

UNITARY INTERPOLATION ON $Ax = y$ IN A TRIDIAGONAL ALGEBRA $\text{Alg}\mathcal{L}$

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Abstract. Given vectors x and y in a separable complex Hilbert space \mathcal{H} , an interpolating operator is a bounded operator A such that $Ax = y$. We show the following: Let $\text{Alg}\mathcal{L}$ be a tridiagonal algebra on \mathcal{H} and let $x = (x_i)$ and $y = (y_i)$ be vectors in \mathcal{H} . Then the following are equivalent:

- (1) There exists a unitary operator $A = (a_{ij})$ in $\text{Alg}\mathcal{L}$ such that $Ax = y$.
- (2) There is a bounded sequence $\{\alpha_i\}$ in \mathbb{C} such that $|\alpha_i| = 1$ and $y_i = \alpha_i x_i$ for $i \in \mathbb{N}$.

1. Introduction

Let \mathcal{A} be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on a Hilbert space \mathcal{H} and let x and y be vectors in \mathcal{H} . An *interpolation question* for \mathcal{A} asks for which x and y is there a bounded operator $A \in \mathcal{A}$ such that $Ax = y$. An n -vectors interpolation problem was considered for a C^* -algebra \mathcal{U} by Kadison[8]. In case \mathcal{U} is a nest algebra, the (one-vector) interpolation problem was solved by Lance[9]: his result was extended by Hopenwasser[2] to the case that \mathcal{U} is a CSL-algebra. Munch[10] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[3] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n -vectors, although necessity was not proved in that paper.

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In [1], authors studied the problem of finding A so that $Ax = y$ and A is required to lie in certain ideals contained in $\text{Alg}\mathcal{L}$ (for a nest \mathcal{N}); in particular, they considered the ideal of compact operators, the Jacobson radical, and Larson's ideal \mathcal{R}^∞ .

Roughly speaking, when an operator maps one thing to another, we think of the operator as the interpolating operator and the equation representing the mapping as the interpolation equation. The equations $Ax = y$ and $AX = Y$ are indistinguishable if spoken aloud, but we mean the change to capital letter to indicate that we intend to look at fixed operators X and Y , and ask under what conditions there will exist an operator A satisfying the equation $AX = Y$.

In this paper, we investigate unitary interpolation problems in a tridiagonal algebra $\text{Alg}\mathcal{L}$: Given vectors x and y on a Hilbert space \mathcal{H} , under what conditions there will exist a unitary operator A in a tridiagonal algebra $\text{Alg}\mathcal{L}$ satisfying the equation $Ax = y$.

We establish some notations and conventions. A commutative subspace lattice \mathcal{L} , or CSL \mathcal{L} is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space \mathcal{H} . We assume that the projections 0 and I lie in \mathcal{L} . We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If \mathcal{L} is CSL, $\text{Alg}\mathcal{L}$ is called a CSL-algebra. The symbol $\text{Alg}\mathcal{L}$ is the algebra of all bounded operators on \mathcal{H} that leave invariant all the projections in \mathcal{L} . Let x and y be two vectors in a Hilbert space \mathcal{H} . Then $\langle x, y \rangle$ means an inner product of the vectors x and y . Let M be a subset of a Hilbert space \mathcal{H} . Then \overline{M} means the closure of M and \overline{M}^\perp the orthogonal complement of \overline{M} . An operator U is *unitary* if $UU^* = U^*U = I$, where I is the identity operator acting on \mathcal{H} . Let \mathbb{N} be the set of all natural numbers and let \mathbb{C} be the set of all complex numbers.

2. Results

Let \mathcal{H} be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \dots\}$. Let x_1, x_2, \dots, x_n be vectors in \mathcal{H} . Then $[x_1, x_2, \dots, x_n]$ means the closed subspace generated by the vectors x_1, x_2, \dots, x_n . Let \mathcal{L} be the subspace lattice generated by the subspaces $[e_{2k-1}, e_{2k-1}, e_{2k}, e_{2k+1}]$ ($k = 1, 2, \dots$). Then the algebra $\text{Alg}\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson[1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let \mathcal{A} be the algebra consisting of all bounded operators acting on \mathcal{H} of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & & * & * & * \\ & & & * & \\ & & & & * & \ddots \\ & & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \dots\}$, where all non-starred entries are zero. It is easy to see that $\text{Alg}\mathcal{L} = \mathcal{A}$.

We consider unitary interpolation problems for the above tridiagonal algebra $\text{Alg}\mathcal{L}$.

Lemma 1. *Let $A = (a_{ij})$ be an operator in the tridiagonal algebra $\text{Alg}\mathcal{L}$. Then the following are equivalent:*

- (1) $A = (a_{ij})$ is unitary.
- (2) A is a diagonal operator with $|a_{ii}| = 1$ for i in \mathbb{N} .

Proof. Suppose that $A = (a_{ij})$ is unitary. Since $AA^* = A^*A = I$, $a_{ij} = 0$ for all $i \neq j$ and $|a_{ii}| = 1$. So A is a diagonal operator with $|a_{ii}| = 1$ for i in \mathbb{N} .

The converse is clear. \square

Theorem 2. *Let $\text{Alg}\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space \mathcal{H} and let $x = (x_i)$ and $y = (y_i)$ be vectors in \mathcal{H} . Then the following are equivalent:*

- (1) *There exists a unitary operator $A = (a_{ij})$ in $\text{Alg}\mathcal{L}$ such that $Ax = y$.*
- (2) *There is a bounded sequence $\{\alpha_n\}$ in \mathbb{C} such that $|\alpha_i| = 1$ and $y_i = \alpha_i x_i$ for all $i \in \mathbb{N}$.*

Proof. Suppose that $A = (a_{ij})$ is a unitary operator in $\text{Alg}\mathcal{L}$ such that $Ax = y$. By Lemma 1, A is a diagonal operator with $|a_{ii}| = 1$ for all i in \mathbb{N} . Let $\alpha_i = a_{ii}$ for $i = 1, 2, \dots$. Since $Ax = y$, $y_i = a_{ii}x_i = \alpha_i x_i$ for $i = 1, 2, \dots$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ in \mathbb{C} such that $|\alpha_i| = 1$ and $y_i = \alpha_i x_i$ for $i = 1, 2, \dots$. Let $A = (a_{ii})$ be a diagonal operator with $a_{ii} = \alpha_i$ for each $i \in \mathbb{N}$. Since $\{\alpha_n\}$ is bounded, A is a bounded operator and unitary. Since $y_i = \alpha_i x_i$ for all $i = 1, 2, \dots$, $Ax = y$. \square

Theorem 3. *Let $\text{Alg}\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space \mathcal{H} and let $x_i = (x_j^{(i)})$ and $y_i = (y_j^{(i)})$ be vectors*

in \mathcal{H} for all $i = 1, 2, \dots, n$, where n is a fixed natural number. Then the following are equivalent:

- (1) There exists a unitary operator $A = (a_{kt})$ in $\text{Alg}\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \dots, n$.
- (2) There is a bounded sequence $\{\alpha_j\}$ in \mathbb{C} such that $|\alpha_j| = 1$ and $y_j^{(i)} = \alpha_j x_j^{(i)}$ for all $i = 1, 2, \dots, n$ and $j \in \mathbb{N}$.

Proof. Suppose that $A = (a_{kt})$ is a unitary operator in $\text{Alg}\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \dots, n$. By Lemma 1, A is a diagonal operator with $|a_{kk}| = 1$ for all k in \mathbb{N} . Let $\alpha_k = a_{kk}$ for $k = 1, 2, \dots$. Then $\{\alpha_n\}$ is bounded. Since $Ax_i = y_i$, $y_j^{(i)} = a_{jj}x_j^{(i)} = \alpha_j x_j^{(i)}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ in \mathbb{C} such that $|\alpha_j| = 1$ and $y_j^{(i)} = \alpha_j x_j^{(i)}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots$. Let A be a diagonal operator with diagonal $\{\alpha_j\}$. Since $\{\alpha_n\}$ is bounded, A is a bounded operator. Since $y_j^{(i)} = \alpha_j x_j^{(i)}$, $Ax_i = y_i$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots$. \square

By the similar way with the above, we have the following.

Theorem 4. Let $\text{Alg}\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space \mathcal{H} and let $x_i = (x_j^{(i)})$ and $y_i = (y_j^{(i)})$ be vectors in \mathcal{H} for $i = 1, 2, \dots$. Then the following are equivalent:

- (1) There exists a unitary operator $A = (a_{kt})$ in $\text{Alg}\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \dots$.
- (2) There is a bounded sequence $\{\alpha_j\}$ in \mathbb{C} such that $|\alpha_j| = 1$ and $y_j^{(i)} = \alpha_j x_j^{(i)}$ for all i and j in \mathbb{N} .

Proof. Suppose that $A = (a_{kt})$ is a unitary operator in $\text{Alg}\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \dots$. By Lemma 1, A is a diagonal operator with $|a_{kk}| = 1$ for all k in \mathbb{N} . Let $\alpha_k = a_{kk}$ for $k = 1, 2, \dots$. Then $\{\alpha_n\}$ is bounded. Since $Ax_i = y_i$, $y_j^{(i)} = a_{jj}x_j^{(i)} = \alpha_j x_j^{(i)}$ for $i = 1, 2, \dots$ and $j = 1, 2, \dots$.

Conversely, assume that there is a bounded sequence $\{\alpha_j\}$ in \mathbb{C} such that $|\alpha_j| = 1$ and $y_j^{(i)} = \alpha_j x_j^{(i)}$ for $i = 1, 2, \dots$ and $j = 1, 2, \dots$. Let A be a diagonal operator with diagonal $\{\alpha_j\}$. Since $\{\alpha_n\}$ is bounded, A is a bounded operator. Since $y_j^{(i)} = \alpha_j x_j^{(i)}$, $Ax_i = y_i$ for $i = 1, 2, \dots$ and $j = 1, 2, \dots$. \square

References

- [1] Gilfeather, F. and Larson, D., *Commutants modulo the compact operators of certain CSL algebras*, Operator Theory: Adv. Appl. **2** Birkhauser, Basel, (1981), 105-120.
- [2] Hopenwasser, A., *The equation $Tx = y$ in a reflexive operator algebra*, Indiana University Math. J. **29** (1980), 121-126.
- [3] Hopenwasser, A., *Hilbert-Schmidt interpolation in CSL algebras*, Illinois J. Math. **33**(4) (1989), 657-672.
- [4] Jo, Y. S., *Isometris of Tridiagonal algebras*, Pacific J. Math. **140** (1989), 97-115.
- [5] Jo, Y. S. and Choi, T. Y., *Isomorphisms of $\text{Alg}\mathcal{L}_n$ and $\text{Alg}\mathcal{L}_\infty$* , Michigan Math. J. **37** (1990), 305-314.
- [6] Jo, Y. S., Joo Ho Kang, Park, Dongwan, *Equations $AX = Y$ and $Ax = y$ in $\text{Alg}\mathcal{L}$* , J. Korean Math. Soc. **43** (2006), 399-411.
- [7] Katsoulis, E., Moore, R. L., Trent, T. T., *Interpolation in nest algebras and applications to operator Corona Theorems*, J. Operator Theory **29** (1993), 115-123.
- [8] Kadison, R., *Irreducible Operator Algebras*, Proc. Nat. Acad. Sci. U.S.A. (1957), 273-276.
- [9] Lance, E. C., *Some properties of nest algebras*, Proc. London Math. Soc. **19**(3) (1969), 45-68.
- [10] Munch, N., *Compact causal data interpolation*, J. Math. Anal. Appl. **140** (1989), 407-418.

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