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Optimal Retirement Time and Consumption/Investment in Anticipation of a Better Investment Opportunity

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ABSTRACT

We investigate an optimal retirement time and consumption/investment policy of a wage earner who expects to find a better investment opportunity after retirement by being freed from other work and participating fully in the financial market. We obtain a closed form solution to the optimization problem by using a dynamic programming method under general time-separable von Neumann-Morgenstern utility. It is optimal for the wage earner to retire from work if and only if his wealth exceeds a certain critical level which is obtained from a free boundary value problem. The wage earner consumes less and takes more risk than he would without anticipation of a better investment opportunity.

Keywords: Retirement time, Consumption, Investment, Utility, Labor income, Investment opportunity

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1. INTRODUCTION

We study an optimal retirement time and consumption/investment policy of an economic agent who is currently a wage earner and seeks to maximize the total expected utility from consumption in an infinite-horizon continuous-time framework. A particular aspect of our problem is that the agent can enlarge his investment opportunity set after retirement. The reason for this aspect is that before retirement he has only limited time and energy available for observing market variables, and therefore, has full information only about the assets in a current investment opportunity set. Hence he manages a portfolio consisting of only the assets in the current investment opportunity set according to the key behavioral assumption as in Merton (1987): *an investor uses a security in constructing his optimal portfolio only if the investor knows about the security*. But after retirement, he can contribute all his time and efforts of labor to gathering full information about more assets in the financial market. For example, studying foreign financial markets or small companies may not be feasible during his time as a laborer, but after retirement he has full freedom and enough time to study these markets.

We obtain closed forms for the optimal retirement policy as well as for the optimal consumption and portfolio policy under a fairly general assumption that the agent has time-separable von Neumann-Morgenstern utility. We show that it is optimal to retire if and only if the agent's wealth exceeds a critical level that is obtained from a free boundary value problem. A wage earner stops his work and becomes a full-time investor as soon as he becomes sufficiently wealthy, an intuitively appealing result. An interesting property of the solution is that the wage earner consumes less and takes more risk if he expects to find a better investment opportunity after retiring from labor than he would if he did not have such an anticipation. Intuitively, he tries to accumulate his wealth fast enough and increase expected growth rate of it to exploit a better investment opportunity by sacrificing his current consumption and taking more risk.

The optimal retirement problem can be investigated in the context of a real option. The economic agent has an option to invest in broader set of assets and the cost associated with exercising the option is his future labor income. He will exercise the option only when the benefit from exercising the option far exceeds the cost. As we said in the above, he will exercise the option only when he is sufficiently rich and the benefit of exploiting a better investment opportunity surpasses the cost of losing his future labor income.

There has been extensive research in consumption and portfolio selection after Merton's pioneering study (Merton, 1969), where closed form solutions are provided under constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) utilities. Karatzas *et al.* (1986) provides a closed form solution for a consumption/ investment problem under general time-separable von Neumann-Morgenstern utility.

There have been also studies on mixed consumption-portfolio-stopping problems. Jeanblac *et al*. (2004) have solved a problem of an agent under obligation to pay a debt at a fixed rate who can declare bankruptcy. Choi and Koo (2005) have studied the effect of a preference change around a discretionary stopping time.

Among the studies on mixed consumption-portfolio-stopping problems, some papers involve retirement time as a control variable. For examples, Choi and Shim (2006) (Farhi and Panageas (2007), resp.) have studied a problem in which a wage earner can choose consumption/investment policies, and the time to retire considering a trade-off between labor income and disutility(leisure, resp.). That is, the agent's motive for retirement in Choi and Shim (2006) is not to suffer from disutility associated with a job and that in Farhi and Panageas (2007) is to enjoy more leisure. In this paper we focus on a rich investor's motive to be freed from other work and devote all his time to management of wealth, that is, to be a full-time investor with a better investment opportunity.

Some papers study the effect of enlargement of the investment opportunity set facing an economic agent. Choi *et al*. (2003) have studied a consumption and investment problem in which the agent's investment opportunity set gets larger if the agent's wealth touches a critical level. The critical wealth level in their model is given exogenously. However, it is endogenously determined as a result of an optimal retirement decision in this paper. Shim (2011) has studied a consumption and investment problem where the agent's investment opportunity set gets enlarged by information gathering for which he is required to pay information cost. Shim (2011) considered only the case where the agent exhibits constant relative aversion (CRRA), while the closed form solution in our model is given under a fairly general assumption that the agent has time-separable von Neumann-Morgenstern utility.

The rest of the paper proceeds as follows. Section 2 sets up the mixture of optimal retirement and optimal consumption/portfolio selection problem. Section 3 characterize a general solution to the problem and Section 4 studies properties of the solution. Section 5 illustrate the special case where the agent has constant relative risk aversion (CRRA) utility. Section 6 concludes.

2. THE OPTIMIZATION PROBLEM

Before stating our problem we state a standard consumption/portfolio selection problem in infinite time horizon which is called *Merton's problem*. Although the existing literature (Merton, 1969, 1971; Karatzas *et al*., 1986) already described and solved *Merton's problem*, we state it again for convenience of exposition in our model:

There are one riskless asset and m risky assets available in the market. That is, the investment opport-unity set consists of these assets. The risk-free rate is a constant $r > 0$ $r > 0$ and the price $p_0(t)$ of the riskless asset follows a deterministic process

$$dp_0(t) = p_0(t)rdt, \quad p_0(0) = p_0.$$

The price $p_j(t)$ of the j-th risky asset follows a geometric Brownian motion

$$dp_j(t) = p_j(t)\{\alpha_j dt + \sum_{k=1}^{m} \sigma_{jk} dw_k(t)\},$$
$$p_j(0) = p_j,\ j = 1, \cdots, m,$$

where $\mathbf{w}(t) = (w_1(t), \cdots, w_m(t))$ is an m-dimensional standard Brownian motion defined on the underlying probability space $(\Omega, \mathrm{F}, \mathrm{P})$. The market parameters, α_j 's and σ_{jk} 's for $j, k = 1, \cdots, m,$ are assumed to be constants. Let $(\mathrm{F}_t)_{t=0}^{\infty}$ be the augmentation under p of the natural filtration generated by the standard Brownian motion $(\mathbf{w}(t))_{t=0}^{\infty}$. We assume that the matrix $D = (\sigma_{ij})_{i,j=1}^{m}$, called the volatility matrix, is nonsingular, i.e., there is no redundant asset among the m risky assets. Hence $\Sigma := DD^{\mathrm{T}}$ and Σ^{-1} are positive definite. Let $\alpha = (\alpha_1, \cdots, \alpha_m)$ be the row vector of returns of the risky assets in the current investment opportunity set and $\mathbf{1}_m = (1, \cdots, 1)$ the row vector of m ones. We assume that $\alpha - r\mathbf{1}_m$ is not the zero vector.

Let $\pi_t = (\pi_{1,t}, \cdots, \pi_{m,t})$ be the row vector of amounts invested in the risky assets at time t and c_t be the consumption rate at time t. The consumption rate process $\mathbf{c} := (c_t)_{t=0}^{\infty}$ is a nonnegative process adapted to $(\mathrm{F}_t)_{t=0}^{\infty}$ and satisfies $\int_0^t c_s ds < \infty$, for all $t \geq 0$, a.s. The portfolio process $\pi := (\pi_t)_{t=0}^{\infty}$ is adapted to $(\mathrm{F}_t)_{t=0}^{\infty}$ and satisfies $\int_0^t \|\pi_s\|^2 ds < \infty$, for all $t \geq 0$, a.s.

The agent's wealth process x_t with initial wealth $x_0 = x \geq 0$ evolves according to

$$dx_t = (\alpha - r\mathbf{1}_m)\,\pi_t^{\mathrm{T}} dt + (rx_t - c_t)dt + \pi_t D d\mathbf{w}^{\mathrm{T}}(t), \ for\ \ t \geq 0. \quad (1)$$

The agent faces the nonnegative wealth constraint

$$x_t \geq 0, \quad for\ all\ t \geq 0\ a.s. \tag{2}$$

A pair of control $(\mathbf{c}, \pi)$ satisfying the constraint (2) is said to be admissible at *x*. Let $A^M(x)$ denote the set of admissible controls at *x*.

Let $R_1 := D^{-1}(\alpha - r\mathbf{1}_m)^{\mathrm{T}}$ be the Sharpe ratio of the investment opportunity set and define the positive constant κ_1 by

$$\kappa_1 := \| R_1 \|^2 / 2 = (\alpha - r\mathbf{1}_m)\Sigma^{-1}(\alpha - r\mathbf{1}_m)^{\mathrm{T}} / 2 > 0,$$

where the inequality comes from the fact that Σ^{-1} is positive definite and $\alpha - r\mathbf{1}_m$ is not the zero vector. The optimization problem (*Merton's problem*) is to maximize the expected total reward

$$V_{(\mathbf{c},\pi)}(x) := \mathrm{E}_x\left[\int_0^\infty \exp(-\beta t)U(c_t)dt\right] \tag{3}$$

over all $(\mathbf{c}, \pi) \in A^M(x)$ where E_x denotes the expectation operator with initial wealth $x_0 = x \geq 0$. The function *U*, called a utility function, is real-valued on $(0, \infty)$, and $\beta > 0$ is a subjective discount rate. We assume that *U* is strictly increasing, strictly concave and three times continuously differentiable on $(0, \infty)$ with $\lim_{c\uparrow\infty} U'(c) = 0$ as in Karatzas *et. al*. (1986) and Choi and Shim (2006).

The quadratic equation of λ

$$\kappa_1\lambda^2 - (r - \beta - \kappa_1)\lambda - r = 0 \tag{4}$$

has two distinct solutions $\lambda_- < -1$ and $\lambda_+ > 0$.
The value function, $V^M(x) = \sup\{V_{(\mathbf{c},\pi)}(x) : (\mathbf{c}, \pi) \in A^M(x)\}$, of *Merton's problem* is finite and attainable by an addmissible strategy for $x > 0$ (See Karatzas *et al*., 1986) if

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} < \infty \quad for\ all\ c > 0. \tag{5}$$

$V^M(0) = U(0)/\beta$ is also attainable where $U(0) := \lim_{c\downarrow 0} U(c)$ which may be $-\infty$.

Remark 2.1: *Note that the value function is uniquely determined by parameters* r, β *and* κ_1. *Therefore, for given* r *and* β, *The value function is uniquely determined by the Sharpe ratio vector* R_1 *or the constant* κ_1 *(See Karatzas et al., 1986).*

Now we turn to our model.

We consider the same economic agent as in the above *Merton's problem* except that he is currently a wage earner(therefore receives labor income) and expects to find a better investment opportunity set after retirement by being freed from other work and participating fully in the financial market.

Before retirement, the agent has full information only about the riskless asset and *m* risky assets described in the above *Merton's problem*. That is, we assume that the current investment opportunity set is the investment opportunity set of the above *Merton's problem*. The agent, before retirement, manages a portfolio consisting of only the assets in the current investment opportunity set according to the key behavioral assumption as in Merton (1987) we mentioned in Section 1.

After retirement, the agent can manage the assets in an enlarged investment opportunity set since he can devote his time and efforts to gathering full information about more risky assets in the market.

The prices of the risky assets in the enlarged investment opportunity set follow geometric Brownian motions with constant market coefficients where the corresponding volatility matrix is invertible as in the current investment opportunity set. Since the market coefficients of the enlarged investment opportunity set are realized after retirement, they are not fully known to the agent until retirement.

Let $\tilde{R}_2$ be the Sharpe ratio vector of the enlarged investment opportunity set and let $\tilde{\kappa}_2 = \left\|\tilde{R}_2\right\|^2/2$. Of course $\tilde{\kappa}_2$ is not fully known to the agent before retirement. However it is assumed to be partially known in the sense that its probability distribution is known to the agent before retirement. Since the enlarged investment opportunity set contains the current investment opportunity set, we have $\mathrm{P}(\tilde{\kappa}_2 \geq \kappa_1) = 1$, which can be proved by using Remark 2.1 in Shim (2011). Since $\tilde{\kappa}_2$ is a function of market parameters in the enlarged investment opportunity set and the current market parameters are constant, it is reasonable to impose the following assumption.

Assumption 2.1: F_∞ *is independent of* $\tilde{\kappa}_2$ *where* $\mathrm{F}_\infty := \sigma(\cup_{t\geq 0}\mathrm{F}_t)$.

Thus the improvement of information about $\tilde{\kappa}_2$ occurs discretely only by the event of retirement which permits the agent to know the realized $\tilde{\kappa}_2$.

Let τ be the retirement time which is an F_t-stopping time and is another control variable. We assume that the random variable $\tilde{\kappa}_2$ is realized to the agent immediately after τ. The agent receives labor income with constant rate $\varepsilon > 0$ until retirement. Since the present value of the future income stream is $\varepsilon / r = \int_0^\infty e^{-rt}\varepsilon\, dt$, the wealth constraint the agent faces before retirement is

$$x_t \geq -\varepsilon / r \quad for \quad 0 \leq t < \tau \quad a.s. \tag{6}$$

After retirement, as in (2), the agent faces the nonnegative wealth constraint

$$x_t \geq 0, \quad for \quad t \geq \tau \quad a.s. \tag{7}$$

In particular, the agent's wealth x_τ at the retirement time τ should satisfy

$$x_\tau \geq 0. \tag{8}$$

Note that after retirement the agent faces the new *Merton's problem* with the enlarged investment opportunity set. The quadratic equation of $\tilde{\eta}$

$$\tilde{\kappa}_2\tilde{\eta}^2 - (r - \beta - \tilde{\kappa}_2)\tilde{\eta} - r = 0 \tag{9}$$

has two distinct solutions $\tilde{\eta}_- < -1$ and $\tilde{\eta}_+ > 0$. We assume (5) about the current opportunity set and the cor-responding condition about the enlarged opportunity set.

Assumption 2.2: *For all* $c > 0$

$$\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} < \infty \text{ and } \mathsf{P}\left\{\int_c^\infty \frac{d\theta}{(U'(\theta))^{\tilde{\eta}_-}} < \infty\right\} = 1.$$

An intuitively obvious fact is that after retirement the optimizing agent will follow the optimal consumption and investment policies of the new *Merton's problem* with the enlarged investment opportunity set. Therefore, we can focus on the retirement time τ and consumption/investment policies $(\mathbf{c}, \pi) := ((c_t)_{t=0}^\tau, (\pi_t)_{t=0}^\tau)$ until τ.

The agent's wealth process $(x_t)_{t=0}^\tau$ until retirement with initial wealth $x_0 = x \geq -\varepsilon/r$ evolves according to

$$dx_t = (\alpha - r\mathbf{1}_m)\pi_t^{\mathrm{T}}dt + (rx_t - c_t + \varepsilon)dt + \pi_t D d\mathbf{w}^{\mathrm{T}}(t). \tag{10}$$

We call a triple of the above control $(\tau, \mathbf{c}, \pi) = (\tau, (c_t)_{t=0}^\tau, (\pi_t)_{t=0}^\tau)$ satisfying (6) and (8) with $x_0 = x \geq -\varepsilon/r$ admissible at *x*. Let $A(x)$ denote the set of admissible controls at *x*. If $x_0 = x = -\varepsilon/r$, the problem becomes trivial. Therefore, we just consider the case where $x_0 = x > -\varepsilon/r$. Letting $\tilde{V}_2(\cdot)$ be the value function of *Merton's problem* with the enlarged opportunity set, our optimization problem is to maximize the expected total reward

$$V_{(\tau,\mathbf{c},\pi)}(x) := \mathsf{E}_x\left[\int_0^\tau \exp(-\beta t)U(c_t)\,dt + \exp(-\beta\tau)\tilde{V}_2(x_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] \tag{11}$$

over all admissible policies $(\tau, \mathbf{c}, \pi) \in A(x)$ with initial wealth $x_0 = x > -\varepsilon/r$. For later use, we let $I(\cdot)$ be the inverse function of $U'(\cdot)$. By Remark 2.1, $\tilde{V}_2(x)$ can be rewritten in a parametric form $\tilde{V}_2(x) = L(x; \tilde{\kappa}_2)$ *for* $x \geq 0$, and we have

$$\begin{aligned}
&\mathsf{E}_x[\exp(-\beta\tau)\tilde{V}_2(x_\tau)\mathbf{1}_{\{\tau<\infty\}}] \\
&= \mathsf{E}_x[\exp(-\beta\tau)L(x_\tau; \tilde{\kappa}_2)\mathbf{1}_{\{\tau<\infty\}}] \\
&= \mathrm{E}_x[\mathrm{E}_x[\exp(-\beta\tau)L(x_\tau; \tilde{\kappa}_2)\mathbf{1}_{\{\tau<\infty\}} \mid \mathrm{F}_\infty]] \\
&= \mathrm{E}_x[\exp(-\beta\tau)\mathbf{1}_{\{\tau<\infty\}}\mathrm{E}_x[L(x_\tau; \tilde{\kappa}_2) \mid \mathrm{F}_\infty]],
\end{aligned}$$

where the second equality comes from the law of iterated expectations (or the tower property) and the third equality from the fact that $\exp(-\beta\tau)\mathbf{1}_{\{\tau<\infty\}}$ is F_∞ measurable. Furthermore, we have

$$\mathrm{E}_x[L(x_\tau; \tilde{\kappa}_2) \mid \mathrm{F}_\infty] = \mathrm{E}_x[L(z; \tilde{\kappa}_2) \mid \mathrm{F}_\infty]|_{z=x_\tau} = \mathrm{E}_x[L(z; \tilde{\kappa}_2)]|_{z=x_\tau},$$

where the first equality holds since x_τ is F_∞ measurable, and the second by Assumption 2.1. Therefore, we have

$$\mathsf{E}_x[\exp(-\beta\tau)\tilde{V}_2(x_\tau)\mathbf{1}_{\{\tau<\infty\}}] = \mathsf{E}_x[\exp(-\beta\tau)\mathbf{1}_{\{\tau<\infty\}}V_2(x_\tau)],$$

where $V_2(\cdot)$ is the function defined by

$$V_2(z) = \mathrm{E}_x[L(z; \tilde{\kappa}_2)] \quad \textit{for } z \geq 0. \tag{12}$$

Note that *z* is a real number and the only involved random variable is $\tilde{\kappa}_2$ when taking the expectation operator E_x in (12). Thus (11) can be rewritten as

$$V_{(\tau,\mathbf{c},\pi)}(x) := \mathsf{E}_x[\int_0^\tau \exp(-\beta t)U(c_t)dt + \exp(-\beta\tau)V_2(x_\tau)\mathbf{1}_{\{\tau<\infty\}}]. \tag{13}$$

As is shown in Karatzas *et al*. (1986), $\tilde{V}_2(z) = L(z; \tilde{\kappa}_2)$ is twice continuously differentiable function of $z > 0$ for each $\tilde{\kappa}_2$ (See Karatzas *et al*., 1986).

Assumption 2.3: *The function* $V_2(z)$ *is twice continuously differentiable for* $z > 0$.

It is obvious that $V_2(z)$ is strictly increasing and strictly concave for $z > 0$ since $\tilde{V}_2(z) = L(z; \tilde{\kappa}_2)$ is strictly increasing and strictly concave function of $z > 0$ for each $\tilde{\kappa}_2$ (See Karatzas *et al*., 1986).

Let $V^*(x)$ be the value function, that is,

$$V^*(x) = \sup\{V_{(\tau,\mathbf{c},\pi)}(x) : (\tau, \mathbf{c}, \pi) \in A(x)\} \quad \textit{for } x > -\varepsilon/r. \tag{14}$$

3. A Solution under a General Utility Class

In this section we solve the problem under general utility class. The Bellman equation for $t < \tau$ and $x > -\varepsilon/r$ is given by

$$\beta V(x) = \max_{c\geq 0,\,\pi}\{(\alpha - r\mathbf{1}_m)\pi^T V'(x) + (rx - c + \varepsilon)V'(x) + \frac{1}{2}\pi\Sigma\pi^T V''(x) + U(c)\}. \tag{15}$$

We first consider the case where $U'(0) = \infty$.

Let

$$X_0(c) = \frac{c}{r} - \frac{1}{\kappa_1(\lambda_+ - \lambda_-)}\left\{\frac{(U'(c))^{\lambda_+}}{\lambda_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}}\right.$$

$$+\frac{(U'(c))^{\lambda_-}}{\lambda_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}\Bigg\} - \frac{\varepsilon}{r}, \quad c > 0 \qquad (16)$$

and let

$$J_0(c) = \frac{U(c)}{\beta} - \frac{1}{\kappa_1(\rho_+ - \rho_-)}\Bigg\{\frac{(U'(c))^{\rho_+}}{\rho_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}}$$
$$+\frac{(U'(c))^{\rho_-}}{\rho_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}\Bigg\}, \quad c > 0 \qquad (17)$$

where $\rho_+ = 1+\lambda_+$ and $\rho_- = 1+\lambda_-$. For $\hat{B} \geq 0$, we define a function $X(c;\hat{B}) = \hat{B}(U'(c))^{\lambda_-} + X_0(c)$ for $c > 0$. We have the following lemma.

Lemma 3.1: *If* $U'(0) = \infty$, *then*

$$\lim_{c\downarrow 0}\frac{U(c)}{U'(c)} = 0, \qquad (18)$$

$$\lim_{c\downarrow 0}(U'(c))^{\lambda_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} = 0 \qquad (19)$$

and

$$\lim_{c\downarrow 0}(U'(c))^{\lambda_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}} = 0. \qquad (20)$$

Proof: When $U(0)$ is finite, (18) trivially holds. When $U(0) = -\infty$, $\limsup_{c\downarrow 0} U(c)/U'(c) \leq 0$. For every $\delta > 0$ and $0 < c < \delta$, $U(c) \geq U(\delta) - U'(c)(\delta - c)$. Therefore,

$$\liminf_{c\downarrow 0} U(c)/U'(c) \geq \liminf_{c\downarrow 0}(U(\delta)/U'(c) - \delta + c) = -\delta.$$

Since $\delta > 0$ is arbitrary, $\liminf_{c\downarrow 0} U(c)/U'(c) \geq 0.$
Hence (18) holds. Since

$$0 \leq \liminf_{c\downarrow 0}(U'(c))^{\lambda_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}}$$
$$\leq \limsup_{c\downarrow 0}(U'(c))^{\lambda_+}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}}$$
$$= \limsup_{c\downarrow 0} c = 0.$$

(19) holds. Finally, since, for every $\delta > 0$,

$$0 \leq \liminf_{c\downarrow 0}(U'(c))^{\lambda_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}$$
$$\leq \limsup_{c\downarrow 0}(U'(c))^{\lambda_-}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}$$
$$\leq \limsup_{c\downarrow 0}\int_c^\delta \Big(\frac{U'(c)}{U'(\theta)}\Big)^{\lambda_-} d\theta + \limsup_{c\downarrow 0}(U'(c))^{\lambda_-}\int_\delta^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}$$
$$\leq \limsup_{c\downarrow 0}(\delta - c)$$
$$= \delta,$$

(20) holds. □

By (19) and (20), we have

$$X(0;\hat{B}) := \lim_{c\downarrow 0} X(c;\hat{B}) = -\varepsilon/r \quad \text{if} \quad U'(0) = \infty.$$

Similarly to (6.11) in Karatzas *et al*. (1986), $\lim_{c\uparrow\infty} X(c;\hat{B}) = \infty$. Using the relation $\lambda_+\lambda_- = -r/\kappa_1$, we have

$$X'(c;\hat{B}) = \lambda_-\hat{B}(U'(c))^{\lambda_- -1}U''(c) -$$
$$\frac{U''(c)}{\kappa_1(\lambda_+ - \lambda_-)}\Bigg\{(U'(c))^{\lambda_+ -1}\int_0^c \frac{d\theta}{(U'(\theta))^{\lambda_+}} + (U'(c))^{\lambda_- -1}\int_c^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}\Bigg\}.$$

Since $U(\cdot)$ is strictly concave, $X'(c;\hat{B}) > 0$ for all $c > 0$. Hence $X(\cdot;\hat{B})$ is strictly increasing and maps $[0,\infty)$ onto $[-\varepsilon/r,\infty)$ so that its inverse function, $C(\cdot;\hat{B})$, exists and is also strictly increasing and maps $[-\varepsilon/r,\infty)$ onto $[0,\infty)$. Define a function $G:(0,\infty)\to R$ by

$$G(z) = \lambda_- V_2'(z)\{X_0(I(V_2'(z))) - z\}$$
$$-\rho_-\{J_0(I(V_2'(z))) - V_2(z)\}. \qquad (21)$$

Assumption 3.1: *There exists* $z^* > 0$ *such that*

$$G(z^*) = 0 \qquad (22)$$
$$\text{and} \quad z^* - X_0(I(V_2'(z^*))) > 0. \qquad (23)$$

Remark 3.1: *As mentioned in Section 2,* $P(\tilde{\kappa}_2 \geq \kappa_1) = 1$. *Assumption 3.1 is equivalent to the assumption that* $P(\tilde{\kappa}_2 > \kappa_1) > 0$ *if the utility function is given in the CRRA class(See Section 5).*

Let

$$\hat{B}^* = (V_2'(z^*))^{-\lambda_-}\{z^* - X_0(I(V_2'(z^*)))\}. \qquad (24)$$

Then, by (23) we have $\hat{B}^* > 0$ and

$$X(I(V_2'(z^*));\hat{B}^*) = z^*. \qquad (25)$$

For $\hat{A} \geq 0$, we define

$$J(c;\hat{A}) = \hat{A}(U'(c))^{\rho_-} + J_0(c) \qquad (26)$$

for $c > 0$.

Now, define a function $V:(-\varepsilon/r,\infty)\to R$ by

$$V(x) = J\Big(C(x;\hat{B}^*);\ \frac{\lambda_-}{\rho_-}\hat{B}^*\Big) \quad for \quad -\frac{\varepsilon}{r} < x < z^*, \qquad (27)$$

and

$$V(x) = V_2(x) \quad for \quad x \geq z^*. \qquad (28)$$

As in Lemma 8.7 of Karatzas *et al*. (1986), we have

$$\lim_{x\downarrow -\frac{\varepsilon}{r}} V(x) = \frac{U(0)}{\beta}. \tag{29}$$

By (25), we have

$$\lim_{x\uparrow z^*} C(x;\hat{B}^*) = C(z^*;\hat{B}^*) = I(V_2'(z^*)), \tag{30}$$

so that

$$\lim_{x\uparrow z^*} V(x) = J(I(V_2'(z^*)); \frac{\lambda_-}{\rho_-}\hat{B}^*). \tag{31}$$

By (27), (29), we have

$$\begin{aligned} V_2(z^*) &= -\frac{\lambda_-}{\rho_-}V_2'(z^*)\{X_0(I(V_2'(z^*))) - z^*\} + J_0(I(V_2'(z^*))) \\ &= \frac{\lambda_-}{\rho_-}\hat{B}^*(V_2'(z^*))^{\rho_-} + J_0(I(V_2'(z^*))) \\ &= J(I(V_2'(z^*)); \frac{\lambda_-}{\rho_-}\hat{B}^*). \end{aligned}$$

Hence by (31), we get

$$\lim_{x\uparrow z^*} V(x) = V_2(z^*). \tag{32}$$

We have the following lemma.

Lemma 3.2: *When* $U'(0)=\infty$, $V(x)$ *defined by (27) and (28) is strictly increasing and strictly concave for* $x > -\varepsilon/r$, *and satisfies the Bellman Equation (15) for* $-\varepsilon/r < x < z^*$.

Proof: By calculation, we have

$$\begin{aligned} \frac{\partial}{\partial x}J(C(x;\hat{B}^*); \frac{\lambda_-}{\rho_-}\hat{B}^*) &= \frac{J'(C(x;\hat{B}^*); \frac{\lambda_-}{\rho_-}\hat{B}^*)}{X'(C(x;\hat{B}^*);\hat{B}^*)} && (33) \\ &= U'(C(x;\hat{B}^*)) && (34) \\ &> 0,\ x > -\varepsilon/r. && (35) \end{aligned}$$

Hence by (32) and the fact that $V_2(\cdot)$ is strictly increasing, $V(x)$ is strictly increasing for $x > -\varepsilon/r$. By (34) and (30) we have

$$\lim_{x\uparrow z^*} V'(x) = U'(I(V_2'(z^*))) = V_2'(z^*) = \lim_{x\downarrow z^*} V'(x). \tag{36}$$

We have

$$V''(x) = U''(C(x;\hat{B}^*))C'(x;\hat{B}^*) < 0 \ \textit{for} - \varepsilon/r < x < z^*. \tag{37}$$

This inequality and (36) imply that $V(\cdot)$ is strictly concave for $x > -\varepsilon/r$ since $V_2(\cdot)$ is also strictly concave. For $-\varepsilon/r < x < z^*$ applying $V(\cdot)$ in the Bellman equation (15) and maximizing over investment in risky assets gives

$$\pi = -\frac{V'(x)}{V''(x)}(\alpha - r\mathbf{1}_m)\Sigma^{-1}.$$

Hence the Bellman equation (15) becomes

$$\beta V(x) = -\kappa_1\frac{(V'(x))^2}{V''(x)} + \max_{c\ge 0}\{(rx - c + \varepsilon)V'(x) + U(c)\}. \tag{38}$$

By (34) and (37), (38) takes the form

$$\begin{aligned} \beta V(x) = &-\kappa_1\frac{(U'(C(x;\hat{B}^*)))^2X'(C(x;\hat{B}^*);\hat{B}^*)}{U''(C(x;\hat{B}^*))} \\ &+(rx - C(x;\hat{B}^*) + \varepsilon)V'(x) + U(C(x;\hat{B}^*)) \end{aligned}$$

for $-\varepsilon/r < x < z^*$, which is equivalent to

$$\begin{aligned} \beta J(c;\frac{\lambda_-}{\rho_-}\hat{B}^*) = &-\kappa_1\frac{(U'(c))^2X'(c;\hat{B}^*)}{U''(c)} \\ &+(rX(c;\hat{B}^*) - c + \varepsilon)U'(c) + U(c) \end{aligned} \tag{39}$$

for $0 < c < I(V_2'(z^*)$ by (30). By calculation and using the relation $\rho_+\rho_- = -\beta/\kappa_1$, (39) can be shown to hold for $0 < c < I(V_2'(z^*))$. Hence $V(\cdot)$ satisfies the Bellman Equation (15) for $-\varepsilon/r < x < z^*$. □

As is shown in (37), $V \in C^2(-\varepsilon/r, z^*)$. By (36) and Assumption 2.3, $V(\cdot) \in C^1(-\varepsilon/r,\infty)\cap C^2((-\varepsilon/r, z^*)\cup(z^*,\infty))$, $\lim_{x\to z^*+}V''(x)$ and $\lim_{x\to z^*-}V''(x)$ exist and finite.

Let's consider the strategy for $t \ge 0$

$$\tau = \infty,\ c_t = C(x_t;\hat{B}^*),\ \pi_t = -\frac{V'(x_t)}{V''(x_t)}(\alpha - r\mathbf{1}_m)\Sigma^{-1}. \tag{40}$$

As in Equation (7.4) in Karatzas *et al*. (1986), the stochastic differential equation for $\{c_t := C(x_t;\hat{B}^*), t \ge 0\}$ becomes

$$dy_t = -(r-\beta)y_t\,dt - y_tR_1^{\mathrm{T}}d\mathbf{w}^{\mathrm{T}}(t), \tag{41}$$

where $y_t := U'(c_t)$. Hence

$$U'(c_t) = y_t = U'(c_0)\exp[-(r-\beta+\kappa_1)t - R_1^{\mathrm{T}}\mathbf{w}^{\mathrm{T}}(t)],$$

so that we get

$$c_t = I(U'(c_0)\exp[-(r-\beta+\kappa_1)t - R_1^{\mathrm{T}}\mathbf{w}^{\mathrm{T}}(t)]),\ t \ge 0.$$

Therefore, if $U'(0)=\infty$, then, with strategy (40),

$$\inf\{t \ge 0 : x_t = -\varepsilon / r\} = \inf\{t \ge 0 : c_t = 0\} \\ = \inf\{t \ge 0 : y_t = \infty\} = \infty,\ a.s. \tag{42}$$

We use the following notation

$$T^{\xi} := \inf\{t \ge 0 : x_t \ge \xi\}. \tag{43}$$

Now we consider the following strategy (τ^*, $\mathbf{c}^*$, π^*):

$$\tau^* = T^{z^*}, \tag{44}$$

$$c_t^* = C(x_t; \hat{B}^*),\ \pi_t^* = -\frac{V'(x_t)}{V''(x_t)}(\alpha - r\mathbf{1}_m)\Sigma^{-1},\ 0 \le t < \tau^*. \tag{45}$$

With this strategy the wealth process does not touch $-\varepsilon / r$ before retirement by (42).

For $c_0^* > 0$ (or equivalently $x > -\varepsilon / r$), let

$$H(c_0^*) := V_{(\tau^*, \mathbf{c}^*, \pi^*)}(x) \\ = \mathrm{E}_x\left[\int_0^{\tau^*} \exp(-\beta t)U(c_t^*)dt + \exp(-\beta\tau^*)V_2(x_{\tau^*})\mathbf{1}_{\{\tau^* < \infty\}}\right]. \tag{46}$$

If $c_0^* \ge I(V_2'(z^*))$ (or equivalently $x \ge z^*$), then $\tau^* = 0$. Thus

$$H(c_0^*) = V_2(x)\ \ for\ c_0^* \ge I(V_2'(z^*))\ (or\ equivalently\ x \ge z^*). \tag{47}$$

For $0 < c_0^* < I(V_2'(z^*))$ (or equivalently $-\varepsilon / r < x < z^*$), (46) can be rewritten as

$$H(c_0^*) = V_{(\tau^*, \mathbf{c}^*, \pi^*)}(x) = \mathrm{E}_x\left[\int_0^{\tau^*} \exp(-\beta t)U(c_t^*)dt \\ + \exp(-\beta\tau^*)V_2(z^*)\right].$$

Note that since $C(\cdot\,; 0)$ is strictly increasing and maps $(-\varepsilon / r, \infty)$ onto $(0, \infty)$, there exists $\hat{x} > -\varepsilon / r$ such that

$$c_0^* = C(x; \hat{B}^*) = C(\hat{x}; 0). \tag{48}$$

When retirement time τ is enforced to be infinite, that is, the agent has no option to retire, it can be shown, as in Karatzas *et al*. (1986), that the optimal consumption strategy $(\hat{c}_t)_{\{t \ge 0\}}$ with initial wealth $\hat{x} > -\varepsilon / r$ satisfies

$$\hat{c}_t = I(U'(C(\hat{x}; 0))\exp[-(r - \beta + \kappa_1)t - R_1^T \mathbf{w}^T(t)])$$

and the value function $V_0(\hat{x}) := \mathrm{E}\left[\int_0^{\infty}\exp(-\beta t)U(\hat{c}_t)\,dt\right]$ in this case is well defined and finite. By (48), $\hat{c}_t = c_t^*$ for all $0 \le t < \tau$. Hence it follows that $H(c_0^*)$ is well-defined and finite for $0 < c_0^* < I(V_2'(z^*))$ (or equivalently $-\varepsilon / r < x < z^*$). Define

$$\Psi(y_0) = H(I(y_0)) = \mathrm{E}_x\left[\int_0^{\tau^*} \exp(-\beta t)U(I(y_t))dt \\ + \exp(-\beta\tau^*)V_2(z^*)\right]$$

for $V_2'(z^*) = U'(I(V_2'(z^*))) < y_0 < U'(0) = \infty$ where $y_t = U'(c_t^*)$ so that y_t satisfies the stochastic differentiable equation (41) for $0 \le t \le \tau^*$ with $y_0 = U'(c_0^*)$. By Theorem 13.16 of Dynkin (1965) (Feynman-Kac formula), Ψ is C^2 on $(V_2'(z^*), \infty)$ and satisfies

$$\beta\Psi(y) = -(r - \beta)y\Psi'(y) + \kappa_1 y^2 \Psi''(y) + U(I(y)) \tag{49}$$

for $V_2'(z^*) < y_0 < \infty$ with $\lim_{y \downarrow V_2'(z^*)}\Psi(y) = V_2(z^*)$. Hence H is C^2 on $(0, I(V_2'(z^*)))$ and satisfies

$$\beta H(c) = -\frac{U'(c)}{U''(c)}\left[r - \beta + \kappa_1\frac{U'(c)U'''(c)}{(U''(c))^2}\right]H'(c) \\ + \kappa_1\left(\frac{U'(c)}{U''(c)}\right)^2 H''(c) + U(c) \tag{50}$$

for $0 < c < I(V_2'(z^*))$ with $\lim_{c \uparrow I(V_2'(z^*))} H(c) = V_2(z^*)$. The general solution to the Equation (50) is

$$A(U'(c))^{\rho_+} + J(c; \hat{A})\ \ for\ \ 0 < c < I(V_2'(z^*)).$$

Hence, for $0 < c < I(V_2'(z^*))$,

$$H(c) = A(U'(c))^{\rho_+} + J(c; \hat{A})$$

for some A and $\hat{A}$ such that

$$\lim_{c \uparrow I(V_2'(z^*))} H(c) = A(U'(I(V_2'(z^*))))^{\rho_+} + J(I(V_2'(z^*)); \hat{A}) = V_2(z^*).$$

As in Theorem 8.8 of Karatzas *et al*. (1986), it is shown that $A = 0$ when $U'(0) = \infty$ so that for $0 < c < I(V_2'(z^*))$, $H(c) = J(c; \hat{A})$ for some $\hat{A}$ such that

$$\lim_{c \uparrow I(V_2'(z^*))} H(c) = J(I(V_2'(z^*)); \hat{A}) = V_2(z^*). \tag{51}$$

Using (22), (25) and (51), we get $\hat{A} = \lambda_- \hat{B}^* / \rho_-$ so that

$$H(c_0^*) = J(c_0^*; \lambda_- \hat{B}^* / \rho_-) = J(C(x; \hat{B}^*); \lambda_- \hat{B}^* / \rho_-) \tag{52}$$

for $0 < c_0^* < I(V_2'(z^*))$ (or equivalently $-\varepsilon / r < x < z^*$). This equality and (47) imply

$$H(c_0^*) = V_{(\tau^*, \mathbf{c}^*, \pi^*)}(x) = V(x)\ \ for\ \ x > -\varepsilon / r. \tag{53}$$

Now we give a solution to the problem when $U'(0) = \infty$ in the following theorem.

Theorem 3.1: *Suppose that* $U'(0) = \infty$. *Assume that*

$$V_2(x) \le V(x) \ for\ 0 \le x < z^*. \tag{54}$$

If

$$\beta V_2(x) \ge \max_{c\ge 0,\ \pi} \Big\{ (\alpha - r\mathbf{1}_m)\ \pi^{\mathrm{T}} V_2'(x) + (rx - c + \varepsilon)V_2'(x) + \frac{1}{2}\pi\Sigma\ \pi^{\mathrm{T}} V_2''(x) + U(c) \Big\} \ for\ x \ge z^*, \tag{55}$$

then the optimal value function is $V(x)$ defined by (27) and (28), and an optimal strategy is given by (44) and (45).

Proof: Fix $x > -\varepsilon / r$. Let $(\tau, \mathbf{c}, \pi) \in A(x)$ be arbitrary. Choose $x < \xi < \infty$ and define $S_n = \inf\left\{t \ge 0 : \int_0^t \|\pi_s\|^2 ds = n\right\}$. With the notation in (43) put $\tau_n = T^\xi \wedge S_n \wedge \tau \wedge n$ so that $\tau_n \uparrow \tau$ as $\xi \uparrow \infty$ and $n \uparrow \infty$. With a $\delta > 0$ let $z_t = x_t + \delta\ for\ t \ge 0$. From the fact that $V(x)$ defined by (27) and (28) satisfies the Bellman equation (15) for $-\varepsilon / r < x < z^*$, by (55), and by using generalized Itô's rule, we get

$$\begin{aligned}
&\mathrm{E}_x\left[\int_0^{\tau_n} \exp(-\beta t) U(c_t)\, dt\right] \\
&\le \mathrm{E}_x\left[\int_0^{\tau_n} \exp(-\beta t)\Big[\beta V(z_t) - (\alpha - r\mathbf{1}_m)\ \pi_t^{\mathrm{T}} V'(z_t) - (rz_t - c_t + \varepsilon)V'(z_t) - \frac{1}{2}\pi_t \Sigma\ \pi_t^{\mathrm{T}} V''(z_t)\Big] dt\right] \\
&= \mathrm{E}_x\left[\int_0^{\tau_n} [-d(\exp(-\beta t)V(z_t)) + \exp(-\beta t)V'(z_t)\ \pi_t D d\mathbf{w}^{\mathrm{T}}(t)\right] \\
&\quad + \mathrm{E}_x\left[\int_0^{\tau_n} - r\delta \exp(-\beta t)V'(z_t)dt\right] \\
&= -\mathrm{E}_x\left[\exp(-\beta\tau_n)V(x_{\tau_n} + \delta)\right] + V(x+\delta) \\
&\quad + \mathrm{E}_x\left[\int_0^{\tau_n} - r\delta \exp(-\beta t)V'(x_t + \delta)dt\right] \\
&\le -\mathrm{E}_x\left[\exp(-\beta\tau_n)V(x_{\tau_n} + \delta)\right] + V(x+\delta),
\end{aligned}$$

where the last inequality comes from the fact that $V'(x) > 0$ for all $x > -\varepsilon / r$. Hence

$$\begin{aligned}
V(x+\delta) &\ge \mathrm{E}_x\left[\int_0^{\tau_n} \exp(-\beta t) U(c_t) dt\right] + \mathrm{E}_x\left[\exp(-\beta\tau_n)V(x_{\tau_n} + \delta)\right] \\
&= \mathrm{E}_x\left[\int_0^{\tau_n} \exp(-\beta t)\ U(c_t)\, dt\right] + \mathrm{E}_x\left[\exp(-\beta\tau_n)V(x_{\tau_n} + \delta)\mathbf{1}_{\{\tau=\infty\}}\right].
\end{aligned}$$

By applying the monotone convergence theorem to

$\mathrm{E}_x\left[\int_0^{\tau_n} \exp(-\beta t) U^{\pm}(c_t)\, dt\right]$, we get

$$\mathrm{E}_x\left[\int_0^{\tau_n} \exp(-\beta t) U(c_t) dt\right] \to \mathrm{E}_x\left[\int_0^{\tau} \exp(-\beta t) U(c_t) dt\right]$$

as $\xi \uparrow \infty$ *and* $n \uparrow \infty$.

Since $V(x_{\tau_n} + \delta) \ge V(-\varepsilon / r + \delta) > -\infty$, by Fatou's lemma,

$$\liminf_{\xi\uparrow\infty, n\uparrow\infty} \mathrm{E}_x\left[\exp(-\beta\tau_n)V(x_{\tau_n} + \delta)\mathbf{1}_{\{\tau<\infty\}}\right] \ge \mathrm{E}_x\left[\exp(-\beta\tau)V(x_\tau + \delta)\mathbf{1}_{\{\tau<\infty\}}\right]$$

and

$$\liminf_{\xi\uparrow\infty, n\uparrow\infty} \mathrm{E}_x\left[\exp(-\beta\tau_n)V(x_{\tau_n} + \delta)\mathbf{1}_{\{\tau=\infty\}}\right] \ge V(-\varepsilon / r + \delta)\mathrm{E}_x\left[\lim_{\xi\uparrow\infty, n\uparrow\infty} \exp(-\beta\tau_n)\mathbf{1}_{\{\tau=\infty\}}\right] = 0.$$

Therefore, we get

$$\begin{aligned}
V(x+\delta) &\ge \mathrm{E}_x\left[\int_0^{\tau} \exp(-\beta t) U(c_t) dt\right] + \mathrm{E}_x\left[\exp(-\beta\tau)V(x_\tau + \delta)\mathbf{1}_{\{\tau<\infty\}}\right] \\
&\ge \mathrm{E}_x\left[\int_0^{\tau} \exp(-\beta t) U(c_t) dt\right] + \mathrm{E}_x\left[\exp(-\beta\tau)V(x_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] \\
&\ge \mathrm{E}_x\left[\int_0^{\tau} \exp(-\beta t) U(c_t) dt\right] + \mathrm{E}_x\left[\exp(-\beta\tau)V_2(x_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] \\
&= V_{(\tau, \mathbf{c}, \pi)}(x),
\end{aligned}$$

where the third inequality comes from (8) and the fact that $V(x) \ge V_2(x)$ for $0 \le x < z^*$ and $V(x) = V_2(x)$ for $x \ge z^*$. Letting $\delta \downarrow 0$ we get $V(x) \ge V_{(\tau, \mathbf{c}, \pi)}(x)$. Since $(\tau, \mathbf{c}, \pi) \in A(x)$ is arbitrary, we get $V(x) \ge V^*(x)$. Since $(\tau^*, \mathbf{c}^*, \pi^*) \in A(x)$ and (53) holds, we have $V(x) = V^*(x)$. □

Remark 3.2: *If the utility function is given in the CRRA class, then the condition (54) holds automatically without further assumption and the condition (55) is equivalent to a restriction on parameter values (See Section 5).*

We now consider the case where $U'(0)$ is finite so that $U(0)$ is also finite. Recall that $I : (0, U'(0)] \to [0, \infty)$ denotes the inverse of U'. We extend I by setting $I \equiv 0$ on $[U'(0), \infty)$. If V is C^2, strictly increasing, and strictly concave, then the Bellman equation (15) for $t < \tau$ becomes, for $x > -\varepsilon / r$,

$$\beta V(x) = -\kappa_1 \frac{(V'(x))^2}{V''(x)} + [rx - I(V'(x)) + \varepsilon]V'(x) + U(I(V'(x))). \tag{56}$$

For $y > 0$, let

$$\mathrm{X}_0(y) := \frac{I(y)}{r} - \frac{1}{\kappa_1(\lambda_+ - \lambda_-)}\left[\frac{y^{\lambda_+}}{\lambda_+}\int_0^{I(y)} \frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{y^{\lambda_-}}{\lambda_-}\int_{I(y)}^{\infty} \frac{d\theta}{(U'(\theta))^{\lambda_-}}\right] - \frac{\varepsilon}{r}, \tag{57}$$

and let

$$\mathrm{J}_0(y) := \frac{U(I(y))}{\beta} - \frac{1}{\kappa_1(\rho_+ - \rho_-)}\left[\frac{y^{\rho_+}}{\rho_+}\int_0^{I(y)}\frac{d\theta}{(U'(\theta))^{\lambda_+}} + \frac{y^{\rho_-}}{\rho_-}\int_{I(y)}^{\infty}\frac{d\theta}{(U'(\theta))^{\lambda_-}}\right]. \quad (58)$$

Define a function $F:(0,\infty)\to R$ by

$$F(z) = \lambda_- V_2'(z)\{\mathrm{X}_0(V_2'(z)) - z\} - \rho_-\{\mathrm{J}_0(V_2'(z)) - V_2(z)\}.$$

Suppose that there exists a $z^* > 0$ such that

$$F(z^*) = 0 \quad (59)$$

and

$$z^* - \mathrm{X}_0(V_2'(z^*)) > 0. \quad (60)$$

Let

$$\hat{B} \equiv (V_2'(z^*))^{-\lambda_-}\left\{z^* - \mathrm{X}_0(V_2'(z^*))\right\}. \quad (61)$$

Then, by (60) we have $\hat{B} > 0.$ With this $\hat{B} > 0,$ we define a function

$$\mathrm{X}(y;\hat{B}) = \hat{B}y^{\lambda_-} + \mathrm{X}_0(y), \quad (62)$$

for $y > 0,$ then we have

$$\mathrm{X}(V_2'(z^*);\hat{B}) = z^* \quad (63)$$

For $c \ge 0,$ we have $c = I(U'(c)),$ hence

$$\mathrm{X}(U'(c);\hat{B}) = X(c;\hat{B}).$$

For $y > 0$ and $y \ne U'(0),$ using the relation $\lambda_+\lambda_- = -r/\kappa_1,$ we get

$$\mathrm{X}'(y) = \hat{B}\lambda_- y^{\lambda_- - 1} - \frac{1}{\kappa_1(\lambda_+ - \lambda_-)}\left[y^{\lambda_+}\int_0^{I(y)}\frac{d\theta}{(U'(\theta))^{\lambda_+}} + y^{\lambda_-}\int_{I(y)}^{\infty}\frac{d\theta}{(U'(\theta))^{\lambda_-}}\right] < 0.$$

Hence $\mathrm{X}(\,\cdot\,;\hat{B})$ is strictly decreasing. Furthermore,

$$\lim_{y\downarrow 0}\mathrm{X}(y;\hat{B}) = \lim_{c\uparrow\infty}\mathrm{X}(U'(c);\hat{B}) = \lim_{c\uparrow\infty}X(c;\hat{B}) = \infty$$

and

$$\lim_{y\uparrow\infty}\mathrm{X}(y;\hat{B}) = \lim_{y\uparrow\infty}\left[\hat{B}y^{\lambda_-} - \frac{1}{\kappa_1(\lambda_+ - \lambda_-)}\frac{y^{\lambda_-}}{\lambda_-}\int_0^{\infty}\frac{d\theta}{(U'(\theta))^{\lambda_-}} - \frac{\varepsilon}{r}\right] = -\varepsilon/r.$$

Therefore $\mathrm{X}(\,\cdot\,;\hat{B})$ maps $(0,\infty)$ onto $(-\varepsilon/r,\infty)$ and has the inverse function $\mathrm{Y}(\,\cdot\,;\hat{B}):(-\varepsilon/r,\infty)\to(0,\infty).$ For $\hat{A}\ge 0,$ we define

$$\mathrm{J}(y;\hat{A}) = \hat{A}y^{\rho_-} + \mathrm{J}_0(y) \;\; for \;\; y > 0. \quad (64)$$

Now, define a function $\mathrm{V}:(-\varepsilon/r,\infty)\to R$ by

$$\mathrm{V}(x) = \mathrm{J}(\mathrm{Y}(x;\hat{B});\lambda_-\hat{B}/\rho_-) \;\; for \;\; -\varepsilon/r < x < z^*, \quad (65)$$

and

$$\mathrm{V}(x) = V_2(x) \;\; for \; x \ge z^*. \quad (66)$$

Then, we have

$$\lim_{x\downarrow -\frac{\varepsilon}{r}}\mathrm{V}(x) = \lim_{y\uparrow\infty}\mathrm{J}(y;\lambda_-\hat{B}/\rho_-) = \lim_{y\uparrow\infty}\left[\frac{\lambda_-}{\rho_-}\hat{B}y^{\rho_-} + \frac{U(0)}{\beta} - \frac{1}{\kappa_1(\rho_+ - \rho_-)}\frac{y^{\rho_-}}{\rho_-}\int_0^{\infty}\frac{d\theta}{(U'(\theta))^{\lambda_-}}\right] = U(0)/\beta.$$

By (63), we have

$$\lim_{x\uparrow z^*}\mathrm{Y}(x;\hat{B}) = \mathrm{Y}(z^*;\hat{B}) = V_2'(z^*),$$

so that

$$\lim_{x\uparrow z^*}\mathrm{V}(x) = \mathrm{J}(V_2'(z^*);\lambda_-\hat{B}/\rho_-).$$

By (59), (61), we have

$$\lim_{x\uparrow z^*}\mathrm{V}(x) = V_2(z^*).$$

We have the following lemma which is similar to Lemma 3.2.

Lemma 3.3: *Suppose that $U'(0) < \infty.$ If there exists a $z^* > 0$ satisfying (59) and (60), then $\mathrm{V}(x)$ defined by (65) and (66)is strictly increasing, strictly concave for $x > -\varepsilon/r$, and satisfies the Bellman equation (15) for $-\varepsilon/r < x < z^*.$*

We get the following theorem which can be proved using an argument similar to that for the case where $U'(0) = \infty.$

Theorem 3.2: *Suppose that $U'(0)$ is finite. Assume that there exists a $z^* > 0$ satisfying (59) and (60) and that $V_2(x) \le V(x)$ for $0 \le x < z^*.$ If*

$$\beta V_2(x) \ge \max_{c\ge 0,\,\pi}\{(\alpha - r\mathbf{1}_m)\,\pi^{\mathrm{T}}V_2'(x) + (rx - c + \varepsilon)V_2'(x)$$

$$+\frac{1}{2}\pi\Sigma\ \pi^{\mathrm{T}}V_2''(x)+U(c)\}\ for\ x\geq z^*,$$

then the optimal value function is $\mathrm{V}(x)$ defined by (65) and (66), and an optimal strategy is given by the following strategy ($\tau^*, \mathbf{c}^*, \pi^*$):

$$\tau^*=T^{z^*},$$

$$c_t^*=I(\mathrm{V}'(x_t),\ \pi_t^*=-\frac{\mathrm{V}'(x_t)}{\mathrm{V}''(x_t)}(\alpha-r\mathbf{1}_m)\Sigma^{-1},\ 0\leq t<\tau^*.$$

4. PROPERTIES OF THE SOLUTION

Now we investigate the properties of optimal policies. If the agent does not have the option to retire from labor, that is, if we restrict τ to be infinite, then the optimal value $V_0(x)$ at x and an optimal strategy are obtained similarly to Karatzas *et al*. (1986) as follows: When $U'(0)=\infty$, $V_0(x)=J_0(C(x;0))$ for $x>-\varepsilon/r$ and an optimal strategy is given by $c_t=C(x_t;0)$, $\pi_t=-\frac{V_0'(x_t)}{V_0''(x_t)}(\alpha-r\mathbf{1}_m)\Sigma^{-1}$ for $t\geq 0$. When $U'(0)<\infty$, $V_0(x)=\mathrm{J}_0(\mathrm{Y}(x;0))$ for $x>-\varepsilon/r$ and an optimal strategy is given by $c_t=I(V_0'(x_t))$, $\pi_t=-\frac{V_0'(x_t)}{V_0''(x_t)}(\alpha-r\mathbf{1}_m)\Sigma^{-1}$ for $t\geq 0$.

The following two propositions illustrate effects of the change of the investment opportunity set resulting from retiring from labor on optimal consumption and portfolio decisions.

The agent consumes less if he expects a better investment opportunity after retiring from labor than he would if he did not have such an option. Intuitively, he tries to accumulate wealth fast enough to exploit a better investment opportunity and sacrifices the current consumption. Proposition 4.1 states this property.

Proposion 4.1: *In Theorem 3.1 which considers the case where* $U'(0)=\infty$, *it holds that*

$$C(x;\hat{B})<C(x;0)\ \ for\ \ -\varepsilon/r<x<z^*. \tag{67}$$

In Theorem 3.2 which treats the case where $U'(0)<\infty$, it holds that if $\mathrm{X}_0(U'(0))<z^*$ then

$$I(\mathrm{V}'(x))=I(V_0'(x))=0\ \ for\ x\leq \mathrm{X}_0(U'(0))$$

and $I(\mathrm{V}'(x))<I(V_0'(x))$ *for* $\mathrm{X}_0(U'(0))<x<z^*$.

Proof: Consider the case where $U'(0)=\infty$: Since $\hat{B}$ is larger than zero, $X(c;\hat{B})>X_0(c)$ for all $c>0$. Hence their inverse functions have the relation $C(x;\hat{B})<C(x;0)$ for all $x>-\varepsilon/r$ since $X(\cdot\,;\hat{B})$ and $X_0(\cdot)$ are increasing functions.

Consider the case where $U'(0)<\infty$: Since $\hat{B}$ is larger than zero, $\mathrm{X}(y;\hat{B})>\mathrm{X}_0(y)$ for all $y>0$. Hence their inverse functions have the relation v for all $x>-\varepsilon/r$ since $\mathrm{X}(\cdot\,;\hat{B})$ and $\mathrm{X}_0(\cdot)$ are decreasing functions. It can be easily checked that $\mathrm{V}'(x)=\mathrm{Y}(x;\hat{B})$ and, as in Karatzas *et al*. (1986), it can be shown that $V_0'(x)=Y(x;0)$. If $x\leq \mathrm{X}_0(U'(0))$, then $\mathrm{Y}(x;\hat{B})>\mathrm{Y}(x;0)\geq U'(0)$. Therefore $I(\mathrm{V}'(x))=I(V_0'(x))=0$ for $x\leq \mathrm{X}_0(U'(0))$ since $I\equiv 0$ on $[U'(0),\infty)$. If $\mathrm{X}_0(U'(0))<x\leq \mathrm{X}(U'(0);\hat{B})$, then $Y(x;0)<U'(0)$ and $\mathrm{Y}(x;\hat{B})\geq U'(0)$. Hence $I(V_0'(x))>0$ and $I(\mathrm{V}'(x))=0$ for $\mathrm{X}_0(U'(0))<x\leq \mathrm{X}(U'(0);\hat{B})$ since $I(y)>0$ for $0<y<U'(0)$ and $I\equiv 0$ on $[U'(0),\infty)$. If $x>\mathrm{X}(U'(0);\hat{B})$, then $0<\mathrm{Y}(x;0)<\mathrm{Y}(x;\hat{B})<U'(0)$. Hence $I(\mathrm{V}'(x))<I(V_0'(x))$ for $x>\mathrm{X}(U'(0);\hat{B})$ since $I(\cdot)$ is strictly decreasing for $0<y<U'(0)$. □

The agent tends to take more risk and thereby tries to increase expected growth rate of his wealth when he expects a better investment opportunity after retiring from labor. This is summarized in Proposition 4.2.

Proposion 4.2: *In Theorem 3.1 which considers the case where* $U'(0)=\infty$, *it holds that*

$$\frac{V'(x)}{-V''(x)}>\frac{V_0'(x)}{-V_0''(x)}\ \ for\ -\varepsilon/r<x<z^*, \tag{68}$$

and in Theorem 3.2 which treats the case where $U'(0)<\infty$, it holds that

$$\frac{\mathrm{VV}'(x)}{-\mathrm{V}''(x)}>\frac{V_0'(x)}{-V_0''(x)}\ for -\varepsilon/r<x<z^*. \tag{69}$$

Proof: We prove only (68), that is, we consider only the case where $U'(0)=\infty$ since (74) is proved similarly.

As in Karatzas *et al*. (1986), it is easily shown that $V_0'(x)=U'(C(x;0))$ for $x>-\varepsilon/r$. Note that $V'(x)=U'(C$ $V'(x)=U'(C(x;\hat{B})$ for $-\varepsilon/r<x<z^*$. We can calculate that

$$\frac{V'(x)}{-V''(x)}=-\lambda_-\left\{x-\frac{C(x;\hat{B})}{r}+\frac{1}{r}(U'(C(x;\hat{B})))^{\lambda_+}\int_0^{C(x;\hat{B})}\frac{d\theta}{(U'(\theta))^{\lambda_+}}\right\}-\frac{\lambda_-\varepsilon}{r}$$

and that

$$\frac{V_0'(x)}{-V_0''(x)}=-\lambda_-\{x-\frac{C(x;0)}{r}+\frac{1}{r}(U'(C(x;0)))^{\lambda_+}\int_0^{C(x;0)}\frac{d\theta}{(U'(\theta))^{\lambda_+}}\}-\frac{\lambda_-\varepsilon}{r}.$$

By differentiation it is easily checked that

$$-\lambda_-\left\{x-\frac{c}{r}+\frac{1}{r}(U'(c))^{\lambda_+}\int_0^c\frac{d\theta}{(U'(\theta))^{\lambda_+}}\right\}$$

is a decreasing function of *c*. Since $C(x;\hat{B})<C(x;0)$, we have $\frac{V'(x)}{-V''(x)}>\frac{V_0'(x)}{-V_0''(x)}$ for $-\varepsilon/r<x<z^*$. □

5. THE SOLUTION UNDER THE CRRA UTILITY CLASS

In this section we consider the case where the utility function is in the CRRA class with CRRA coefficient γ. That is, the utility function is given by

$$U(c)=\frac{c^{1-\gamma}}{1-\gamma}\ for\ c>0\ if\ \ 0<\gamma\neq 1. \tag{70}$$

(The log utility, $U(c)=\log c\ for\ c>0$, corresponds to the case where $\gamma=1$, which we will not consider here since similar results holds in this case).

If the utility function is given by (70), then we can easily check that Assumption 2.2 can be rewritten (as mentioned in Karatzas *et al*. (1986), Koo *et al*. (2003), and Choi and Shim (2006)) as

$$K_1>0\ and\ \mathrm{P}(\tilde{K}_2>0)=1, \tag{71}$$

where $K_1:=r+(\beta-r)/\gamma+(\gamma-1)\kappa_1/\gamma^2$ and $\tilde{K}_2:=r+(\beta-r)/\gamma+(\gamma-1)\tilde{\kappa}_2/\gamma^2$.

Remark 5.1: *As mentioned in Koo et al. (2003) and Choi and Shim (2006),* $K_1>0\Leftrightarrow 1+\gamma\lambda_-<0.$

Assumption 3.1 is equivalent to (see the arguments below)

$$\mathrm{P}(\tilde{\kappa}_2>\kappa_1)>0, \tag{72}$$

which implies that the agent expects a better investment opportunity after retirement (see Remark 2.1 in Shim (2011)): As in Karatzas *et al*. (1986) and Merton (1969),

$$\tilde{V}_2(z)=\frac{\tilde{K}_2^{-\gamma}}{(1-\gamma)}z^{1-\gamma}\ for\ z\geq 0.$$

so that

$$V_2(z)=\mathrm{E}_x\left[\tilde{V}_2(z)\right]=\mathrm{E}_x\left[\tilde{K}_2^{-\gamma}\right]\frac{z^{1-\gamma}}{(1-\gamma)}\ \ for\ \ z\geq 0.$$

By using (71) and (72), we can easily that there exists κ_2 such that $\kappa_2>\kappa_1$, $K_2:=r+(\beta-r)/\gamma+(\gamma-1)\kappa_2/\gamma^2>0$, and $\mathrm{E}_x[\tilde{K}_2^{-\gamma}]=K_2^{-\gamma}$. That is κ_2 is the *certainty equivalent* of $\tilde{\kappa}_2$ in the sense that

$$V_2(z)=\frac{K_2^{-\gamma}}{(1-\gamma)}z^{1-\gamma}.$$

By calculation, we have

$$X_0(c)=\frac{c}{K_1}-\frac{\varepsilon}{r}\ and\ J_0(c)=\frac{1}{(1-\gamma)K_1}c^{1-\gamma},\ for\ c>0.$$

Hence the function $G:(0,\infty)\to R$ defined by (21), in this case, becomes

$$G(z)=-z^{-\gamma}\left\{\frac{(1+\gamma\lambda_-)(K_2-K_1)}{(1-\gamma)K_1}K_2^{-\gamma}z+\frac{\lambda_-\varepsilon}{r}K_2^{-\gamma}\right\}.$$

Let

$$g(z)=\frac{(1+\gamma\lambda_-)(K_2-K_1)}{(1-\gamma)K_1}K_2^{-\gamma}z+\frac{\lambda_-\varepsilon}{r}K_2^{-\gamma},\quad z>0.$$

Then $g(0):=\lim_{z\to 0}g(z)=\frac{\lambda_-\varepsilon}{r}K_2^{-\gamma}<0$ and $\lim_{z\to\infty}g(z)=\infty$. Furthermore *g* is a strictly increasing function. The $z^*>0$ such that $g(z^*)=0$ (equivalently $G(z^*)=0$) becomes

$$z^*=\frac{-\lambda_-\varepsilon(1-\gamma)K_1}{r(1+\gamma\lambda_-)(K_2-K_1)}>0. \tag{73}$$

The constant $\hat{B}^*$ in (29) becomes

$$\hat{B}^*=(K_2z^*)^{\gamma\lambda_-}\frac{1}{K_1}[K_1(z^*+\frac{\varepsilon}{r})-K_2z^*]. \tag{74}$$

Remark 5.2: *$\hat{B}^*$ is positive, that is, we have* $K_2z^*<K_1(z^*+\varepsilon/r)$

Proof: When $0<\gamma<1$ it is clear. When $\gamma>1$, it is easily checked that if $0<z<\frac{K_1}{K_2-K_1}\frac{\varepsilon}{r}$, then $K_2z<K_1(z+\frac{\varepsilon}{r})$ and if $z>\frac{K_1}{K_2-K_1}\frac{\varepsilon}{r}$, then $K_2z>K_1\left(z+\frac{\varepsilon}{r}\right)$. However $g\left(\frac{K_1}{K_2-K_1}\frac{\varepsilon}{r}\right)=\frac{1+\lambda_-}{1-\gamma}K_2^{-\gamma}\frac{\varepsilon}{r}>0$. Hence $0<z^*<\frac{K_1}{K_2-K_1}\frac{\varepsilon}{r}$ so that $K_2z^*<K_1\left(z^*+\frac{\varepsilon}{r}\right)$. □

The function $X(\cdot;\hat{B}^*):(0,\infty)\to(-\varepsilon/r,\infty)$ becomes

$$X(c;\hat{B}^*)=\hat{B}^*c^{-\gamma\lambda_-}+\frac{c}{K_1}-\frac{\varepsilon}{r}. \tag{75}$$

For $\hat{A}\geq 0$, the function $J(\cdot;\hat{A})$ in (26) becomes

$$J(c;\hat{A})=\hat{A}c^{-\gamma\rho_-}+\frac{1}{(1-\gamma)K_1}c^{1-\gamma},$$

and the function $V:(-\varepsilon/r,\infty)\to R$ defined in (27) and (28) becomes

$$V(x)=J(C(x;\hat{B}^*);\frac{\lambda_-}{\rho_-}\hat{B}^*)\quad for\ -\frac{\varepsilon}{r}<x<z^*, \tag{76}$$

and

$$V(x)=\frac{K_2^{-\gamma}}{1-\gamma}x^{1-\gamma}\quad for\ x\geq z^*. \tag{77}$$

Lemma 5.1: *The condition (54) is automatically satisfied if the utility function is given by (70).*

Proof: Since $V_2(z^*)=V(z^*)$, the condition (54) is satisfied if $V_2'(x)\geq V'(x)$ for $0\leq x\leq z^*$, which is equivalent to $I(V_2'(x))\leq I(V'(x))=C(x;\hat{B}^*)$ for $0\leq x\leq z^*$ where the equality comes from (39). Hence it suffices to show that

$X(I(V_2'(x);\hat{B}^*)\leq x$ for $0\leq x\leq z^*$. Let, for $0\leq x\leq z^*$,

$$\begin{aligned}\phi(x)&=X(I(V_2'(x);\hat{B})-x\\&=\hat{B}^*(I(V_2'(x))^{-\gamma\lambda_-}+\frac{I(V_2'(x))}{K_1}-\frac{\varepsilon}{r}-x\\&=(K_2z^*)^{\gamma\lambda_-}\frac{1}{K_1}[K_1(z^*+\frac{\varepsilon}{r})-K_2z^*](K_2x)^{-\gamma\lambda_-}\\&\quad+\frac{K_2x}{K_1}-\frac{\varepsilon}{r}-x\\&=(z^*)^{\gamma\lambda_-}\frac{1}{K_1}[K_1(z^*+\frac{\varepsilon}{r})-K_2z^*](x)^{-\gamma\lambda_-}+\frac{K_2x}{K_1}-\frac{\varepsilon}{r}-x.\end{aligned}$$

Then we have

$$\phi(z^*)=0,\quad \phi(0)=-\varepsilon/r<0, \tag{78}$$

and

$$\phi'(x)=-\gamma\lambda_-(z^*)^{\gamma\lambda_-}\frac{1}{K_1}[K_1(z^*+\frac{\varepsilon}{r})-K_2z^*]x^{-\gamma\lambda_--1}+\frac{K_2}{K_1}-1.$$

By Remarks 5.1 and 5.2, $\phi'(\cdot)$ is an increasing function so that $\phi(\cdot)$ is convex. Therefore, by (78) we have $\phi(x)\leq 0$ for $0\leq x\leq z^*$. □

Lemma 5.2: *A necessary and sufficient condition for (55) is* $1-r/\lambda_-K_1\gamma-r/K_1\geq 0$.

Proof: By calculation we have

$$\begin{aligned}&\beta V_2(x)-\max_{c\geq 0,\,\pi}\{(\alpha-r\mathbf{1}_m)\,\pi^{\mathrm{T}}V_2'(x)\\&+(rx-c+\varepsilon)V_2'(x)+\frac{1}{2}\pi\Sigma\,\pi^{\mathrm{T}}V_2''(x)+U(c)\}\\&=-\frac{\kappa_2-\kappa_1}{\gamma}K_2^{-\gamma}x^{1-\gamma}+\varepsilon K_2^{-\gamma}x^{-\gamma}\quad for\ x\geq z^*.\end{aligned}$$

Therefore the condition (55) is equivalent to $-\frac{\kappa_2-\kappa_1}{\gamma}x+\varepsilon\leq 0$ for $x\geq z^*$, which is again equivalent to $-\frac{\kappa_2-\kappa_1}{\gamma}z^*+\varepsilon\leq 0$ since $\kappa_2>\kappa_1$. By the fact that $g(z^*)=0$, this inequality is equivalent to $1-r/\lambda_-K_1\gamma-r/K\geq 0$. □

Figure 1 compares the consumption rates before retirement for the two cases: (1) τ is enforced to be infinite and (2) the agent has an option to retire. As explained in Proposition 4.1, the figure shows that the wage earner consumes less before touching the critical wealth level in the latter case than in the former case.

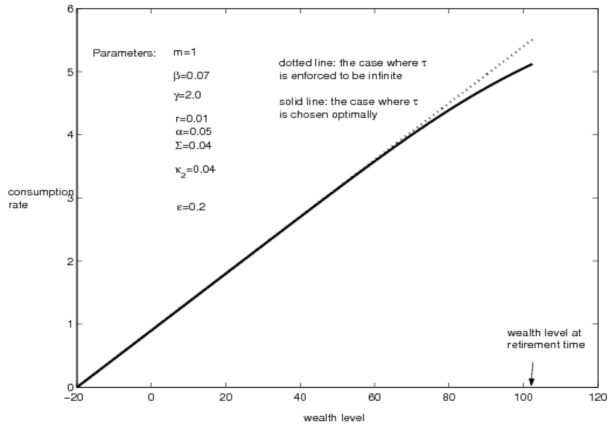

Figure 1. Comparison of consumption rates with a CRRA utility function

Figure 2 compares amount of wealth invested in the risky assets in the two cases. As explained in Proposition 4.2, the figure shows that, before retirement, the agent invests more in the risky assets in the latter case than in the former case.

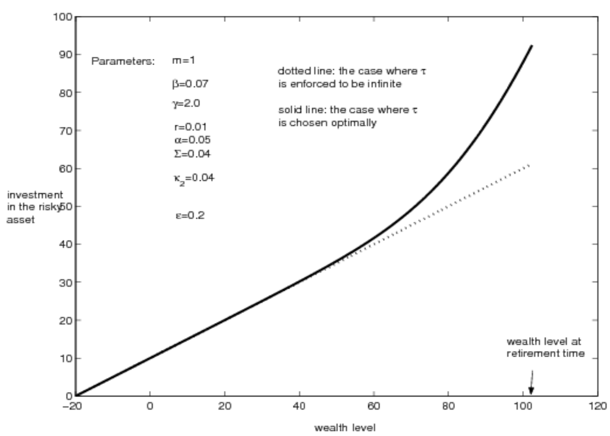

Figure 2. Comparison of amount of wealth invested in the risky asset with a CRRA utility function

6. CONCLUSION

We have studied an optimal retirement, consumption and portfolio selection problem of a wage earner. We have obtained a closed form solution to the optimization problem by using a dynamic programming method under general time-separable von Neumann-Morgenstern utility. We have shown that the wage earner retires from his work and becomes a full-time investor as soon as his wealth exceeds a critical wealth level that is obtained from a free boundary value problem. We have also shown that the agent consumes less and takes more risk if the agent expects a better investment oppor-tunity after retiring from labor than he would if he did not have such an option

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