Korean J. Math. **22** (2014), No. 4, pp. 591–598
http://dx.doi.org/10.11568/kjm.2014.22.4.591

FINITELY t-VALUATIVE DOMAINS

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Abstract. Let D be an integral domain with quotient field K. In [1], the authors called D a finitely valuative domain if, for each $0 \neq u \in K$, there is a saturated chain of rings $D = D_0 \subsetneq D_1 \subsetneq \cdots \subseteq D_n = D[x]$, where $x = u$ or u^{-1}. They then studied some properties of finitely valuative domains. For example, they showed that the integral closure of a finitely valuative domain is a Prüfer domain. In this paper, we introduce the notion of finitely t-valuative domains, which is the t-operation analog of finitely valuative domains, and we then generalize some properties of finitely valuative domains.

1. Introduction

Let D be an integral domain with quotient field K. Let R be an *overring* of D, i.e., a ring between D and K. As in [1], we say that R is *within n steps of D* if there is a saturated chain of overrings $D = D_0 \subsetneq D_1 \subsetneq D_1 \subsetneq \cdots \subsetneq D_m = R$ where $m \leq n$. We say that R is *within finitely many steps* of D if R is within n steps of D for some integer $n \geq 1$. An $x \in K$ is said to be *within n steps of D* if $D[x]$ is within n steps of D. An integral domain D is an *n valuative domain* if, for each $0 \neq u \in K$, at least one of u or u^{-1} is within n steps of D, while D is a *finitely valuative domain* if, for each $0 \neq u \in K$, at least one of u or u^{-1} is

Received September 6, 2014. Revised September 16, 2014. Accepted September 16, 2014.

2010 Mathematics Subject Classification: 13A15, 13A18, 13B02, 13G05.

Key words and phrases: Finitely t-valuative domain, PvMD, t-operation, $D[X]_{N_v}$.

This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0007069).

within n steps of D for some integer $n = n(u) \geq 1$. Clearly, an n valuative domain is a finitely valuative domain. In this paper, we introduce the notion of finitely t-valuative domains, which is the t-operation analog of finitely valuative domains, and we then generalize some results of finitely valuative domains.

To facilitate the reading of introduction, we first review the definitions related to the t-operation. Let $\overline{D}$ be the integral closure of D in K, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D. For a polynomial $f \in K[X]$, we denote by $c_D(f)$ (simply, $c(f)$) the fractional ideal of D generated by the coefficients of f. Let $\mathbf{F}(D)$ (resp., $\mathbf{f}(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D; so $\mathbf{f}(D) \subseteq \mathbf{F}(D)$. For $I \in \mathbf{F}(D)$, let $I^{-1} = \{u \in K \mid uI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \cup\{J_v \mid J \in \mathbf{f}(D)$ and $J \subseteq I\}$. Clearly, if $I \in \mathbf{f}(D)$, then $I_v = I_t$. We say that $I \in \mathbf{F}(D)$ is a *t-ideal* if $I_t = I$; a t-ideal is a *maximal t-ideal* if it is maximal among proper integral t-ideals; and t-Max(D) is the set of maximal t-ideals of D. It is well known that each maximal t-ideal is a prime ideal and t-Max$(D) \neq \emptyset$ when D is not a field. An $I \in \mathbf{F}(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$. We say that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. An *upper to zero* in $D[X]$ is a nonzero prime ideal Q of $D[X]$ with $Q \cap D = (0)$. A domain D is called a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t-ideal. It is known that D is a UMT-domain if and only if $\overline{D_P}$ is a Prüfer domain for all $P \in t$-Max(D) [5, Theorem 1.5]. In particular, $\overline{D}$ is a Prüfer domain if and only if D is a UMT-domain whose maximal ideals are t-ideal [4, Theorem 1.1 and Corollary 1.3]. It is also known that D is a PvMD if and only if D is an integrally closed UMT-domain [6, Proposition 3.2]. Recall that D is a GCD-domain if and only if I_v is principal for all $I \in \mathbf{f}(D)$; so GCD-domains are PvMDs. An overring R of D is said to be *t-linked over* D if $I^{-1} = D$ for $I \in \mathbf{f}(D)$ implies $(IR)^{-1} = R$. For an overring R of D, let $R_w = \{x \in K \mid xJ \subseteq R$ for some $J \in \mathbf{f}(D)$ with $J^{-1} = D\}$. It is known that R_w is the smallest t-linked overring of D that contains R [2, Remark 3.3]; hence R is t-linked over D if and only if $R_w = R$. Also, if we let $N_v = \{f \in D[X] \mid c(f)_v = D\}$, then $R[X]_{N_v} \cap K = R_w$, and hence R is t-linked over D if and only if $R[X]_{N_v} \cap K = R$ [2, Lemma 3.2].

Let R be a t-linked overring of D. We say that R is *within t-linked n steps of* D if there is a saturated chain of t-linked overrings $D = D_0 \subsetneq$

$D_1 \subsetneq D_1 \subsetneq \cdots \subsetneq D_m = R$ where $m \leq n$. We say that R is *within t-linked finitely many steps* of D if R is within t-linked n steps of D for some integer $n \geq 1$. We say that a nonzero $u \in K$ is *within t-linked finitely many steps* of D if $(D[u])_w$ is within t-linked finitely many steps of D. We say that D is a *finitely t-valuative domain* if, for each nonzero $u \in K$, at least one of u or u^{-1} is within t-linked finitely many steps of D. Our first result of this paper shows that if there is an integer $n \geq 1$ such that for each $0 \neq u \in K$, at least one of u or u^{-1} is within t-linked n steps of D, then D is an n-valuative domains, which shows why we don't need to define the t-operation analog of n valuative domains. We prove that if D is a finitely t-valuative domain, then D is a UMT-domain, and hence an integrally closed finitely t-valuative domain is a PvMD. It is also shown that (i) Krull domains are finitely t-valuative; (ii) if D is a GCD-domain, then D is finitely t-valuative if and only if $D[X]$ is finitely t-valuative, if and only if $D[X]_{N_v}$ is finitely valuative; and (iii) if D is an integrally closed n valuative domain for an integer $n \geq 1$, then $D[X]$ is a finitely t-valuative domain.

2. Finitely t-valuative domains

Throughout D is an integral domain with quotient field K, X is an indeterminate over D, $D[X]$ is the polynomial ring over D, and $N_v = \{f \in D[X] \mid c(f)_v = D\}$.

Proposition 1. *Let n be a positive integer. If, for each $0 \neq u \in K$, either u or u^{-1} is within t-linked n steps of D, then $|t\text{-Max}(D)| \leq 2n+1$. Hence t-Max(D) = Max(D), the set of maximal ideals of D, and thus D is an n-valuative domain.*

Proof. Assume $|t\text{-Max}(D)| \geq 2n+2$. Let $\{P_i | i = 1, \ldots, 2n+2\}$ be a set of maximal t-ideals of D, and set $S = D \setminus \cup_{i=1}^{2n+2} P_i$. Then $\text{Max}(D_S) = \{P_i D_S | i = 1, \ldots, 2n+2\}$. Let $0 \neq u \in K$, and let $x = u$ or u^{-1}. Note that $(D[x]_w)_S = D[x]_S = D_S[x]$; hence if A is a ring such that $D_S \subseteq A \subseteq D[x]_S$, then $A = (A \cap D[x]_w)_S$ and $A \cap D[x]_w$ is t-linked over D (note that both A and $D[x]_w$ are t-linked over D). Hence, either u or u^{-1} is within n steps of D_S. Thus, D_S is an n-valuative domain, and so by [1, Theorem 2.6], D_S has at most $2n+1$ maximal ideals, a contradiction. Therefore, $|t\text{-Max}(D)| \leq 2n+1$. Moreover, if M is a maximal ideal of D, then $M \subseteq \cup_{P \in t\text{-Max}(D)} P$, and since $|t\text{-Max}(D)| \leq 2n+1$, we have

$M \subsetneq P$ or $M = P$ for some $P \in t\text{-Max}(D)$. Thus, each maximal ideal of D is a t-ideal, which means that $t\text{-Max}(D) = \text{Max}(D)$ and each overring of D is t-linked over D. □

As we prove in Proposition 1, if there is a positive integer n such that, for each $0 \neq u \in K$, either u or u^{-1} is within t-linked n steps of D, then D is an n-valuative domain. So, in this paper, we focus on finitely t-valuative domains. Our next result shows the relationship between finitely valuative domains and finitely t-valuative domains.

Proposition 2. *D is finitely valuative if and only if D is finitely t-valuative and each maximal ideal of D is a t-ideal.*

Proof. Assume that D is finitely valuative. Then the integral closure of D is a Prüfer domain [1, Theorem 3.4], and hence D is a UMT-domain in which each maximal ideal of D is a t-ideal. Moreover, note that if each maximal ideal of D is a t-ideal, then every overring of D is t-linked over D. Thus, D is finitely t-valuative. The converse is clear. □

We next give the finitely t-valuative domain analog of [1, Theorem 3.4] that the integral closure of a finitely valuative domain is a Prüfer domain.

Theorem 3. *If D is a finitely t-valuative domain, then D is a UMT-domain. In particular, an integrally closed finitely t-valuative domain is a PvMD.*

Proof. Let P is a maximal t-ideal of D. It suffices to show that the integral closure of D_P is a Prüfer domain [5, Theorem 1.5]. To show this, let $0 \neq u \in K$. Then at least one of u or u^{-1}, for convenience, say u, is within t-linked finitely many steps of D. Hence there exists a saturated chain of t-linked overrings of D, say, $D = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_n = (D[u])_w$. Clearly, $D_P = (D_0)_P \subsetneq (D_1)_{D\setminus P} \subsetneq \cdots \subsetneq (D_n)_{D\setminus P} = ((D[u])_w)_{D\setminus P} = (D[u])_{D\setminus P} = D_P[u]$ is a chain of overrings of D_P. Let R be a ring such that $(D_i)_{D\setminus P} \subsetneq R \subsetneq (D_{i+1})_{D\setminus P}$. Note that $R = (R \cap D_{i+1})_{D\setminus P}$; $D_i \subseteq R \cap D_{i+1} \subseteq (R \cap D_{i+1})_w \subseteq (D_{i+1})_w = D_{i+1}$; and $(R \cap D_{i+1})_w$ is t-linked over D. Hence, either $(R \cap D_{i+1})_w = D_i$ or $(R \cap D_{i+1})_w = D_{i+1}$, and thus $R = (R \cap D_{i+1})_{D\setminus P} = ((R \cap D_{i+1})_w)_{D\setminus P} = (D_i)_{D\setminus P}$ or $R = ((R \cap D_{i+1})_w)_{D\setminus P} = (D_{i+1})_{D\setminus P}$. Therefore, the chain $D_P = (D_0)_P \subsetneq (D_1)_{D\setminus P} \subsetneq \cdots \subsetneq (D_n)_{D\setminus P}$ is saturated. Hence D_P is a finitely valuative domain, and thus the integral closure of D_P is a Prüfer domain [1, Theorem 3.4]. The "in particular" part follows because an integrally closed UMT-domain is a PvMD. $\square$

By Theorem 3, an integrally closed finitely t-valuative domain is a PvMD. Thus, it is reasonable to study PvMDs that are finitely t-valuative domains. Let $N_v = \{f \in D[X] \mid c(f)_v = D\}$. It is well known that D is a PvMD if and only if $D[X]_{N_v}$ is a Prüfer domain, if and only if each ideal of $D[X]_{N_v}$ is extended from D [7, Theorems 3.1 and 3.7]; in this case, $fD[X]_{N_v} = c(f)D[X]_{N_v}$ for each $f \in D[X]$.

LEMMA 4. *Let D be a PvMD and $\{D_\alpha\}$ be the set of t-linked overrings of D.*

1. *The mapping $D_\alpha \mapsto D_\alpha[X]_{N_{v_\alpha}}$ is a bijection from the set $\{D_\alpha\}$ onto the set of overrings of $D[X]_{N_v}$, where $N_{v_\alpha} = \{f \in D_\alpha[X] \mid c_{D_\alpha}(f)_v = D_\alpha\}$.*
2. *If $0 \neq u \in K$, then u is within t-linked n steps of D if and only if u is within n steps of $D[X]_{N_v}$.*
3. *If $D[X]_{N_v}$ is a finitely valuative domain, then D is a finitely t-valuative domain.*

Proof. (1) This follows directly from [3, Lemma 2 and Corollary 6]. (2) This is an immediate consequence of (1), because $D[u]_w = D[u][X]_{N_v} \cap K$ and $D[u][X]_{N_v} = (D[X]_{N_v})[u]$. (3) This is an immediate consequence of (2). $\square$

We say that D is of *finite character* (resp., *finite t-character*) if each nonzero nonunit of D is contained in a finite number of maximal ideals (resp., maximal t-ideals) of D. The t-dimension of a PvMD D, denoted by t-dim(D), is $\sup\{\text{ht}P \mid P \in t\text{-Max}(D)\}$. It is clear that if D is a Krull domain, then D is a PvMD of t-dim$(D) = 1$ and finite t-character.

COROLLARY 5. *If D is a PvMD of t-dim$(D) < \infty$ and finite t-character, then D is a finitely t-valuative domain. Hence a Krull domain is finitely t-valuative.*

Proof. Clearly, $D[X]_{N_v}$ is a finite dimensional Prüfer domain of finite character, and hence $D[X]_{N_v}$ is a finitely valuative domain [1, Corollary 4.15]. Thus, D is a finitely t-valuative domain by Lemma 4(3). $\square$

Let I be an ideal of D. As in [1], we say that I is *finitely light* if I is contained in finitely many prime ideals of D. Similarly, we say that I is *finitely t-light* if the number of prime t-ideals of D containing I is finite.

Recall that if P is a nonzero prime ideal of a PvMD D, then $P_t \subsetneq D$ if and only if P is a t-ideal; so if $I_t \subsetneq D$, then I is finitely t-light if and only if $ID[X]_{N_v}$ is finitely light.

Corollary 6. *The following are equivalent for an integrally closed domain D.*

1. *D is a finitely t-valuative domain.*
2. *D is a PvMD such that for $0 \neq b, c \in D$, letting $I = bD + cD$, at least one of bI^{-1} or cI^{-1} is finitely t-light.*

Proof. (1) $\Rightarrow$ (2) First, note that D is a PvMD by Theorem 3, and hence $D[X]_{N_v}$ is a Prüfer domain. Let $u = \frac{b}{c}$. Then either u or u^{-1} is within t-linked n steps of D for some integer $n = n(u) \geq 1$, and thus either u or u^{-1} is within n steps of $D[X]_{N_v}$ by Lemma 4(2). Hence, by [1, Corollary 1.15], either $(D[X]_{N_v} :_{D[X]_{N_v}} u) = c \cdot (ID[X]_{N_v})^{-1} = (cI^{-1})D[X]_{N_v}$ or $(D[X]_{N_v} :_{D[X]_{N_v}} u^{-1}) = (bI^{-1})D[X]_{N_v}$ is contained in exactly n primes. Thus, either bI^{-1} or cI^{-1} is contained in exactly n prime t-ideals of D. Hence at least one of bI^{-1} or cI^{-1} is finitely t-light.

(2) $\Rightarrow$ (1) By assumption, $D[X]_{N_v}$ is a Prüfer domain and either $(cI^{-1})D[X]_{N_v}$ or $(bI^{-1})D[X]_{N_v}$ is finitely light. Hence if $u = \frac{b}{c}$, then u or u^{-1} is within finitely many steps of $D[X]_{N_v}$ [1, Lemma 4.4], and so by Lemma 4(2), u or u^{-1} is within t-linked finitely many steps of D. Thus, D is finitely t-valuative. □

It is known that if D is an integrally closed n-valuative domain, then D is a Prüfer domain with at most $2n+1$ maximal ideals [1, Proposition 4.2]. Hence, an integrally closed n-valuative domain is a Bezout domain (and so a GCD-domain). This is why we next study GCD-domains that are finitely t-valuative domains.

Corollary 7. *The following are equivalent for a GCD-domain D.*

1. *D is a finitely t-valuative domain.*
2. *$D[X]_{N_v}$ is a finitely valuative domain.*
3. *$D[X]$ is a finitely t-valuative domain.*
4. *For each pair of t-comaximal elements $a, b \in D$, i.e., $(aD + bD)_t = D$, at least one of a or b is finitely t-light.*
5. *For each pair of t-comaximal finitely generated ideals I and J of D, i.e., $(I + J)_t = D$, at least one of I or J is finitely t-light.*

Proof. (1) $\Rightarrow$ (4) Corollary 6.

(4) $\Leftrightarrow$ (5) This follows because A_t is principal for all nonzero finitely generated ideals A of a GCD-domain and $(I + J)_t = (I_t + J_t)_t$.

(5) $\Rightarrow$ (2) Let $f, g \in D[X]$ be nonzero such that $fD[X]_{N_v} + gD[X]_{N_v} = D[X]_{N_v}$. Then $fD[X]_{N_v} = c(f)D[X]_{N_v}$; $gD[X]_{N_v} = c(g)D[X]_{N_v}$; and $(c(f) + c(g))_t = D$. Hence by (5), at least one of $c(f)$ or $c(g)$ is finitely t-light, and thus either f or g is finitely light. Thus, $D[X]_{N_v}$ is finitely valuative [1, Theorem 4.5].

(2) $\Rightarrow$ (1) Lemma 4(3).

(3) $\Rightarrow$ (4) Note that $a, b \in D$ are t-comaximal in D if and only if a, b are t-comaximal in $D[X]$ and that $P[X]$ is a prime t-ideal of $D[X]$ for all prime t-ideals P of D. Thus, the proof is completed by the equivalence of (1) and (4).

(5) $\Rightarrow$ (3) Let $f, g \in D[X]$ be t-comaximal elements of $D[X]$. Then $c(f)$ and $c(g)$ are t-comaximal finitely generated ideals of D, and hence at least one of $c(f)$ or $c(g)$ is finitely t-light. Note that if Q is a prime t-ideal of $D[X]$, then $Q \cap D = (0)$ or $Q = (Q \cap D)[X]$ and $Q \cap D$ is a prime t-ideal of D (cf. [7, Theorem 3.1] and [6, Theorem 1.4]). Clearly, each nonzero element of $D[X]$ is contained in only finitely many prime t-ideals Q of $D[X]$ with $Q \cap D = (0)$, because $D[X]_{D \setminus \{0\}}$ is a principal ideal domain. Thus, either f or g is finitely t-light. Therefore, $D[X]$ is a finitely t-valuative domain by the equivalence of (1) and (4). □

Corollary 8. *If D is an integrally closed n-valuative domain for some integer $n \geq 1$, then $D[X]$ is a finitely t-valuative domain.*

Proof. Recall from [1, Proposition 4.2] that D is a Bezout domain (hence GCD-domain). Thus, by Corollary 7, $D[X]$ is a finitely t-valuative domain. □

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