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An Alternative Approach to the Robust Inventory Control Problem

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ABSTRACT

The robust inventory control problem was proposed and solved by Bertsimas and Thiele (2006). Their results are very interesting in that the problem can be solved easily and also the solution possesses nice properties of those found in the traditional stochastic inventory control problem. However, their formulation is shown to be incorrect, which invalidates all of the results given there. In this paper, we propose an alternative formulation of the problem which uses a different but practically applicable uncertainty set. Under the newly proposed model, all of the useful properties given in Bertsimas and Thiele (2006) will be shown to be valid.

Keywords: Robust Optimization, Inventory Control, Inventory Policy

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1. INTRODUCTION

Designing optimal inventory control policy is an important issue in supply chain management and has been studied extensively. The seminal paper by Clark and Scarf (1960) showed that the base-stock policy is optimal for the traditional inventory control problem, which seeks to find the optimal order quantities over a finite planning horizon to minimize the total cost.

Following Bertsekas (1995), the inventory control problem can be defined as follows. For each period (decision epoch) $t \in \{0, 1, \cdots, T-1\}$, a demand w_t is given. In each period, we can order as much as we want and the cost of ordering the quantity $u \geq 0$ is given by $D(u) = K + cu$ when $u > 0$ and $D(0) = 0$. The unit inventory holding cost and the unit penalty cost due to shortage are given by h and p, respectively, where it is assumed that $p \geq h$ holds. For each period t, let x_t be the initial stock level of the period (x_0 is assumed to be given) and u_t the order quantity. Then the following equation holds:

$$x_{t+1} = x_t + u_t - w_t = x_0 + \sum_{k=0}^{t}(u_k - w_k) . \tag{1}$$

For a given order quantity vector $u \in R_+^T$, the total cost can be written as follows:

$$C(u) = \sum_{t=0}^{T-1}[cu_t + K1_{\{u_t>0\}} + \max(hx_{t+1}, -px_{t+1})] . \tag{2}$$

The problem is to find the optimal order quantity which minimizes the cost function (2) subject to the constraints (1). Note that both the initial stock level (1) and the total cost (2) only depend on the cumulative sum of demands up to each period.

Usually there exists inherent uncertainty in the demand. If we know the distribution of the random demand, we can use dynamic programming approach to minimize the expected cost, which is a traditional approach to the optimal inventory control problem. However, if only the limited information is available on the future demand, we cannot use the traditional dynamic programming approach. In addition, finding the optimal inventory control policy using the stochastic dynamic programming requires much computational effort, which makes it impractical to apply in practice.

Robust optimization approach (Ben-Tal *et al*., 2009; Bertsimas and Sim, 2004) provides an alternative way to handle the uncertainty in demand. In brief, the approach finds the robust optimal solution which performs best in the worst case. Instead of the full knowledge on the stochastic nature of the demand, it requires only limited information which is sufficient to define the set of all possible demand realizations (called an uncertainty set).

Bertsimas and Thiele (hereafter, abbreviated as B&T (Bertsimas and Thiele, 2006)) for the first time applied the robust optimization approach to the inventory control problem. They showed that the robust inventory control problem can be formulated as a compact MIP problem and the robust optimal policy can be obtained by solving a deterministic (nominal) inventory control problem. In addition, they presented various interesting properties of the robust optimal inventory policy which are managerially insightful.

However, as pointed out by Bienstock and Özbay (2008), the formulation presented by B&T is only an approximation of the true robust inventory control problem. Hence all the results presented in the paper are not technically correct since they are based on the proposed (inaccurate) formulation. Bienstock and Özbay (2008) presented an exact solution approach to the similar robust inventory control problem (simple base-stock policy model) which requires an iterative procedure. However, in this case, the solution procedure requires rather long computation time and all the interesting properties on the robust optimal policy are lost.

This paper presents an alternative approach to the problem presented by B&T, which not only allows an efficient solution procedure but also preserves all the interesting properties on the robust optimal policy. To this end, we propose an alternative uncertainty set which differs from that used by them. The proposed uncertainty set only requires an estimation of the cumulative sums of demands up to each period, which has practical advantages in many cases. Then we show that the robust inventory problem under the proposed uncertainty set can be formulated as the MIP model presented by B&T, which leads to the conclusion that all the results presented in their paper are valid for the newly proposed uncertainty set.

To elaborate our motivation of the study, let us first briefly review the robust inventory theory presented by B&T.

1.1 Review on the Robust Inventory Theory

To model the uncertainty of the demands, B&T assumed that the demand at each period t can be represented as follows:

$$w_t = \bar{w}_t + \hat{w}_t z_t \text{, where}$$

$$z \in U_{bt} = \{|z_t| \le 1, \textstyle\sum_{i=0}^{t} |z_i| \le \Gamma_t, t = 0, 1, \cdots, T-1\}. \qquad (3)$$

Hence the demand of each period is defined by the nominal value ($\bar{w}_t$) and its possible deviation ($\hat{w}_t z_t$), where z is a member of the uncertainty set U_{bt}. The parameters Γ_t are used to control the conservativeness of the solution, where it is assumed that $\Gamma_t \le \Gamma_{t+1} \le \Gamma_t + 1$.

By letting

$$A_t = \max\{\textstyle\sum_{k=0}^{t} \hat{w}_k z_k \mid \sum_{i=0}^{t} z_k \le \Gamma_t,\ 0 \le z_k \le 1, \text{ for all } k\},$$

B&T showed that the problem can be formulated as follows:

$$\text{(RIP\_bt)} \quad \min \sum_{t=0}^{T-1} (cu_t + Kv_t + y_t)$$

$$y_t \ge h(x_0 + \sum_{k=0}^{t} (u_k - \bar{w}_k) + A_k), \quad t = 0, 1, \cdots, T-1 \qquad (4)$$

$$y_t \ge p(-x_0 - \sum_{i=0}^{t} (u_i - \bar{w}_i) + A_t), \quad t = 0, 1, \cdots, T-1 \qquad (5)$$

$$0 \le u_t \le Mv_t, \quad v_t \in \{0, 1\}, \quad t = 0, 1, \cdots, T-1,$$

where M is a sufficiently large number.

In the above formulation, the variable y_t denotes either the inventory cost or the penalty cost and v_t is a binary variable denoting whether an order is made or not.

The following example shows why the above formulation (RIP_bt) is only an approximation of the true robust inventory control problem.

Example 1: Consider a two-period problem defined by the following parameters:

$$\bar{w}_0 = \bar{w}_1 = 100, \quad \hat{w}_0 = 10, \hat{w}_1 = 20, \quad \Gamma_0 = \Gamma_1 = 1.$$

Hence the possible realization of demand lies in the following set:

$$\{w = (100 + 10z_0, 100 + 20z_1) \mid |z_0| \le 1, |z_0| + |z_1| \le 1\}. \qquad (6)$$

In this case, $A_0 = 10$ and $A_1 = 20$, which correspond to the different realizations in the set (6). It will be shown in Section 2 that the true uncertainty set corresponding to the formulation (RIP_bt) is the following box uncertainty set:

$$\{(w_0, w_1) \mid 90 \le w_0 \le 110, 180 \le w_0 + w_1 \le 220\}. \qquad (7)$$

Note that the above uncertainty set (7) properly contains the initially assumed uncertainty set (6). Hence the (RIP_bt) only gives an approximation of the true robust inventory cost.

1.2 Motivations of the Research

As noted above, the formulation (RIP_bt) is only an approximation of the true robust inventory control problem. To exactly formulate the problem, let us define

$\bar{x}_{t+1} = x_0 + \sum_{k=0}^{t}(u_k - \bar{w}_k)$. Then the exact formulation of the robust inventory problem is as follows:

$$\min_{u\geq 0} \max_{z\in U_{bt}} C(u, z), \tag{8}$$
$$C(u, z) = \sum_{t=0}^{T-1}[cu_t + K1_{\{u_t \geq 0\}}$$
$$+\max(h(\bar{x}_{t+1} - \sum_{k=0}^{t}\hat{w}_t z_t),\ p(\sum_{k=0}^{t}\hat{w}_t z_t - \bar{x}_{t+1}))].$$

Note that the problem (8) is a non-convex optimization problem. In this paper, we present an alternative approach to the robust optimization problem, which not only preserves all the properties of the robust optimal policy presented by B&T but also results in a very efficient computational procedure. The motivation of the research comes from careful observation on the problem (8). Note that in the (true) robust optimization problem (8), we don't need data on the individual demand. Instead, the information on the cumulative demand up to period t (that is, $\sum_{k=0}^{t} w_k$) is sufficient. This has a significant managerial implication, since in many cases the sum of the demands is much easier to estimate than the individual demands.

2. ALTERNATIVE APPROACH TO THE ROBUST INVENTORY PROBLEM

2.1 Uncertainty Set

Let us assume the cumulative demand of periods up to t, D_t, is defined as follows:

$$D_t = \sum_{k=0}^{t} w_k = \bar{W}_t + \xi_t, \quad t = 0, 1, \cdots, T-1. \tag{9}$$

Note that $\bar{W}_t$ denotes the nominal sum (deterministic term) of demands of the periods up to t and ξ_t represents the (cumulative) deviation from the nominal sum. We naturally assume that $\{\bar{W}_t\}_{t=0}^{T-1}$ is a nondecreasing sequence. Further, we assume the vector $\xi = (\xi_t)_{t=0}^{T-1}$ belongs to an uncertainty set U defined as follows:

$$U \equiv \{\xi = (\xi_{0,} \cdots, \xi_{T-1}) \mid \xi_t \in [-A_t, A_t],\ t = 0, 1, \cdots, T-1\}. \tag{10}$$

The sequence $\{A_t\}_{t=0}^{T-1}$ is also assumed to be nondecreasing, which reflects the fact that the level of uncertainty increases as the time horizon expands

There are many alternative methods to define the uncertainty set U. One possible approach is to derive $\{\bar{W}_t\}_{t=0}^{T-1}$ and $\{A_t\}_{t=0}^{T-1}$ by using the B&T's uncertainty set U_{bt}. In this case, we can set $\bar{W}_t = \sum_{i=0}^{t}\bar{w}_i$ and $A_t = \max \{\sum_{i=0}^{t}\hat{w}_i z_i \mid z \in U_{bt}\}$. Another possible approach is to model the cumulative demand as a normal distribution and then use a suitably chosen confidence interval. Specifically if we model the cumulative demand up to period t as a normal distribution with mean μ_t and standard deviation σ_t, then we can set

$$\bar{W}_t = \mu_t, \quad A_t = \beta_t\sigma_t,$$
$$\text{where} \quad \Pr\{\mu_t - \beta_t\sigma_t \leq W_t \leq \mu_t + \beta_t\sigma_t\} = \alpha,$$

for some suitable value of α. This approach may be more useful than the first one when the information on the individual demands is limited.

Usually, it is much easier (and more accurate) to estimate the sum of the demands than the individual demands (Simchi-Levi *et al*., 2007). Also when forecasting the sum of the demands, we can more easily consider the correlations among the individual demands. Thus defining the uncertainty set as above has two practical advantages over the B&T's approach.

(1) The estimation of the uncertainty set can be much easier since we only need the sum of the demands.
(2) If we have some historical information, the estimation can be more accurate since the sum of the demands can absorb fluctuations of individual demands.

One possible critic on the proposed uncertainty set U is that it lacks a mechanism to control conservativeness of the solution. However, we think that the conservativeness can be controlled in many ways. For example, if we use the normal distribution framework, the conservativeness can be controlled by the value of α. Note that increasing the value of α results in more conservativeness.

2.2 Robust Inventory Problem and Its Formulation

Let us define $\bar{x}_{t+1} = x_0 + \sum_{k=0}^{t} u_k - \bar{W}_t$. The robust inventory control problem can be stated as follows:

$$\text{(RIP)} \quad \min_{u\geq 0} \max_{\xi\in U} C(u, \xi), \tag{11}$$

Where $C(u, \xi) = \sum_{t=0}^{T-1}[cu_t + K1_{\{u_t>0\}} + \max(h(\bar{x}_{t+1} - \xi_t),\ p(\xi_t - \bar{x}_{t+1}))]$. As noted in the previous section, the above problem is a non-convex optimization problem. However, we will show that (RIP) can be solved very efficiently.

To show that, first we consider the inner maximization problem in (11). That is, for a given $u \geq 0$,

$$\max_{\xi\in U} C(u, \xi). \tag{12}$$

The problem (12) is to find the worst-case realization for a given feasible solution u. By ignoring the constant terms, it is equivalent to the following problem:

$$\max_{\xi\in U} \sum_{t=0}^{T-1}[\max(h(\bar{x}_{t+1} - \xi_t),\ p(\xi_t - \bar{x}_{t+1}))]. \tag{13}$$

Note that the following relation holds:

$$\max_{\xi\in U} \sum_{t=0}^{T-1}[\max(h(\bar{x}_{t+1} - \xi_t), p(\xi_t - \bar{x}_{t+1}))] \tag{14}$$
$$\leq \sum_{t=0}^{T-1}\max_{\xi\in U}\max[h(\bar{x}_{t+1} - \xi_t), p(\xi_t - \bar{x}_{t+1}))]$$

$$\leq \sum_{t=0}^{T-1} \max[\max_{\xi \in U} h(\bar{x}_{t+1} - \xi_t),\ \max_{\xi \in U} p(\xi_t - \bar{x}_{t+1}))]$$
$$= \sum_{t=0}^{T-1} \max[h(\bar{x}_{t+1} + A_t),\ p(A_t - \bar{x}_{t+1}))].$$

Now, we will show that the above relation (14) holds at equality. For a given $u \geq 0$, let us define an element in the uncertainty set U as follows:

$$\xi(u)_t = \begin{cases} -A_t, \text{ if } \bar{x}_{t+1} \geq \dfrac{p-h}{p+h} A_t \\ A_t, \text{ otherwise} \end{cases}. \tag{15}$$

Then note that for all t, if $\bar{x}_{t+1} \geq \dfrac{p-h}{p+h} A_t$,

$$\max(h(\bar{x}_{t+1} - \xi(u)_t),\ p(\xi(u)_t - \bar{x}_{t+1}))$$
$$= \max(h(\bar{x}_{t+1} + A_t),\ p(-A_t - \bar{x}_{t+1}))$$
$$= h(\bar{x}_{t+1} + A_t) = \max(h(\bar{x}_{t+1} + A_t),\ p(A_t - \bar{x}_{t+1})).$$

Similar result holds for the other case. Hence we have the following relation:

$$\max_{\xi \in U} C(u, \xi) \geq C(u, \xi(u)) \tag{16}$$
$$= \sum_{t=0}^{T-1} [\max(h(\bar{x}_{t+1} - \xi(u)_t,\ p(\xi(u)_t - \bar{x}_{t+1}))]$$
$$= \sum_{t=0}^{T-1} [\max(h(\bar{x}_{t+1} + A_t,\ p(A_t - \bar{x}_{t+1}))],$$

which together with (14) results in the following:

$$\max_{\xi \in U} \sum_{t=0}^{T-1} [\max(h(\bar{x}_{t+1} - \xi_t),\ p(\xi_t - \bar{x}_{t+1}))] \tag{17}$$
$$= \sum_{t=0}^{T-1} \max[h(\bar{x}_{t+1} + A_t),\ p(A_t - \bar{x}_{t+1}))].$$

By using the above result (17), we can reformulate the problem (RIP) as the following MIP problem:

$$\text{(RIP)} \quad \min \sum_{t=0}^{T-1} (cu_t + Kv_t + y_t)$$
$$y_t \geq h(x_0 + \sum_{i=0}^{t} u_i - \bar{W}_t + A_t), \quad \text{for all } t,$$
$$y_t \geq p(-x_0 - \sum_{i=0}^{t} u_i + \bar{W}_t + A_t), \quad \text{for all } t,$$
$$0 \leq u_t \leq Mv_t, \quad \text{for all} \quad t = 0, 1, \cdots, T-1.$$

Note that the above formulation is the exact (not an approximation as in the case B&T) formulation of (RIP). The results are summarized in the following theorem.

Theorem 1: The robust inventory problem (11) with the uncertainty set (9) and (10) can be formulated as an MIP problem (RIP).

Since the formulation (RIP) is essentially the same as (RIP_bt), we can get the following corollary.

Corollary 1: The uncertainty set corresponding to (RIP_bt) is that defined by (9) and (10), where

$$A_t = \max\{\sum_{k=0}^{t} \hat{w}_k z_k \mid \sum_{k=0}^{t} z_k \leq \Gamma_t,\ 0 \leq z_k \leq 1, \text{ for all } k\},$$
$$\bar{W}_t = \sum_{i=0}^{t} \bar{w}_t, \text{ for all } t.$$

3. PROPERTIES OF THE ROBUST OPTIMAL INVENTORY POLICY

Based on the MIP formulation (RIP_bt), B&T showed the following properties of the robust inventory policy.

P1: The optimal policy for the formulation (RIP_bt) corresponds to the optimal policy for the nominal (deterministic) problem with the modified demand defined as follows:

$$w_t' = \bar{w}_t + \frac{p-h}{p+h}(A_t - A_{t-1}), \tag{18}$$

where $A_{-1} \equiv 0$.

P2: The robust optimal policy is (s, S) policy.

Since the formulation (RIP) is essentially the same as (RIP_bt), we can expect that the same results hold for (RIP). However, in the case of our newly defined uncertainty set U, there is no information on the individual demands. To get the result corresponding to P1, we can define the individual (nominal) demand as follows:

$$\bar{w}_t = \bar{W}_t - \bar{W}_{t-1}, \tag{19}$$

where $\bar{W}_{-1} \equiv 0$.

Then by using the same approach to prove the property (P1) used in B&T, we can get the following result, of which proof is omitted here.

Theorem 2: The robust optimal policy for (RIP) is the optimal policy for the nominal problem with the modified demand

$$w_t' = (\bar{W}_t - \bar{W}_{t-1}) + \frac{p-h}{p+h}(A_t - A_{t-1}), \tag{20}$$

where $\bar{W}_{-1} = A_{-1} \equiv 0$.

Hence we can compute the robust optimal ordering quantity by solving a deterministic inventory control problem with the demand given as (20). The corresponding problem can be solved efficiently by a dynamic programming algorithm (see Park and Lee, 2010).

The property (P2) follows from the property (P1), see Bertsimas and Thiele (2006). Hence in our model, it also remains to be valid by Theorem 2.

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Corollary 2: The robust optimal policy is of the form (s, S) policy.

4. CONCLUDING REMARKS

This paper presents an alternative approach to the robust inventory problem first introduced by Bertsimas and Thiele (2006). Specifically, by proposing an alternative uncertainty set, we show that we can correctly formulate the problem and preserve all the useful results presented in their paper.

The proposed uncertainty set only requires the estimation of the cumulative sum of demands. Since it can be much easier and accurate to estimate the sum of demands than the individual demand, the proposed uncertainty set can have practical advantages in many cases. In addition, it results in a very efficient solution procedure which is readily adopted in practice.

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