicki, Alessandro Trombetta, and Giulio Trombetta

Abstract. We prove the admissibility of the space $L_0(A, X)$ of vector-valued measurable functions determined by real-valued finitely additive set functions defined on algebras of sets.

The notion of admissibility introduced by Klee [7] guarantees that a compact mapping into an admissible Hausdorff topological vector space $E$ can be approximated by compact finite dimensional mappings. This notion is very important in degree theory and fixed point theory. It is known that locally convex spaces are admissible (see [10]). There are some classes of nonlocally convex spaces which are admissible. Riedrich in [13] proved the admissibility of the space $S(0, 1)$ of measurable functions and in [12] the admissibility of the space $L_p(0, 1)$ for $0 < p < 1$. The admissibility of other function spaces has been proved by Mach [6] and Ishii [8]. In [14] it is proved the admissibility of spaces of Besov-Triebel-Lizorkin type.

Definition 1 ([7]). Let $E$ be a Hausdorff topological vector space. A subset $Z$ of $E$ is said to be admissible if for every compact subset $K$ of $Z$ and for every neighborhood $V$ of zero in $E$ there exists a continuous mapping $H : K \to Z$ such that $\dim(\text{span } [H(K)]) < +\infty$ and $x - Hx \in V$ for every $x \in K$. If $Z = E$ we say that the space $E$ is admissible.

In this paper we deal with spaces of vector-valued measurable functions and, as a major fact, instead of $\sigma$-additive measures we consider finitely additive set functions defined on algebras of sets.

Let $X$ be a Banach space, $\Omega$ a nonempty set, $A$ a subalgebra of the power set $\mathcal{P}(\Omega)$ of $\Omega$ and $\mu : A \to \mathbb{R}$ a finitely additive set function. We prove...
the admissibility of the space $L_0(A, X)$ of all $X$-valued $\mu$-measurable functions defined on $\Omega$ (see [3, Chp III]).

It is important to notice that in [2] Cauty provides an example of a metric linear space in which the admissibility fails. Moreover, it is known that in general $L_0(A, X)$ is not homeomorphic to the classical space $L_0([0, 1], X)$ of all Lebesgue measurable functions from $[0, 1]$ to $X$ endowed with the topology generated by the convergence in measure, and to our knowledge the question if all the spaces $L_0(A, X)$ are homeomorphic or not to an Hilbert space is open. Some results in this latter direction have appeared in [11, Theorem 4.9] in the case where $\mu$ is a finite nonatomic measure.

1. Preliminaries and notations

Let $(X, \| \cdot \|_X)$ be a real or complex Banach space, $\Omega$ a nonempty set, $A$ a subalgebra of the power set $\mathcal{P}(\Omega)$ of $\Omega$ and $\mu : A \to \mathbb{R}$ a finitely additive set function. Then for every $A \in A$ the total variation $|\mu|(A)$ of $\mu$ on $A$ is defined by $|\mu|(A) = \sup \sum_{i=1}^{n} |\mu(A_i)|$ where the supremum is taken over all finite sequences $(A_i)$ of disjoint sets in $A$ with $A_i \subseteq A$. Then $|\mu|$ induces the submeasure $\eta : \mathcal{P}(\Omega) \to [0, +\infty]$ defined by $\eta(E) = \inf\{|\mu|(A) : A \in A \text{ and } E \subseteq A\}$ for $E \subseteq \Omega$. We denote by

$$S(A, X) = \left\{ \sum_{i=1}^{n} x_i \chi_{A_i} : n \in \mathbb{N}, x_i \in X, A_i \in A \right\},$$

the space of all $X$-valued simple functions on $\Omega$; where $\chi_A$ denotes the characteristic function of the set $A$ defined on $\Omega$. Let $X^\Omega$ denote the set of all functions $f$ from $\Omega$ to $X$. For a function $f \in X^\Omega$ we set

$$\|f\|_0 = \inf \{\alpha \geq 0 : \eta(\{\|f\|_X \geq \alpha\}) \leq \alpha\},$$

where $\|f\|_X$ denotes the function $t \to \|f(t)\|_X$ and $\{\|f\|_X \geq \alpha\} = \{t \in \Omega : \|f(t)\|_X \geq \alpha\}$, with the convention $\inf \emptyset = +\infty$. Then $\| \cdot \|_0$ has the following properties:

$$\begin{align*}
\|0\|_0 &= 0, \\
\|f + g\|_0 &\leq \|f\|_0 + \|g\|_0, \\
\|f\|_X &\leq \|g\|_X \quad \text{implies} \quad \|f\|_0 \leq \|g\|_0 \quad \text{for} \quad f, g \in X^\Omega, \\
\|y \chi_A\|_0 &= \min\{\eta(A), \|y\|_X\} \quad \text{for} \quad A \subseteq \Omega, \ y \in X \text{ and } f \in X^\Omega.
\end{align*}$$

(1)

A function $f \in X^\Omega$ is said to be a $\mu$-null function if $\eta(\{\|f\|_X \geq a\}) = 0$ for any $a > 0$. Then by $L_0(A, X)$ we denote the $F$-normed space (in the sense of [9]) given by the closure of the space $S(A, X)$ in $(X^\Omega, \| \cdot \|_0)$, where it is understood that we identify functions differing by a $\mu$-null function.

We briefly recall the definitions of integrable function and integral for an integrable function, with respect to $\mu$, of a function $f$ of $L_0(A, X)$ as introduced
in [3]. Let \( s = \sum_{i=1}^{n} x_i \chi_{A_i} \) be a simple function in \( S(A, X) \) and \( A \in A \), then the integral over \( A \) of \( f \) is defined by
\[
\int_{A} f(t) \, \mu(dt) = \sum_{i=1}^{n} x_i \, \mu(A_i).
\]

Let \( L_1(X) \) denote the Lebesgue space of all functions \( f \in L_0(A, X) \) for which there is a sequence \((s_n)\) in \( S(A, X) \) converging to \( f \) with respect to \( \| \cdot \|_0 \) such that
\[
\lim_{m,n} \int_{\Omega} \|s_m(t) - s_n(t)\|_X \, |\mu|(dt) = 0.
\]
The sequence \((s_n)\) is said to be a determining sequence for \( f \) and the integral over \( A \) of \( f \) is defined by
\[
\int_{A} f(t) \, \mu(dt) = \lim_{n \to \infty} \int_{A} s_n(t) \mu(dt), \quad A \in A.
\]
For each \( f \in L_1(X) \), \( \|f\|_1 = \int_{\Omega} \|f(t)\|_X |\mu|(dt) \) and we also have
\[
\|f\|_1 = \lim_{n \to \infty} \int_{\Omega} \|s_n(t)\|_X |\mu|(dt).
\]
Obviously \( \eta(A) = |\mu|_A = \|\chi_A\|_1 \) for \( A \in A \). In the sequel we will use the property given in the following lemma.

**Lemma 1.** Let \( f \in L_1(X) \). Then \( \|f\|_0 \leq \|f\|_1^{1/2} \).

**Proof.** Let \( s \in S(A, X) \setminus \{0\} \). Assume on the contrary that \( \|s\|_0 > \|s\|_1^{1/2} \). Take \( \alpha = \|s\|_1^{1/2} \), then
\[
\|s\|_1 = \int_{\Omega} \|s(t)\|_X |\mu|(dt) \geq \int_{\{\|s\|_X \geq \alpha\}} \|s(t)\|_X |\mu|(dt) \geq \alpha \eta(\{\|s\|_X \geq \alpha\})
\]
which is a contradiction.

Next let \( f \in L_1(X) \) and \((s_n)\) a sequence in \( S(A, X) \) determining \( f \). Then we have both \( \lim_{n \to \infty} \|s_n\|_0 = \|f\|_0 \) and \( \lim_{n \to \infty} \|s_n\|_1 = \|f\|_1 \), which imply the assert. \( \square \)

Let \( B_a(X) \) denote the closed ball of radius \( a > 0 \). We denote by \( \rho_a \) the radial projection of \( X \) onto \( B_a(X) \) defined by
\[
\rho_a(x) = \begin{cases} 
  \frac{x}{\|x\|_X} & \text{if } \|x\|_X \leq a, \\
  \frac{a}{\|x\|_X} x & \text{if } \|x\|_X > a.
\end{cases}
\]
Then we define the mapping \( T_a : L_0(A, X) \to X^\Omega \) by setting
\[
(T_a f)(t) = \rho_a(f(t)), \quad t \in \Omega.
\]
The function \( T_a s \) is a simple function for each simple function \( s \in S(A, X) \), and moreover it can be easily seen that \( T_a(L_0(A, X)) \subseteq L_0(A, X) \). The projection
$\rho_a$ is Lipschitz with constant 2 (cf. [4]), thus, since $X$ is a Banach space, by (1) for $f, g \in L_0(A, X)$ we have
\[ \|T_a f - T_a g\|_0 \leq 2\|f - g\|_0. \]

**Lemma 2.** Let $K$ be a subset of $L_0(A, X)$. Then

(i) $T_a(K) \subseteq L_1(X)$,

(ii) the $\|\cdot\|_0$-topology and the $\|\cdot\|_1$-topology coincide on $T_a(K)$.

**Proof.** (i) Since $\|T_a f\|_X \leq a\chi_\Omega$ for $f \in K$, by [3, Theorem III.2.22] we have $T_a f \in L_1(X)$.

(ii) As $T_a(K) \subseteq \{f \in L_1(X) : \|f\|_X \leq a\chi_\Omega\}$ the assert follows from [3, Theorem III.3.6].

Next we introduce the operator $P_\pi$ which will be used for the proof of our main result. Given a partition $\pi = \{A_1, \ldots, A_n\}$ of $\Omega$ with $\eta(A_i) > 0$ for $i = 1, \ldots, n$ we consider $P_\pi : L_1(X) \to \mathcal{S}(A, X)$ the linear operator defined by setting
\[ P_\pi f = \sum_{i=1}^{n} \frac{\int_{A_i} f(t) \mu(dt)}{\eta(A_i)} \chi_{A_i}. \]
Then for each $f \in L_1(X)$ we have
\[ \|P_\pi f\|_1 \leq \|f\|_1. \]
Indeed, if, for each $i$, we put $s_i(f) = \int_{A_i} f(t) \mu(dt) / \eta(A_i)$ applying Jensen’s inequality, we have
\[ \|s_i(f)\|_X \leq \frac{\int_{A_i} \|f(t)\|_X \mu(dt)}{\eta(A_i)}. \]
Consequently we get
\[ \|P_\pi f\|_1 = \int_\Omega \| \sum_{i=1}^{n} s_i(f) \chi_{A_i} \| \mu|dt) = \sum_{i=1}^{n} \int_{A_i} \|s_i(f)\|_X \mu(dt) \]
\[ = \sum_{i=1}^{n} \eta(A_i) \|s_i(f)\|_X \leq \sum_{i=1}^{n} \int_{A_i} \|f(t)\|_X \mu(dt) \]
\[ = \int_\Omega \|f(t)\|_X \mu|dt) = \|f\|_1. \]

2. **Admissibility of $L_0(A, X)$**

We recall that for a bounded subset $A$ of $X$ the Hausdorff measure of non-compactness $\gamma(A)$ of $A$ is the infimum of all $\epsilon > 0$ such that $A$ has an $\epsilon$-net in $X$ ([5]). Moreover for each bounded subsets $K$ of $L_0(A, X)$ we consider the quantitative characteristic $\sigma(K)$, introduced in [1], defined by setting $\sigma(K) = \inf \{\epsilon > 0 : \exists M \subseteq X$ with $\gamma(M) \leq \epsilon$ such that, $\forall f \in K$, there exists $D_f \subseteq \Omega$ with $\eta(D_f) \leq \epsilon$ and $f(\Omega \setminus D_f) \subseteq M\}$. In order to prove the admissibility of $L_0(A, X)$ we need the following two lemmas.
Lemma 3. Let $K$ be a bounded subset of $L_0(A, X)$. If $\sigma(K) = 0$, then for all $\varepsilon > 0$ there is $\alpha > 0$ such that

$$\|f - T_uf\|_0 \leq \varepsilon \quad \text{for each } f \in K.$$ 

Proof. Let $\varepsilon > 0$ be given. Since $\sigma(K) = 0$ there is a subset $M$ of $X$, with $\gamma(M) \leq \varepsilon/2$, such that for all $f \in K$ there is $D_f \subseteq \Omega$ with $\eta(D_f) \leq \varepsilon/2$ and $f(\Omega \setminus D_f) \subseteq M$. Fix $y_1, \ldots, y_m \in X$ such that $M \subseteq \bigcup_{j=1}^m (y_j + B_{\varepsilon/2}(X)).$

Then for each $f \in K$ and $t \in \Omega \setminus D_f$ there exists $j \in \{1, \ldots, m\}$ such that $f(t) \in y_j + B_{\varepsilon/2}(X)$. Therefore

$$\|f(t)\|_X \leq \|f(t) - y_j\|_X + \|y_j\|_X \leq \frac{\varepsilon}{2} + \|y_j\|_X.$$ 

Set $\alpha = \varepsilon/2 + \max_j \|y_j\|_X$. Then $f(\Omega \setminus D_f) \subseteq B_\alpha(X)$, which implies $f = T_uf$ on $\Omega \setminus D_f$. Since $\|T_uf\|_X \leq \|f\|_X$, by (1), we have $\|(T_uf)_X D_f\|_0 \leq \|f_X D_f\|_0$. Moreover $\|f_X D_f\|_0 \leq \eta(D_f)$, and thus we find

$$\|f - T_uf\|_0 = \|(f - T_uf)_X D_f\|_0 \leq \|f_X D_f\|_0 + \|(T_uf)_X D_f\|_0 \leq 2\|f_X D_f\|_0 \leq \varepsilon,$$

which gives the result. \qed

Lemma 4. Let $\pi = \{A_1, \ldots, A_n\}$ be a finite partition of $\Omega$. Then the subspace

$$S(\pi) = \left\{ s \in S(A, X) : s = \sum_{i=1}^n x_i \chi_{A_i}, \ x_i \in X \right\}$$

of $L_0(A, X)$ is admissible.

Proof. Let $W$ be a compact subset of $S(\pi)$ and $\varepsilon > 0$ be given. For each $u \in W$ we can write

$$u = \sum_{i=1}^n x_i(\mathbb{X}_{A_i})$$

for suitable elements $x_i(u)$ of $X$. For any fixed $i = 1, \ldots, n$, the set $C_i = \{x_i(u) : u \in W\}$ is a compact subset of $X$, and consequently $C = \bigcup_{i=1}^n C_i$ is also a compact subset of $X$.

Let $\delta = \varepsilon/l$. Then by the admissibility of the Banach space $X$, there exist a finite dimensional space $Z_\varepsilon = \text{span}\{z_1, \ldots, z_m\}$ in $X$ and a continuous mapping $H_\varepsilon: C \to Z_\varepsilon$ such that

$$\|x - H_\varepsilon(x)\|_X \leq \delta \quad \text{for all } x \in C.$$ 

Then for each $i \in \{1, \ldots, n\}$ and for suitable $x_j^i(u) \in X$, with $j = 1, \ldots, m$, we can write

$$H_\varepsilon(x_i(u)) = \sum_{j=1}^m x_j^i(u) z_j.$$
As no confusion can arise, we denote again by $H_\varepsilon$ the continuous mapping $H_\varepsilon : W \to S(\pi)$ defined by

$$H_\varepsilon u = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i(u) x_j(u) \chi_{A_i}.$$ 

Then $H_\varepsilon(W) \subseteq \text{span}[\chi_{A_i}, z_j; \ i = 1, \ldots, n; \ j = 1, \ldots, m]$ and $\dim(\text{span}[H_\varepsilon(W)]) < +\infty$.

On the other hand, for each $u \in W$ we have

$$\|u - H_\varepsilon u\|_0 = \sum_{i=1}^{n} \sum_{j=1}^{m} |x_i(u) - x_j(u)| \chi_{A_i} \leq \sum_{i=1}^{n} \sum_{j=1}^{m} |x_i(u) - x_j(u)| \chi_{A_i}.$$ 

Next by (4) we have

$$\left\| \left( x_i(u) - \sum_{j=1}^{m} x_j(u) z_j \right) \chi_{A_i} \right\|_X \leq \delta \chi_{A_i},$$

hence for a fixed $y \in X$ with $\|y\|_X = 1$ we find

$$\left\| \left( x_i(u) - \sum_{j=1}^{m} x_j(u) z_j \right) \chi_{A_i} \right\|_X \leq \|\delta y \chi_{A_i}\|_X.$$

Consequently

$$\left\| \left( x_i(u) - \sum_{j=1}^{m} x_j(u) z_j \right) \chi_{A_i} \right\|_0 \leq \|\delta y \chi_{A_i}\|_0 = \min\{\eta(A_i), \delta\} \leq \delta.$$

From (5) we get $\|u - H_\varepsilon u\|_0 \leq \varepsilon$ which completes the proof. \hfill $\Box$

Now we are in the position to prove our main result.

**Theorem 1.** The space $L_0(A, X)$ is admissible.

**Proof.** Fix $K$ a compact set in $L_0(A, X)$, and $\varepsilon > 0$. Since $K$ is compact, by [1, Theorem 2.1 and Proposition 2.1] we have $\sigma(K) = 0$. Thus by Lemma 3 there is $a > 0$ such that

$$\|f - T_a f\|_0 \leq \frac{\varepsilon}{3}.$$ 

Next we show that there is a partition $\pi$ of $\Omega$ such that

$$\|g - P_\pi g\|_0 \leq \frac{\varepsilon}{3} \quad \text{for each } g \in T_a(K).$$ 

Let $\delta > 0$ be given. Since by (2) $T_a$ is continuous with respect to $\|\cdot\|_0$, we have that $T_a(K)$ is compact in $(L_1(X), \|\cdot\|_0)$. Moreover by Lemma 2, the $\|\cdot\|_0$-topology and the $\|\cdot\|_1$-topology coincide on $T_a(K)$. So $T_a(K)$ is compact.
in \((L_1(X), \| \cdot \|_1)\). Hence we can choose \(g_1, \ldots, g_n\) in \(T_a(K)\) such that \(T_a(K) \subseteq \bigcup_{i=1}^n (g_i + B_{\delta/3}(X))\). For each fixed \(i = 1, \ldots, n\) let \(s_i\), say \(s_i = \sum_{j=1}^{k_i} x_j \chi_{A_j}\), be a simple function such that \(\|g_i - s_i\|_1 \leq \delta/6\). Set \(\pi(g_i) = \{A_1, \ldots, A_{k_i}\}\), then \(P_{\pi(g_i)}s_i = s_i\), therefore having in mind (3) we find

\[
\|g_i - P_{\pi(g_i)}g_i\|_1 \leq \|g_i - s_i\|_1 + \|P_{\pi(g_i)}s_i - P_{\pi(g_i)}g_i\|_1 \leq \frac{\delta}{3}. 
\]

Denote \(\pi\) the partition generated by all \(\pi(g_i)\) \((i = 1, \ldots, n)\). Let \(g \in T_a(K)\), then there exists \(i \in \{1, \ldots, n\}\) such that \(g = g_i + h\) and \(\|h\|_1 < \frac{\delta}{3}\). Therefore

\[
\|g - P_{\pi}g\|_1 \leq \|g_i - P_{\pi}g_i\|_1 + \|h - P_{\pi}h\|_1 \leq \frac{\delta}{3} + 2\|h\|_1 \leq \delta. 
\]

By Lemma 1, for \(\delta = (\varepsilon/3)^2\) the assert (7) follows.

Assume \(\pi = \{A_1, \ldots, A_l\}\), then the set \(W = P_{\pi}(T_a(K))\) is a compact set included in \(S(\pi) = \left\{s \in S(A, X) : s = \sum_{i=1}^l x_i \chi_{A_i}, \quad x_i \in X\right\}\).

Hence by Lemma 4 there is \(H_x : W \rightarrow L_0(A, X)\) such that \(\text{span}[H_x(W)]\) is finite dimensional and

\[
\|u - H_x u\|_0 \leq \frac{\varepsilon}{3} \quad \text{for each } u \in W. 
\]

Then the continuous mapping \(H : K \rightarrow L_0(A, X)\) defined by \(H = H_x \circ P_{\pi} \circ T_a\) satisfies \(\text{span}[H(K)] \subseteq +\infty\). Moreover by (6), (7) and (8) we have

\[
\|f - Hf\|_0 \leq \|f - T_a f\|_0 + \|T_a f - P_{\pi}T_a f\|_0 + \|P_{\pi}T_a f - Hf\|_0 \leq \varepsilon 
\]

and the admissibility of \(L_0(A, X)\) is proved.

\[\square\]

References


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Diana Caponetti
Dipartimento di Matematica e Informatica
Università di Palermo
90123 Palermo, Italy
E-mail address: diana.caponetti@math.unipa.it

Grzegorz Lewicki
Department of Mathematics and Computer Science
Jagiellonian University
30-348 Krakow, Lojasiewicza 4, Poland
E-mail address: Grzegorz.Lewicki@im.uj.edu.pl

Alessandro Trombetta
Dipartimento di Matematica
Università della Calabria
87036 Arcavacata di Rende, Cosenza, Italy
E-mail address: aletromb@unical.it

Giulio Trombetta
Dipartimento di Matematica
Università della Calabria
87036 Arcavacata di Rende, Cosenza, Italy
E-mail address: trombetta@unical.it